Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
DOI : 10.47974/JDMSC-2321

r-well covered graphs of L (+) N

Manal Al-Labadi [§]
Department of Mathematics
Faculty of Arts and Sciences
University of Petra
Amman
Jordan

Eman Almuhur [†]
Department of Mathematics
Faculty of Science
Applied Science Private University
Amman
Jordan

Shuker Khalil*
Department of Mathematics
College of Science
University of Basrah
Basrah
Iraq

Abstract

This paper presents an efficiently covered property of the zero-divisor graph of L (+) N, with L a commutative ring with the identity element and N an L−module. we present a complete characterization of when Γ (L (+) N) is well covered in terms of the annihilator structure of N. Moreover, new concept is introduced its r−well covered graphs: as far as a well-covered graph has all maximal independent sets of size r. They can further prove that if L is an integral domain, ann(N) = 0 and if the graph Γ (L (+) N) is well covered in the sense of (|N|− 1) −well covered. All this goes to show that at least a meaningful link must exist between the algebraic structural condition of idealization rings and the corresponding

[§] *E-mail:* **manal.allabadi@uop.edu.jo**
[†] *E-mail:* **almuhur@asu.edu.jo**
* *E-mail:* **shuker.khalil@uobasrah.edu.iq (Corresponding Author)**

graph-theoretic properties of their associated zero-divisor graphs: something quite different from the former studies on clique numbers and independence numbers in these structures.

Subject Classification: *Primary 05Cxx, Secondary 05C38.*

Keywords: *Mathematical modeling, Ordinary differential equations, A duopoly economy, Market share.*

1. Introduction

Beck [1] introduced the concept of zero divisors graph (Z. D. G) in 1988; later refined by Anderson and Livingston [2], they have obtained considerable information about the structure of commutative rings (C. R). The more recent study of idealization rings $L\ (+)\ N$, where L is a commutative ring (C. R) with unity and N is an L-module, has commanded attention because its algebraic properties and graph-theoretic behavior are so rich. However, a simple module of integers generally has virtually no discernible measure. This concept, first introduced by Plummer [3], has significant theoretical implications for both graph theory and its branches. Some studies on permutations in groups are given [4-6]. Our research broadens the area to include Z. D. G of idealization rings, investigating their well covered properties, thus, extending earlier work by Al-Labadi et. al. [7], [8], and Anderson, Axtell, and Stickles [9] on various graph-theoretic properties of these structures. The ring of idealization $L(+)N$ is the set of elements (r_1, m_1) where $r_1 \in L$ and $m_1 \in N$, with operations defined as follows: Since $(r_1, m_1) + (r_1', m_1') = (r_1 + r_1', m_1 + m_1')$ & $(r_1, m_1)(r_1', m_1') = (r_1 r_1', r_1 m_1' + r_1' m_1)$. Thus the zero-divisor graph of idealization ring is denoted by $\Gamma(L(+)N)$, as its vertices the non-zero zero-divisors of $L(+)N$, two distinct vertices in the graph are adjacent if and only if their product is zero. The result of conducting trials on some well-known Z. D. G of pseudo-QF-rings is extremely significant if one takes this problem seriously [10, 11]. The annihilator of N is the set of all elements in the ring k ∈L such that kn=0 for all n∈ N.

2. Well covered of $\boldsymbol{\Gamma(L(+)N)}$

Let L be an integral domain *(I. D)* and *N* be an *L*-module. We study the well covered property of Γ (L(+) N) . A graph is well covered denoted by W_C, if all its maximal independent sets have the same cardinality.

Theorem 2.1: *If L is an I. D and N is an L-module. Then: (1) If* $n(N) = 0$, *then* $\Gamma(L(+)N)$ *is trivially well covered, (2) If* $ann(N) \neq 0$, *then* $\Gamma(L(+)N)$ *is not well covered.*

Proof: If L is an I. D and N is an L-module. Then: (1) If $ann(N) = 0$, then $Z^*(L(+)N) = \{(0, f) : f \in N^*\}$ so, every M_S is a Ma_S, trivially well covered W_C. (2) If $ann(N) \neq 0$, then $Z^*(L(+)N) = \{(0, f), (r, t) : r \in ann(N), f \in N^*, t \in N\}$. Then we construct two maximally independent sets of different sizes such that S_1 = {(0, 1)} and S_2 = {(r, 0), (r, f): $r \in ann(N)$, $f \in N^*$}. Therefore, the graph is not well covered W_C. □

