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r-well covered graphs of L (+) N

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Abstract

This paper presents an efficiently covered property of the zero-divisor graph of L (+) N, with L a commutative ring with the identity element and N an L–module. we present a complete characterization of when Γ (L (+) N) is well covered in terms of the annihilator structure of N. Moreover, new concept is introduced its r–well covered graphs: as far as a well-covered graph has all maximal independent sets of size r. They can further prove that if L is an integral domain, ann(N) = 0 and if the graph Γ (L (+) N) is well covered in the sense of (|N|– 1) –well covered. All this goes to show that at least a meaningful link must exist between the algebraic structural condition of idealization rings and the corresponding

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graph-theoretic properties of their associated zero-divisor graphs: something quite different from the former studies on clique numbers and independence numbers in these structures.

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1. Introduction

Beck [1] introduced the concept of zero divisors graph (Z. D. G) in 1988; later refined by Anderson and Livingston [2], they have obtained considerable information about the structure of commutative rings (C. R). The more recent study of idealization rings $L(+)N$, where L is a commutative ring (C. R) with unity and N is an L-module, has commanded attention because its algebraic properties and graph-theoretic behavior are so rich. However, a simple module of integers generally has virtually no discernible measure. This concept, first introduced by Plummer [3], has significant theoretical implications for both graph theory and its branches. Some studies on permutations in groups are given [4-6]. Our research broadens the area to include Z. D. G of idealization rings, investigating their well covered properties, thus, extending earlier work by Al-Labadi et. al. [7], [8], and Anderson, Axtell, and Stickles [9] on various graph-theoretic properties of these structures. The ring of idealization $L(+)N$ is the set of elements (r_1, m_1) where $r_1 \in L$ and $m_1 \in N$, with operations defined as follows: Since $(r_1, m_1) + (r_1', m_1') = (r_1 + r_1', m_1 + m_1')$ & $(r_1, m_1)(r_1', m_1') = (r_1 r_1', r_1 m_1' + r_1' m_1)$. Thus the zero-divisor graph of idealization ring is denoted by $\Gamma(L(+)N)$, as its vertices the non-zero zero-divisors of $L(+)N$, two distinct vertices in the graph are adjacent if and only if their product is zero. The result of conducting trials on some well-known Z. D. G of pseudo-QF-rings is extremely significant if one takes this problem seriously [10, 11]. The annihilator of N is the set of all elements in the ring k ∈L such that kn=0 for all n∈ N.

2. Well covered of $\Gamma(L(+)N)$

Let L be an integral domain *(I. D)* and N be an L-module. We study the well covered property of Γ (L(+) N) . A graph is well covered denoted by W_C, if all its maximal independent sets have the same cardinality.

Theorem 2.1: *If L is an I. D and N is an L-module. Then: (1) If* $n(N) = 0$*, then* $\Gamma(L(+)N)$ *is trivially well covered, (2) If* $ann(N) \neq 0$*, then* $\Gamma(L(+)N)$ *is not well covered.*

Proof: If L is an I. D and N is an L-module. Then: (1) If $ann(N) = 0$, then $Z^*(L(+)N) = \{(0, f) : f \in N^*\}$ so, every M_S is a Ma_S, trivially well covered W_C. (2) If $ann(N) \neq 0$, then $Z^*(L(+)N) = \{(0, f), (r, t) : r \in ann(N), f \in N^*, t \in N\}$. Then we construct two maximally independent sets of different sizes such

that $S_1 = \{(0, 1)\}$ and $S_2 = \{(r, 0),\ (r, f): r \in ann(N),\ f \in N^*\}$. Therefore, the graph is not well covered W_C. □

Theorem 2.2: *Let $\Gamma(Z_{n_1}(+) Z_{m_1})$ be the Z. D. G of the idealization ring $Z_{n_1}(+) Z_{m_1}$. Then: (1)If $n_1 = d$ and $m_1 = d$ where d is prime, then $\Gamma(Z_{n_1}(+) Z_{m_1})$ is well covered, (2) If $n_1 = d^2$ and $m_1 = d$ where d is prime, then $\Gamma(Z_{n_1}(+) Z_{m_1})$ is well covered, (3) If $n_1 = d^\alpha$ and $m_1 = d$ where $\alpha > 2$, then $\Gamma(Z_{n_1}(+) Z_{m_1})$ is not well covered, (4) If $n_1 = dq$ and $m_1 = d$ where d and q are primes, then $\Gamma(Z_{n_1}(+) Z_{m_1})$ is not well covered, (5) If $n_1 = d^2q$ and $m_1 = d$ where d and q are primes, then $\Gamma(Z_{n_1}(+) Z_{m_1})$is not well covered, (6) If $n_1 = d^\alpha q$ and $m_1 = d$ where d and q are primes and $\alpha > 2$, then $\Gamma(Z_{n_1}(+) Z_{m_1})$ is not well covered.*

