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Purely-lifting modules

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Abstract

This article primary goal is to introduce the idea of purely lifting module, after that we proved some properties of this concept the second aim is to studying relations among purely lifting module and many others module. The third aim is to introduce other notions such as cofinitely-purely-lifting modules and purely-hollow-lifting modules, as well as purely supplemented modules and H - purely supplemented modules.

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1. Introduction

In this work, R is commutative and has an identity. A left R-modules M are unitary. the sub-module N for M be define as pure (shortly $N \leq_P M$) if $IM \cap N = IN$ [1] for every ideal I for R. A proper submodule N for M namely a small submodule ($N \ll M$), where all submodules K for M s.t $M = N + K$, so $K = M$, [2]. Al-Bahraany [3] developed a new submodule that is a generalization of a small and is called a purely small submodule (Pu-small) sub-module. For all pure proper submodule K for M, a sub-modules N for the M named the purely small submodule $N \ll_{Pu} M$, if $N + K \neq M$. In this search, as in [1], [4], and [5] we will use purely- small submodules to present purely-lifting module, we introduce some characterization properties, and define co finitely purely-lifting and purely-hollow-lifting modules. Also introduces some characterization properties. Also define Rad_{Pu} -lifting and introducing some characterization and properties. Also investigate characterizations, examples, and the relationship

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between purely-lifting modules and other classes of modules as semi simple, purely-hollow, purely-supplemented, and H-purely-supplemented modules. We give some of their basic properties and give some remarks and propositions of these concepts.

2. Purely-Lifting Modules.

This unit is devoted to introducing details for lifting-modules with some properties. Recall that the module M named lifting when each submodule A for M $\exists$ the direct summand N for M with $N \leq A$ & $\frac{A}{N} \ll \frac{M}{N}$ ($N \leq_{ce} A$). See [1], [6].

Definition 2.1: The left R-module M named purely lifting-module, for short(Pu-lifting) moduleS. When the submodule N for M, t$\exists D \leq N$ s.t $M = D \oplus D_1$, $D_1 \leq M$ also $N \cap D_1 \ll_{Pu} M$. For example, Z as Z -module.

Lemma 2.2 *if M the R-modules with PIP property and $N,$ & K are sub - module* ***for the*** *M, and K represent to the pure in the M. If $N \ll_{Pu} K$ so we have $N \ll_{Pu} M$.*

Proof: Let L be the pure sub module for the M s.t $= L + N$. To illustration $M = L.$ because a sub-module*s* L be pure in the M also an sub-modules K be pure in the M , M has PIP property, thus $L \cap K$ the pure in the M, but $L \cap K \leq K$ by[3, Remark 2.1.18]. Then $L \cap K$ be the pure in the K. Thus, we get $\ll_{Pu} K$, thus $K = L \cap K$, i.e $K \leq L$ therefore $N \ll_{Pu} M$.

A named co-essential (or co-small) for the B in the M, if $\frac{\mathrm{B}}{\mathrm{A}} \ll \frac{\mathrm{M}}{\mathrm{A}}$. See [4]. Assume M the R-modules & it belongs to M as the sub-module and is named purely co-essential for B in summary (Pu-coessential) if $\frac{\mathrm{B}}{\mathrm{A}} \ll_{\mathrm{Pu}} \frac{\mathrm{M}}{\mathrm{A}}$ ($\mathrm{A} \leq_{\mathrm{Pu.ce}} \mathrm{B}$), [7].

Theorem 2.3 *the modules M with (PSP) property, the resulting equivalent:*

1. *M be Pu-lifting module.*
2. *For any submodules for M, $\exists$ the decomposition $A = A_1 \oplus A_2$ s.t A_1 direct summand forM also $A_2 \ll_{Pu} M$.*
3. *For the submodules A for M, where the direct summand A_1 for M s.t $A_1 \leq A$ & $\frac{A}{A_1} \ll_{Pu} \frac{M}{A_1}$ i.e ($A_1 \leq_{P.ce} A$).*

Proof: **1 $\Longrightarrow$ 2**) Assume that A submodule*s* for M. so $\exists A_1$ of A s.t $M = A_1 \oplus B$ also $A \cap B \ll_{Pu} M$. So $A = A \cap (A_1 \oplus B)$ so $A = A_1 \oplus (A \cap B)$. Therefore, there is the decomposition A=$A_1 \oplus A_2$ s.t A_1 is the direct summand for M also $A \cap B = A_2 \ll_{Pu} M$.