Theorem 2.2: *Let Γ ($Z_{n_1}(+)\,Z_{m_1}$) be the Z. D. G of the idealization ring $Z_{n_1}(+)\,Z_{m_1}$. Then: (1)If n_1 = d and m_1 = d where d is prime, then Γ ($Z_{n_1}(+)\,Z_{m_1}$) is well covered, (2) If $n_1 = d^2$ and m_1 = d where d is prime, then Γ ($Z_{n_1}(+)\,Z_{m_1}$) is well covered, (3) If $n_1 = d^\alpha$ and m_1 = d where α > 2, then Γ($Z_{n_1}(+)\,Z_{m_1}$) is not well covered, (4) If n_1 = dq and m_1 = d where d and q are primes, then Γ ($Z_{n_1}(+)\,Z_{m_1}$) is not well covered, (5) If $n_1 = d^2q$ and m_1 = d where d and q are primes, then Γ($Z_{n_1}(+)\,Z_{m_1}$)is not well covered, (6) If $n_1 = d^\alpha q$ and m_1 = d where d and q are primes and α > 2, then Γ($Z_{n_1}(+)\,Z_{m_1}$) is not well covered.*

Proof: Let Γ ($Z_{n_1}(+)\,Z_{m_1}$) be the Z. D. G of the idealization ring $Z_{n_1}(+)\,Z_{m_1}$.Then (1)If n_1 = d and m_1 =d where d is prime, then Z*($Z_{n_1}(+)\,Z_{m_1}$) = { (0, f) , (kd, t) : $f \in Z_d^*, t \in Z_d, gcd(k,d) = 1$} all maximal independent sets of order 1, Γ($Z_{n_1}(+)\,Z_{m_1}$) is well covered, (2) If $n_1 = d^2$ and m_1 = d where d is prime, then Z*($Z_{n_1}(+)\,Z_{m_1}$) = {(0, f), (kd, t): f ∈ Z_d^*, t ∈ Z_d,gcd (k, d) = 1}, all maximal independent sets have order 1, Γ($Z_{n_1}(+)\,Z_{m_1}$) is well covered, (3) If $n_1 = d^\alpha$ and m_1 = d where α > 2, then Z*($Z_{n_1}(+)\,Z_{m_1}$) = {$(0,f), (kd^j, t)$: f ∈ Z_d^*, t ∈ Z_d, ,gcd (k, $d^{\alpha-j}$) = 1}, we have two independent set S_1 = {(0, 1)} and S = {(kd^j, f) : f ∈ Z_d,1 ≤ j ≤ $\lfloor\frac{\alpha-1}{2}\rfloor$, gcd (k, $d^{\alpha-j}$)=1}∪ $\left\{\left(d^{\lfloor\frac{\alpha}{2}\rfloor}, 0\right)\right\}$. Therefore, it Γ ($Z_{n_1}(+)\,Z_{m_1}$) is not well covered, (4) If n_1 = dq and m_1 = d where d and q are primes, then Z*($Z_{n_1}(+)\,Z_{m_1}$) = {(0, f): f∈ Z_d^*}∪ {(kd, t) : t ∈ Z_p, gcd(k, q) = 1}∪{(kq, t): t ∈ Z_d, gcd(k,d) = 1}, we have mor than two maximal independent sets of different sizes, S_1 = {(0, 1)} ∪ {(kq, t): t ∈ Z_d, gcd(k, d) = 1} and S_2 ={(kd, t):t ∈ Z_d ,gcd(k, q) = 1}, so Γ($Z_{n_1}(+)\,Z_{m_1}$) is not well covered. (5) If $n_1 = d^2$q and m_1 = d where d and q are primes, then Z*($Z_{n_1}(+)\,Z_{m_1}$) = {(0, f): f∈ Z_d^*}∪ {(kd, t) : t ∈ Z_p, gcd(k, qd) = 1}∪{(kq, t): t ∈ Z_d, gcd(k, d^2) = 1} ∪{(kdq, t): t ∈ Z_d, gcd(k, d) = 1}∪{(kd^2, t): t ∈ Z_d, gcd(k, q) = 1}, we have mor than two maximal independent sets of different sizes, S_1 = {(0, 1)} ∪ {(kq, t): t ∈ Z_d, gcd(k, d) = 1} and S_2 ={(kd, t): t ∈ Z_d, gcd(k, dq) = 1}∪ {(kq, t): t ∈ Z_d, gcd(k, d^2) = 1}, then Γ($Z_{n_1}(+)\,Z_{m_1}$) is not well covered. (6) If $n_1 = d^\alpha$q and m_1 = d where d and q are primes and α > 2, then Z*($Z_{n_1}(+)\,Z_{m_1}$) = {(0, f): f∈ Z_d^*}∪ {(kd^j, t) : t ∈Z_d, gcd(k, $d^{\alpha-j}$ q) = 1}∪{(kq, t): t ∈ Z_d, gcd(k, d^α) = 1} ∪{(kd^jq, t): t ∈ Z_d , gcd(k, $d^{\alpha-j}$)=1}, we have mor than two maximal independent sets of different sizes, S_1 = {(0, 1)} ∪ {(kq, t): t ∈ Z_d, gcd(k, d) = 1} and S_2 ={$(d^{\alpha-1}q, 0)$}, then Γ($Z_{n_1}(+)\,Z_{m_1}$) is not well covered.