Proof: Let $\Gamma(Z_{n_1}(+) Z_{m_1})$ be the Z. D. G of the idealization ring $Z_{n_1}(+) Z_{m_1}$.Then (1)If n_1 = d and m_1 =d where d is prime, then $Z^*(Z_{n_1}(+) Z_{m_1}) = \{(0, f),\ (kd, t) : f \in Z_d^*, t \in Z_d, gcd(k, d) = 1\}$ all maximal independent sets of order 1, $\Gamma(Z_{n_1}(+) Z_{m_1})$ is well covered, (2) If $n_1 = d^2$ and m_1 = d where d is prime, then $Z^*(Z_{n_1}(+) Z_{m_1}) = \{(0, f), (kd, t): f \in Z_d^*, t \in Z_d$,gcd (k, d) = 1}, all maximal independent sets have order 1, $\Gamma(Z_{n_1}(+) Z_{m_1})$ is well covered, (3) If $n_1 = d^\alpha$ and m_1 = d where $\alpha > 2$, then $Z^*(Z_{n_1}(+) Z_{m_1}) = \{(0, f), (kd^j, t)$: $f \in Z_d^*$, $t \in Z_d$, ,gcd (k, $d^{\alpha-j}$) = 1}, we have two independent set $S_1 = \{(0, 1)\}$ and $S = \{(kd^j, f) : f \in Z_d, 1 \le j \le \lfloor \frac{\alpha-1}{} \rfloor$, gcd (k, $d^{\alpha-j}$)=1}$\cup \left\{\left(d^{\left\lfloor\frac{\alpha}{2}\right\rfloor}, 0\right)\right\}$. Therefore, it $\Gamma(Z_{n_1}(+) Z_{m_1})$ is not well covered, (4) If n_1 = dq and m_1 = d where d and q are primes, then $Z^*(Z_{n_1}(+) Z_{m_1}) = \{(0, f): f \in Z_d^*\} \cup \{(kd, t) : t \in Z_p$, gcd(k, q) = 1}$\cup$\{(kq, t): $t \in Z_d$, gcd(k,d) = 1}, we have mor than two maximal independent sets of different sizes, $S_1 = \{(0, 1)\} \cup \{(kq, t): t \in Z_d$, gcd(k, d) = 1} and S_2 =\{(kd, t):$t \in Z_d$,gcd(k, q) = 1}, so $\Gamma(Z_{n_1}(+) Z_{m_1})$ is not well covered. (5) If $n_1 = d^2q$ and m_1 = d where d and q are primes, then $Z^*(Z_{n_1}(+) Z_{m_1}) = \{(0, f): f \in Z_d^*\} \cup \{(kd, t) : t \in Z_p$, gcd(k, qd) = 1}$\cup$\{(kq, t): t $\in Z_d$, gcd(k, d^2) = 1} $\cup$\{(kdq, t): $t \in Z_d$, gcd(k, d) = 1}$\cup$\{(kd^2, t): $t \in Z_d$, gcd(k, q) = 1}, we have mor than two maximal independent sets of different sizes, S_1 = \{(0, 1)} $\cup$ \{(kq, t): $t \in Z_d$, gcd(k, d) = 1} and S_2 =\{(kd, t): $t \in Z_d$, gcd(k, dq) = 1}$\cup$ \{(kq, t): $t \in Z_d$, gcd(k, d^2) = 1}, then $\Gamma(Z_{n_1}(+) Z_{m_1})$ is not well covered. (6) If $n_1 = d^\alpha q$ and m_1 = d where d and q are primes and $\alpha > 2$, then $Z^*(Z_{n_1}(+) Z_{m_1}) = \{(0, f): f \in Z_d^*\} \cup \{(kd^j, t) : t \in Z_d$, gcd(k, $d^{\alpha-j}$ q) = 1}$\cup$\{(kq, t): $t \in Z_d$, gcd(k, d^α) = 1} $\cup$\{(kd^jq, t): $t \in Z_d$, gcd(k, $d^{\alpha-j}$)=1}, we have mor than two maximal independent sets of different sizes, S_1 = \{(0, 1)} $\cup$ \{(kq, t): $t \in Z_d$, gcd(k, d) = 1} and S_2 =\{$(d^{\alpha-1}q, 0)$}, then $\Gamma(Z_{n_1}(+) Z_{m_1})$ is not well covered.