2 $\Longrightarrow$ 3) Assume that $A \leq M$. Now, by (2) A_1 is a sub-module of A, such that $M = A_1 \oplus B$ Let $\frac{H}{A_1}$ be a puresubmodule of $\frac{M}{A_1}$. Since $A_1 \leq_P M$ hence by

[8, Remark 3.1.5, (4)] $H \leq_P M$, since M (PSP) property and H is the pure sub module for the M. so $H + A_1 \leq_P M$. Therefore, we are done.

3 ⟹ 1) $A = A \cap (A_1 \oplus B) = A_1 \oplus (A \cap B)$, $\frac{M}{A_1} = \frac{A_1 \oplus B}{A_1}$ by (The First Isomorphism Theorem), [9] $\frac{M}{A_1} = \frac{A_1 \oplus B}{A_1} \simeq \frac{B}{A_1 \cap B} \simeq B$ and $\frac{\boldsymbol{A}}{\boldsymbol{A_1}} = \frac{\boldsymbol{A_1 \oplus A \cap B}}{\boldsymbol{A_1}} \simeq A \cap B$, so $\boldsymbol{A \cap B} \ll_{Pu} \boldsymbol{B} \leq M$. Therefore, by [Lemma 2.2] $A \cap B \ll_{Pu} M$ and M $be\ the\ Pu$-lifting.

The sub-module N for the M named purely-supplement for short (Pu-supplement) of V , if $N + V = M$ implies that $N \cap V \ll_{Pu} N$.

Recall, the R- module D is named hollow where each proper sub-module for D is small, [11].

Recall that, an R–module $D \neq 0$ defined purely hollow for Brief Pu -hollow, where each proper submodule for D be Pu -small. If all for R's ideals are Pu-hollow, then a ring R is said to be Pu-hollow, [12].

Examples and Remarks 2.4

1. each lifting be the Pu-lifting. The Z-module Z_6 is Pu-lifting modules. see [12].
2. Every Pu-lifting is an Pu- supplemented module from (1). See example and remarks (4.2) (4), [7].
3. Every semi simple module is the lifting module, hence is purely-lifting-modules. the Z_6 as the Z-modules. However, generally the opposite not exact see the Z as the Z-modules is purely lifting, but not lifting. See [12].
4. Every hollow is lifting, hence purely hollow is purely lifting. Z_4 as the Z-modules be the lifting and pure lifting also Z_8 as the Z-modules lifting & purely lifting.
5. Pu-lifting is an amply Pu-supplemented module, Z_{24} as the Z-modules Pu-lifting, also amply Pu-supplemented. See example and remarks (3.7) (1), [14].
6. The lifting and Pu-lifting modules coincide if M is F-regular modules. the Z_6 as the Z - modules in (1) is lifting, also is Pu- lifting.

The following shows under which condition makes the Pu-lifting hereditary property.

Proposition 2.5 Each Pu -small direct summand for the Pu-lifting- module be Pu-lifting.

Proof: Assume that A be Pu-small diract summend in M, where M be the Pu-lifting. assume $A_1 \leq A \leq M$, By [13, proposition 2.6]. We have $\mathrm{A}_1 \cap \mathrm{A} \cap A_3 \ll_{Pu} A$ Therefore, A is an Pu - lifting module.

Lemma 2.6 *Suppose that M and N are the R-modules, & $f : M \to N$ epimorphisms. If $A \ll_{Pu} M$ and $A \geq Kerf$ with $Kerf$ is pure in M then $f(A) \ll_{Pu} N$.*

Proof: From (First Isomorphism Theorem) $M/Kerf \simeq f(M)$ then $A/(ker\, f) \simeq f(A)$. so by [15, proposition 2.4]. $A/Kerf \ll_{Pu} M/kerf$ since f is epimorphism then $f(A) \ll_{Pu} f(M) = N$, implies that $f(A) \ll_{Pu} N$.