3. r-well covered of Γ(L(+)N):

Here, we define a new notion is called *r*-well covered graph (r-W_C).

Definition 3.1; A graph is called r-well covered (r - W_C) if the graph is well covered and the cardinality of the maximal independent set is r.

Theorem 3.2: *If L is an I. D and N is an L-module and ann $(N) = 0$, then $\Gamma(L(+)N)$ is $(|N| - 1)$-well covered.*

Proof: If L is an I. D and N is an L-module, then If *ann* $(N) = 0$, then $Z^*(L(+)N) = \{(0, f) : f \in N^*\}$ so, the number of maximal independent sets, $S_1 = \{(0, f) : f \in N^*\}$, is well covered by $(|N| - 1)$.

Theorem 3.3: *If $n_1 = d \,\&\, m_1 = d$ where d is prime, then $\Gamma(Z_{n_1}(+)\, Z_{m_1})$ is $1-$well covered.*

Proof: If $n_1 = d$ and $m_1 = d$ where d is prime, then $Z^*(Z_{n_1}(+)Z_{m_1}) = \{(0, f) : f \in Z_d^*\}$, all maximal independent sets of order 1, $\Gamma(Z_{n_1}(+)Z_{m_1})$ is $1-$well covered.

Theorem 3.4: *If $n_1 = d^2$ and $m_1 = d$ where d is prime, then* $\Gamma(Z_{n_1}(+)Z_{m_1})$ is $1-$well covered.

Proof: If $n_1 = d^2$ and $m_1 = d$ where d is prime, then $Z^*(Z_{n_1}(+)Z_{m_1}) = \{(0, f), (kd, t) : f \in Z_d^*, t \in Z_d, \gcd(k, d) = 1\}$, all maximal independent sets have order 1, $\Gamma(Z_{n_1}(+)Z_{m_1})$ is $1-$well covered.

Theorem 3.5; *If L is (I.D) and N is an L-module. Then; (1) If ann(N) = 0, then $g_e(\Gamma(L(+)N)) = 1$, (2) If ann(N) $\neq 0$, then $g_e(\Gamma(L(+)N)) = |ann(N)||N|$.*

Proof: By Theorem 2.1. Then:(1) If *ann*$(N) = 0$, then every M_S is a Ma_S of size 1, $g_e(\Gamma(L(+)N)) = 1$, , see figure (1). (2) If *ann*$(N) \neq 0$, then $S = \{(r, m): r \in ann(N), m \in N\}$ is a M_S: (a) Any two vertices (r_1, m_1) and (r_2, m_2) in S are not adjacent; the product; $(r_1, m_1)(r_2, m_2) \neq (0, 0)$. Therefore, S is an I_S. (b) Any set contains S must be form; $S_1 = \{(r, m): r \in ann(N), m \in N\} \cup \{(0, m) : m \in N^*\}$ that is not an I_S. Therefore, S is a M_St. (c) Any vertex not in S is adjacent to the vertices in S. Therefore, $g_e(\Gamma\ (L\ (+)N\ )) = |ann(N)||N|$. See figure (2) when $|N| = 4$ and $|ann(N)| = 1$. We start by given the following theorem to find $g_e((\Gamma\ (L\ (+)N))$ when $L = Z_{n_1}, N = Z_{m_1}$.