3. r-well covered of $\Gamma(L(+)N)$:

Here, we define a new notion is called *r*-well covered graph (r-W_C).

Definition 3.1; A graph is called *r*-well covered (r - W_C) if the graph is well covered and the cardinality of the maximal independent set is *r*.

Theorem 3.2: *If L is an I. D and N is an L-module and ann (N) = 0, then* $\Gamma(L(+)N)$ *is* $(|N|-1)$*-well covered.*

Proof: If *L* is an I. D and *N* is an *L*-module, then If *ann* (*N*) = 0, then $Z^*(L(+)N) = \{(0,f) : f \in N^*\}$ so, the number of maximal independent sets, $S_1 = \{(0,f) : f \in N^*\}$, is well covered by $(|N|-1)$.

Theorem 3.3: *If* $n_1 = d \& m_1 = d$ *where d is prime, then* $\Gamma(Z_{n_1}(+)Z_{m_1})$ *is* 1−*well covered.*

Proof: If $n_1 = d$ and $m_1 = d$ where *d* is prime, then $Z^*(Z_{n_1}(+)Z_{m_1}) = \{(0,f) : f \in Z_d^*\}$, all maximal independent sets of order 1, $\Gamma(Z_{n_1}(+)Z_{m_1})$ is 1−well covered.

Theorem 3.4: *If* $n_1 = d^2$ *and* $m_1 = d$ *where d is prime, then* $\Gamma(Z_{n_1}(+)Z_{m_1})$ is 1−well covered.

Proof: If $n_1 = d^2$ and $m_1 = d$ where *d* is prime, then $Z^*(Z_{n_1}(+)Z_{m_1}) = \{(0, f), (kd, t) : f \in Z_d^*, t \in Z_d, \gcd(k,d) = 1\}$, all maximal independent sets have order 1, $\Gamma(Z_{n_1}(+)Z_{m_1})$ is 1−well covered.

Theorem 3.5; *If L is (I.D) and N is an L-module. Then; (1) If ann(N) = 0, then* $g_e(\Gamma(L(+)N)) = 1$*, (2) If ann(N)* $\neq 0$*, then* $g_e(\Gamma(L(+)N)) = |ann(N)||N|$.

Proof: By Theorem 2.1. Then:(1) If *ann*(*N*) = 0, then every M_S is a Ma_S of size 1, $g_e(\Gamma(L(+)N)) = 1$, , see figure (1). (2) If *ann*(*N*) ≠0, then $S = \{(r,m): r \in ann(N), m \in N\}$ is a M_S: (a) Any two vertices (r_1, m_1) and (r_2, m_2) in *S* are not adjacent; the product; $(r_1, m_1)(r_2, m_2) \neq (0, 0)$. Therefore, *S* is an I_S. (b) Any set contains *S* must be form; $S_1 = \{(r,m): r \in ann(N), m \in N\} \cup \{(0,m) : m \in N^*\}$ that is not an I_S. Therefore, *S* is a M_St. (c) Any vertex not in *S* is adjacent to the vertices in *S*. Therefore, $g_e(\Gamma(L(+)N)) = |ann(N)||N|$. See figure (2) when $|N| = 4$ and $|ann(N)| = 1$. We start by given the following theorem to find $g_e(\Gamma(L(+)N))$ when $L = Z_{n_1}, N = Z_{m_1}$.

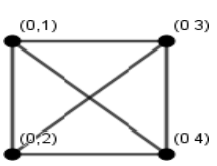


Figure 1
$g_e(\Gamma(Z_5(+)Z_5))=1$

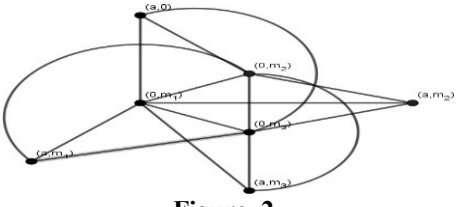


Figure 2
Γ (L (+) N)