Theorem 2.7: *If M the Pu -lifting modules with (PSP)property and Y be the pure sub-modules for the M. Now M/Y the Pu -lifting modules in each for followed:*

1. For any direct summand, K for M, $(K+Y)/Y$ be the direct summand for M/Y.
2. M be the distributive modules.

Proof: If K the direct summand in the M. Implies, exists asubmodule in M, suchthat $M= K \oplus L$. So $\frac{M}{Y} = \frac{K \oplus L}{Y} = \frac{k+Y}{Y} + \frac{L+Y}{Y}$. Since M distributive module. Hence$(K+Y) \cap (L+Y) = Y$. Thus $\frac{M}{Y} = \frac{K+Y}{Y} \oplus \frac{L+Y}{Y}$. Therefore, by (1) $\frac{M}{Y}$ is Pu -lifting.

3. For any $\Psi^2 = \Psi \in End\,(M)$, $\Psi Y \leq Y$. In actual, Y be fully invariant sub-module of M.

Next, we will see the relationship between Pu-lifting and the other concepts. Semi-simple, Pu-hollow, and H- Pu - supplemented modules.

Theorem 2.8: *Every Pu-hollow modules be thePu-lifting*

Proof: Assume A is proper submodule for M. Then, there is sub-module. $\langle \bar{0} \rangle \leq A$, s.t $M = \langle \bar{0} \rangle \oplus M$ then $A \cap M = A \ll_{P_u} M$. If $M = A$, thereexists a sub-module M in M, suchthat $\langle \bar{0} \rangle \oplus M = M$, also $\langle \bar{0} \rangle \cap M = \langle \bar{0} \rangle \ll_{P_u} M$. Therefore, M is Pu-lifting.

Remark 2.9: The opposite at the above proposition, generally may not be valid. At instance, the Z_4- modules Z be the Pu- lifting, but not Pu -hollow. See [13].

Next, we will see when the opposite of proposition (2.8) is holding.

Proposition 2.10 if M the indecomposable Pu - lifting -module. Then M be the Pu -hollow.

Proposition 2.11 If $Rad_{P_u}(M) = 0$ & M the Pu -lifting–modules. Now M is semi simple.

Proposition 2.12 Assume that M is a Pu-liftingmodule. Now $\frac{M}{Rad_{Pu}(M)}$ is a semi simple module.

3. Co finitely Purely Lifting and Purely Hollow-Lifting Modules.

This section aims to introduce new concepts in modules M over rings R. The first one called nonfinitely Pu-lifting the generalizations for Pu-lifting module. The second concept is called Pu-hollow-lifting, which is generalizations for Pu-lifting module. So, we prove some properties for these concepts.

An sub-modules A for modules M named nonfinitely purely lifting, for short (Pu-lifting), $\frac{M}{A}$ is finitely generated. See [1]. As in [17], [18, 19].

Definition 3.2: Assume that M be the modules. Then M is called coffinite purely lifting, for short (coffinite Pu -lifting), if every coffinite submodule A for$M$, there isthe directs mm and $K \leq A$ s.t $M = K \oplus K_1$, where $K_1 \leq M$ & $A \cap K_1 \ll_{P_u} M$.

Examples and Remarks 3.3

1. Every Pu –lifting module is co finitly Pu -lifting. while inverse not accurate. For Example, Z-module Z_{12} be co finitely Pu -lifting since the only co finite sub module of Z_{12} is Z_{12}. So there is $Z_{12} \leq Z_{12}$ such that $\langle 0 \rangle \oplus Z_{12} = Z_{12}$ and $\langle 0 \rangle \cap Z_{12} = \langle 0 \rangle \ll_{P_u} Z_{12}$ but the Z-module Z_{12} is not Pu -lifting. See examples and remarks (2.4) (5).
2. The Z-modules Z_6 is not cofinitly Pu -lifting. Since $\frac{Z_6}{2Z_6} \simeq 3Z_6$, so $3Z_6$ is co finite and the only direct summand contains in $3Z_6$ is $\langle 0 \rangle$ and $3Z_6$ is not Pu -small in Z_6.
3. From (1) and (2), we see that the sub module of the co finitely Pu –lifting module may not be co finitely Pu -lifting. The Z-module Z_{12} is co finitely Pu-lifting, but the sub module $3Z_{12}$.