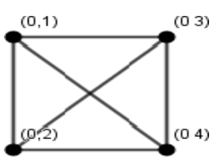

Figure 1
$\boldsymbol{g_e(\Gamma(Z_5(+)Z_5))=1}$

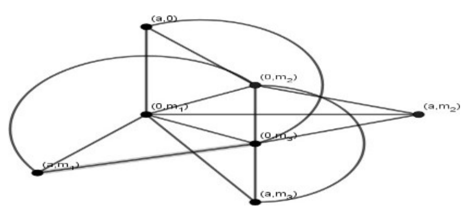

Figure 2
Γ (L (+) N)

Theorem 3.5: *Let Γ ($Z_{n_1}(+)\, Z_{m_1}$) be the Z. D. G of the idealization ring $Z_{n_1}(+)\, Z_{m_1}$. Then:(1) If n_1 = d and m_1 = d where d is prime, then g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) =1.(2) If $n_1 = d^2$ and m_1 = d where d is prime, then g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) =1.(3) If n_1 = d^{α} and m_1 = d where $\alpha > 2$, then g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) = $(d\sum_{j=1}^{\frac{\alpha-1}{2}} \phi(d^{\alpha-j})) + 1$. (4) If $n_1 = dq$ and $m_1 = d$ where d and q are primes $d < q$, then g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) = $(d-1)\phi(q) + d\phi(d)$. (5) If n_1 =d^2q and m_1 = d where d and q are primes, then g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) = $(d-1)\phi(d)\phi(q) + d\phi(d^2) + d\phi(q)$. (6) If n_1 =$d^{\alpha}q$ and $m_1 = d$ where d and q are primes and $\alpha > 2$, then g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) $\geq d\phi(d^{\alpha-j})\phi(q) + d\phi(d^{\alpha})$.*

Proof: By Theorem 2.2. Then by the same way it is held (1) If $n_1 = d$ and m_1 =d where d is prime, then $g_e(\Gamma(L(+)N)) = 1$. (2) If $n_1 = d^2$ and $m_1 = d$ where d is prime, then g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) =1. (3) Let S = $\{(kd^j, f) : f \in Z_d,\ 1 \leq j \leq \lfloor \frac{\alpha-1}{2} \rfloor,\ \gcd(k, d^{\alpha-j})=1\} \cup \left\{\left(d^{\lfloor \frac{\alpha}{2} \rfloor}, 0\right)\right\}$ is a maximum independent set: (a) Any two vertices (r_1, m_1) and (r_2, m_2) in S are not adjacent; the product; (r_1, m_1) (r_2, m_2) $\neq$ (0, 0). So, S is an independent set. (b)Any set contains S must be form; $S_1 = \{(0,f): f \in Z_d^*\} \cup \{(kd^j, f): f \in Z_d,\ j \geq \lfloor \frac{\alpha}{2} \rfloor\}$ that is not an I_S. So, S is a Ma_S. (c) Any vertex not in S is adjacent to the vertices in S. So, g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) = $(d\sum_{j=1}^{\frac{\alpha-1}{2}} \phi(d^{\alpha-j})) + 1$. (3) If $n_1 = dq$ and $m_1 = d$ where d and q are primes, $d < q$, then $Z^*(Z_{n_1}(+)\, Z_{m_1}) = \{(0,f): f \in Z_d^*\} \cup \{(kd, t) : t \in Z_p,\ gcd(k, q) = 1\} \cup \{(kq, t): t \in Z_d,\ gcd(k, d) = 1\}$. Let S = $\{(kd, f): f \in Z_d^*, \gcd(k, q)=1\} \cup \{(kq, t): t \in Z_d, gcd(k, d) = 1\}$ is a maximum independent set:(a) Any two vertices (r_1, m_1) and (r_2, m_2) in S are not adjacent; the product; (r_1, m_1) (r_2, m_2) $\neq$ (0, 0). So, S is an independent set. (b)Any set contains S must be form; $S_1 = \{(0,f): f \in Z_d^*\} \cup \{(kd, 0), \gcd(k,q)=1\} \cup S$ that is not an I_S. So, S is a Ma_S. (c) Any vertex not in S is adjacent to the vertices in S. So, g_e((Γ ($Z_{n_1}(+)\, Z_{m_1}$)) = $(d-1)\phi(q) + d\phi(d)$. (4) Let S = $\{(kd, t) : t \in Z_d^*,\ gcd(k, dq) = 1\} \cup \{(kq, t): t \in Z_d,\ gcd(k, d^2) = 1\} \cup \{(kd^2, t): t \in Z_d,\ gcd(k, q) = 1\}$ is a maximum independent set: By the same way it is held. So S is Ma_S, g_e((Γ ($\Gamma(Z_{n1}(+)Z_{m1})$)) = (d-1)$\phi(d)\phi(q) + d\phi(d^2) + d\phi(q)$. (5) Let S = $\{(kd^j, t): t \in Z_d,\ gcd(k, d^{\alpha-j} q) = 1,\ 1 \leq j \leq \alpha\} \cup \{(kq, t): t \in Z_d,\ gcd(k, d^{\alpha}) = 1\}$ is a maximal independent set: By the same way it is held S is a M_S. So, $g_e(\Gamma(Z_{n1}(+)Z_{m1})) \geq d\phi(d^{\alpha-j})\phi(q) + d\phi(d^{\alpha})$.