Theorem 3.5: *Let Γ ($Z_{n_1}(+)\,Z_{m_1}$) be the Z. D. G of the idealization ring $Z_{n_1}(+)\,Z_{m_1}$. Then:(1) If n_1 = d and m_1 = d where d is prime, then g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) =1.(2) If $n_1 = d^2$ and m_1 = d where d is prime, then g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) =1.(3) If n_1 = d^α and m_1 = d where α > 2, then g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) = $(d\sum_{j=1}^{\frac{\alpha-1}{2}}\phi(d^{\alpha-j}))+1$. (4) If n_1 = dq and m_1 = d where d and q are primes $d < q$, then g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) = $(d-1)\phi(q)+d\phi(d)$. (5) If n_1 =d^2q and m_1 = d where d and q are primes, then g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) = $(d-1)\phi(d)\phi(q)+d\phi(d^2)+d\phi(q)$. (6) If n_1 =$d^\alpha q$ and m_1 = d where d and q are primes and α > 2, then g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) ≥ $d\phi(d^{\alpha-j})\phi(q)+d\phi(d^\alpha)$.*

Proof: By Theorem 2.2. Then by the same way it is held (1) If n_1 = *d* and m_1 =d where *d* is prime, then $g_e(\Gamma(L(+)N))=1$. (2) If $n_1 = d^2$ and $m_1 = d$ where *d* is prime, then g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) =1. (3) Let S = {(kd^j, f) : $f \in Z_d$, $1 \le j \le \lfloor \frac{\alpha-1}{2} \rfloor$, gcd(k, $d^{\alpha-j}$)=1}∪ $\left\{\left(d^{\lfloor\frac{\alpha}{2}\rfloor}, 0\right)\right\}$ is a maximum independent set: (a) Any two vertices (r_1, m_1) and (r_2, m_2) in *S* are not adjacent; the product; (r_1, m_1) (r_2, m_2) ≠ (0, 0). So, *S* is an independent set. (b)Any set contains *S* must be form; S_1 = {$(0, f)$: $f \in Z_d^*$}∪{(kd^j, f): $f \in Z_d$, $j \ge \lfloor\frac{\alpha}{2}\rfloor$} that is not an I_S. So, *S* is a Ma_S. (c) Any vertex not in *S* is adjacent to the vertices in *S*. So, g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) = $(d\sum_{j=1}^{\frac{\alpha-1}{2}}\phi(d^{\alpha-j}))+1$. (3) If n_1 = *dq* and m_1 = *d* where *d* and *q* are primes, $d < q$, then Z^*($Z_{n_1}(+)\,Z_{m_1}$) = {$(0, f)$: $f \in Z_d^*$}∪ {(kd, t) : $t \in Z_p$, $gcd(k, q) = 1$}∪{(kq, t): $t \in Z_d$, $gcd(k, d) = 1$}. Let S = {(kd, f): $f \in Z_d^*$, gcd(k, q)=1}∪ {(kq, t): $t \in Z_d, gcd(k, d) = 1$} is a maximum independent set:(a) Any two vertices (r_1, m_1) and (r_2, m_2) in *S* are not adjacent; the product; (r_1, m_1) (r_2, m_2) ≠ (0, 0). So, *S* is an independent set. (b)Any set contains *S* must be form; S_1 = {$(0, f)$: $f \in Z_d^*$}∪{$(kd, 0)$, gcd(k,q)=1}∪S that is not an I_S. So, *S* is a Ma_S. (c) Any vertex not in *S* is adjacent to the vertices in *S*. So, g_e((Γ ($Z_{n_1}(+)\,Z_{m_1}$)) = $(d-1)\phi(q)+d\phi(d)$. (4) Let S = {(kd, t) : $t \in Z_d^*$, $gcd(k,d\,q) = 1$}∪{(kq, t): $t \in Z_d$, $gcd(k, d^2) = 1$} ∪{(kd^2, t): $t \in Z_d$, $gcd(k, q) = 1$} is a maximum independent set: By the same way it is held. So S is Ma_S, g_e((Γ (Γ($Z_{n1}(+)Z_{m1}$))) = (d-1)$\phi(d)\phi(q)+d\phi(d^2)+d\phi(q)$. (5) Let S = {$(kd^j, t)$: $t \in Z_d$, $gcd(k, d^{\alpha-j}\,q) = 1$, $1 \le j \le \alpha$}∪{(kq, t): $t \in Z_d$, $gcd(k, d^\alpha) = 1$} is a maximal independent set: By the same way it is held *S* is a M_S. So, $g_e(\Gamma(Z_{n1}(+)Z_{m1})) \ge d\phi(d^{\alpha-j})\phi(q)+d\phi(d^\alpha)$.

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