Theorem 3.4 *For the module M with (PSP) property, the resulting are equivalents:*

1. *M be the coffinite Pu -lifting.*
2. *each co finite sub module A in M, exist the decompositions $A = A_1 \oplus A_2$ s.t A_1 be direct summand for $M \& A_2 \ll_{P_u} M$.*
3. *For each co finite sub-module A in M, the direct summand. A_1 of M s.t $A_1 \leq A$ & $\frac{A}{A_1} \ll_{P_u} \frac{M}{A_1}$ (i.e $A_1 \leq A_{P.ce}$).*

Proof: It follows from Proposition (2.3).

Next, we study the following question: will sub module, direct summand, and factor modules inherit the co finite Pu --lifting condition?

Proposition 3.5 Each co finite direct-summand of the co finitely Pu - lifting - module with (PSP) property is co finitely Pu -lifting.

Proof: Assume that A be the co finite directsummand of the modules M where M be the co finitely Pu-lifting. so $M = A \oplus A_1$, for some $A_1 \leq M$ & $\frac{M}{A}$ be finite generated submodules. *if* B be co finite submodule of A, now $\frac{\boldsymbol{A}}{\boldsymbol{B}}$ is finite generated submodule. So $A = A \cap M = A \cap (K \oplus K_1) = K \oplus (A \cap K_1)$. Claim that $\boldsymbol{B \cap K_1 \cap A \ll_{P_u} A}$, now, A is directsummand by [13, Preposition 2.6], we have $B \cap K_1 \cap A \ll_{P_u} A$. Therefore A is a cofinitely Pu -lifting module.

Lemma 3.6 *Assume that $MandN$ be R-modules, & f: $M \to N$ epimorphisms and $A \ll_{Pu} M$& $A \geq Kerf$ with $Kerf$ is pure in M then $f(A) \ll_{Pu} N$.*

Proof: By (first isomorphisms theorems) $M/Kerf \simeq f(M)$ but $A \leq M$ and $A \geq Kerf$ then $A/(\,ker\, f) \simeq f(A)$. Since $A \ll_{\text{Pu}} M$ then by [14, proposition 2.4]. $A/Kerf \ll_{\text{Pu}} M/kerf$ Now, $f(A) \simeq A/Kerf \ll_{\text{pu}} M/kerf \simeq f(M)$, hence $f(A) \ll_{\text{pu}} f(M)$, since f is epimorphism then $f(A) \ll_{\text{Pu}} f(M) = N$ implies that $f(A) \ll_{\text{pu}} N$.

Proposition 3.7 if M be thecoma finitely Pu-lifting -module. so $\frac{M}{A}$ also be the co finitely Pu -lifting for each A be the fully invariant submodules for the M.

Proof: Suppose $\frac{N}{A}$ is co finite submodule for $\frac{M}{A}$. so $\frac{\frac{M}{A}}{\frac{N}{A}} \simeq \frac{M}{N}$ is finitely generated. thus N is the co-finite sub module for the M. so M be the co finitely Pu-lifting modules, there is $B \leq N$ s.t $M = B \oplus B_1$, for somesubmodule B_1 of M, and $\frac{N}{B} \ll_{P_u} \frac{M}{B_1}$, using [4, Lemma 3.3.5]. we have $\frac{\boldsymbol{M}}{\boldsymbol{A}} = \frac{B+A}{A} \oplus \frac{B_1+A}{A}$ with $\frac{B+A}{A} \leq \frac{N}{A}$, so $\frac{\frac{N}{A}}{\frac{B+A}{A}} \simeq \frac{N}{B+A}$ also $\frac{\frac{M}{A}}{\frac{B+A}{A}} \simeq \frac{M}{B+A}$, Since $\frac{N}{B} \ll_{P_u} \frac{M}{B}$ by Lemma (3.6) $\frac{N}{B+A} \ll_{P_u} \frac{M}{B+A}$. Therefore, $we\ are\ done$.

Corollary 3.8 *Assume that M be the coffinite Pu-liftingmodule. Then also $\frac{M}{Rad_{\ Pu}(M)}$ be the coffinite Pu-lifting.*

Proof: Clear, since $Rad_{\ P_u}(M)$ be the fully invariant submodule for M.