References

[1] Istvan Beck, "Coloring of commutative rings," *Journal of Algebra*, vol. 116, no. 1, pp. 208–226 (1988).

[2] David Anderson and Philip Livingston, "The zero-divisor graph of a commutative ring," *Journal of Algebra*, vol. 217, no. 2, pp. 434–447 (1999).

[3] Michael Plummer, "Some covering concepts in graphs," *Journal of Com binatorial Theory*, vol. 8, no. 1, pp.91–98 (1970).

[4] Tariq Alraqad, Hicham Saber and Rashid Abu-Dawwas, "Intersection graphs of graded ideals of graded rings, *AIMS Mathematics*, vol. 6, no. 10, pp. 10355-10368 (2021).

[5] Mulay Shashikant, "Cycles and symmetries of zero-divisors, " *Communications in Algebra*, vol. 30, no. 7, pp. 3533-3558 (2002).

[6] Malik Bataineh and Azza Alshehry, " Rashid Abu-Dawwas, Zariski topologies on graded ideals," *Tatra Mountains Mathematical Publications*, vol. 78, pp. 215-224 (2021).

[7] Manal Al-Labadi, Shuker Khalil, Ahmed Hassan and Wasim Audeh, "New structure of bipolar fuzzy rho-ideals in rho-algebraic semigroups," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 4-B, pp.1145–1151 (2025). DOI: 10.47974/JDMSC-2120

[8] Manal Al-Labadi and Shuker Khalil, The idealization ring. AIP Conf. Proc.; 3097 (1): 080008 (7 May 2024). https://doi.org/10.1063/5.0209920

[9] D. F. Anderson, M. C. Axtell, and J. A. Stickles, "Zero-divisor graphs in commutative rings," in Commutative Algebra, M. Fontana, S. E. Kabbaj, B. Olberding, and I. Swanson, Eds. New York, NY, USA: Springer, pp. 23–45 (2011). doi: 10.1007/978-1-4419-6990-3_2.

[10] Ahmed Maatallah, "On z s -ideals of commutative rings, " *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 27, no. 8, pp. 2273–2287 (2024).

[11] Israa Sattar Ahmed, Ahmed Al-Fayadh, Hassan Hadi Ebrahim, and Luma Sabah Abdalbaqi, "Implementation of the neutrosophic sets in measurable space with respect to neutrosophic ring," *International Journal of Neutrosophic Science*, vol. 25, no. 3, pp. 187–193 (2025), doi: 10.54216/IJNS.250317.

Received June, 2025