Recall, the modules M is has (SSP) where sum for the two direct summand is also the direct summand for M. See [6, 16]

Theorem 3.9 *Assume that M the coffinite Pu-lifting -modules also A is the directsummand for the M. assume M has the (SSP), now $\frac{M}{A}$ the co finitely Pu -lifting.*

Proof: Assume that $\frac{N}{A}$ be the co finite sub-module of $\frac{M}{A}$. Now $\frac{\frac{M}{A}}{\frac{N}{A}} \simeq \frac{M}{N}$ is finitely generated, so N be co finite submodule of M. *S*ince M be co finitely Pu-lifting module, there is $B \leq N$ s.t $M = B \oplus B_1$, and $\frac{N}{B} \ll_{P_u} \frac{M}{B}$. Then M has the SSP, $B + A$ is the direct summand for M also $\frac{B+A}{A}$ is direct summand for $\frac{\boldsymbol{M}}{\boldsymbol{A}}$ since $\frac{N}{B} \ll_{P_u} \frac{M}{B}$ by Proposition 3.6 $\frac{N}{B+A} \ll_{P_u} \frac{M}{B+A}$. Therefore, $\frac{M}{A}$ is co finitely Pu -lifting.

Recall the R-modules D $\neq 0$ is defined to be purely hollow for Brief Pu -hollow when each proper submodule for D be Pu -small. If all of R's ideals are Pu-hollow, then a ring R is said to be Pu-hollow. See [10, 13]

Now, as in [5] we will introduce the following:

Definition 3.10 Assume M the modules. so M named Pu-hollow-lifting module if each sub-module A of M with $\frac{M}{A}$ is Pu- hollow, $\exists$ directsummand $K \leq A$ s.t $M = K \oplus K_1$, for $K_1 \leq M$, & $A \cap K_1 \ll_{P_u} M$.

Next, we provide some characterization of Pu-hollowliftingmodules.

Theorem 3.11 *the following equivalent.*

1) *M the Pu-hollow-lifting*
2) *For the sub-module A in the M with $\frac{M}{A}$ is Pu-hollow, there is the decomposition $A = A_1 \oplus A_2$ s.t A_1, be direct summand for M also $A_2 \ll_{P_u} M$.*
3) *For any sub module A for M with $\frac{M}{A}$ is Pu-hollow, there is direct summand A_1 for M s.t $A_1 \leq A$ & $\frac{A}{A_1} \ll_{P_u} \frac{M}{A_1}$ ($i.e\ A_1 \leq_{P_u.ce} A$).*

Proof: Direct by Proposition (2.3).

Examples and Remarks 3.12

1. Every module with no Pu-hollow factor module is Pu -hollow-lifting.
2. Each Pu-lifting- module is Pu-hollow lifting. While the inverse is not right. For Example, if M is non-zero indecomposable modules with no Pu-hollow factor. Hence M is Pu-hollow-lifting. The claim is that M is not Pu-lifting. If not, we have that M the indecomposable Pu-lifting modules. from Proposition 2.12, M thePu-hollow, and by [13], $\frac{M}{A}$ is Pu-hollow for the proper submodules A in M, which is a contradiction.
3. The Z-modules Z_6 not Pu-hollow-lifting, a submodules $3Z_6$, since $\frac{Z_6}{3Z_6} \simeq Z_3$ the Pu-hollow and the only direct summand contains $3Z_6$ is $\langle \bar{0} \rangle$, $3Z_6$ is not Pu- small in Z_6.
4. Every semi-simple module is not Pu-hollow-lifting.

The following proposition gives a condition to make a factor of Pu-hollow-lifting a Pu -hollow-lifting module.

Proposition 3.13 Suppose that M be a Pu-hollowlifting- modules. Now $\frac{M}{A}$ is Pu-hollow -lifting for each fully invariant sub-module A of M.

4. Conclusions

In our work on the topic of purely-lifting modules and cofinitely purely lifting and purely hollow-lifting modules**.** the most important results we reached were:

- All Pu-small direct summand of a Pu-liftingmodule is also Pu-lifting.

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