

On k -Noetherian and k -Artinian semirings

Jawad Abuhlail *

*Department of Mathematics
College of Computing and Mathematics
King Fahd University of Petroleum & Minerals
Dhahran 31261
Saudi Arabia*

Rangga Ganzar Noegraha [†]

*Department of Actuarial Science
College of Sciences and Computer
Universitas Pertamina
Jakarta 12220
Indonesia*

Abstract

We investigate left k -Noetherian and left k -Artinian semirings. We characterize such semirings using i -injective semimodules. We prove, in particular, a partial version of the celebrated *Bass-Papp Theorem* for semirings. We illustrate our main results by examples and counter examples.

Subject Classification (2020): Primary 16Y60, Secondary 16P40, 16P20.

Keywords: Semirings, Semimodules, Injective semimodules, Noetherian semirings, Artinian semirings.

1. Introduction

Semirings are, roughly, rings not necessarily with subtraction, and generalize both rings and distributive bounded lattices. Semirings, and their *semimodules* (defined, roughly, as modules not necessarily with subtraction), have many important applications in several aspects of Computer Science and Mathematics, e.g., Automata Theory [14], Tropical Geometry [10] and Idempotent Analysis [18]. Our main reference for semirings and their semimodules is Golan's book [11] and for rings and modules Wisbauer's book [22].

* E-mail: abuhlail@kfupm.edu.sa (Corresponding Author)

[†] E-mail: rangga.gn@universitaspertamina.ac.id

Left (right) Noetherian rings, whose lattices of left (right) ideals satisfy the Ascending Chain Condition, are well studied due to the role it plays in simplifying the ideal structure of such rings. On the other hand, left (right) Artinian rings, whose lattices of left (right) ideals satisfy the Descending Chain Condition, generalize simultaneously finite rings and rings that are finite-dimensional vector spaces over fields. Several properties of left (right) modules are valid only over rings with the ACC or the DCC. Some of these properties characterize such rings, *e.g.* the closure of the class of left (right) injective modules under arbitrary direct sums characterizes left (right) Noetherian rings [19, 3.39], and a ring R is left (right) Artinian if and only if every finitely generated left (right) R -module is finitely cogenerated [22, 31.4].

A left (right) ideal I of a semiring S is called a k -ideal, iff $I \leq_s S$ is *subtractive* [13] (equivalently, $I = \text{Ker}(S \xrightarrow{\pi_I} S/I)$, where π_I is the canonical projection). In this paper, we consider the so called *left k -Noetherian semirings* (*left k -Artinian semirings*), whose lattice of *subtractive* left ideals satisfies the ACC (DCC). We generalize several results known for left Noetherian (left Artinian) rings to left k -Noetherian (left k -Artinian) semirings.

The paper is divided into two sections.

In Section 1, we collect the basic definitions, examples and preliminaries used in this paper. In particular, we recall the definitions and basic properties of *exact sequences* introduced by the first author Abuhlail [3] (see also [2]).

In Section 2, we investigate *left k -Noetherian* (resp., *left k -Artinian*) *semirings*, i.e. semirings satisfying the ACC (resp., the DCC) on left k -ideals. In Example 2.10, we show that $S := M_2(\mathbb{R}^+)$ is left k -Noetherian but *not* left Noetherian, and is left k -Artinian but *not* left Artinian. In Theorem 2.13, we show that if every subtractive left ideal of a semiring S is a direct summand, then S is left k -Artinian and left k -Noetherian. In Theorem 2.19, we provide a partial version of the celebrated Bass-Papp Theorem for semirings: we show that if S is a semiring with enough left S - i -injective semimodules and every direct sum of S - i -injective left S -semimodules is S - i -injective, then S is left k -Noetherian.

1. Preliminaries

In this section, we provide the basic definitions and preliminaries used in this work. Any notions from the theory of semirings and semimodules that are not defined here can be found in our main reference [11]. We refer to [22] for the foundations of the theory of module and rings.

Definition 1.1: ([11]) A **semiring** is a datum $(S, +, 0, \cdot, 1)$ consisting of a commutative monoid $(S, +, 0)$ and a monoid $(S, \cdot, 1)$ such that $0 \neq 1$ and

$$\begin{aligned} a \cdot 0 &= 0 = 0 \cdot a \text{ for all } a \in S; \\ a(b + c) &= ab + ac \text{ and } (a + b)c = ac + bc \text{ for all } a, b, c \in S. \end{aligned}$$

1.2 [11] Let S and T be semirings. The categories ${}_S\mathbf{SM}$ of **left S -semimodules** with arrows the S -linear maps, $\mathbf{SM}_T$ of **right S -semimodules** with arrows the T -linear maps, and ${}_S\mathbf{SM}_T$ of (S, T) -bisemimodules are defined

in the usual way (as for modules and bimodules over rings). For a left S -semimodule M , we write $L \leq_S M$ to indicate that L is an S -**subsemimodule** of M .

Definitions 1.3: ([11]) Let $(S, +, 0, \cdot, 1)$ be a semiring.

- If the monoid $(S, \cdot, 1)$ is commutative, we say that S is a **commutative semiring**.
- We say that the semiring S is **additively idempotent**, iff $s + s = s$ for every $s \in S$.
- The set of *cancellative elements* of a left S -semimodules M is defined as
$$K^+(M) = \{x \in M \mid x + y = x + z \Rightarrow y = z \text{ for any } y, z \in M\}.$$
We say that M is a *cancellative semimodule*, iff $K^+(M) = M$.

Example 1.4: ([11])

- Every ring is a cancellative semiring.
- Any *distributive bounded lattice* $\mathcal{L} = (L, \vee, 0, \wedge, 1)$ is a additively idempotent commutative semiring.
- The set $(\mathbb{Z}^+, +, 0, \cdot, 1)$ (resp. $(\mathbb{Q}^+, +, 0, \cdot, 1)$, $(\mathbb{R}^+, +, 0, \cdot, 1)$) of non-negative integers (resp. non-negative rational numbers, non-negative real numbers) is a cancellative commutative semiring which is not a ring.
- $M_n(S)$, the set of all $n \times n$ matrices over a semiring S , is a semiring.
- $\mathbb{B} := \{0, 1\}$ with $1 + 1 = 1$, is a an additively idempotent commutative semiring called the **Boolean semiring**.
- The *max-plus algebra* $\mathbb{R}_{\max, +} := (\mathbb{R} \cup \{-\infty\}, \max, -\infty, +, 0)$ is a semiring.
- The *log algebra* $(\mathbb{R} \cup \{-\infty, \infty\}, \oplus, \infty, +, 0)$ is a semiring, where
$$x \oplus y = -\ln(e^{-x} + e^{-y})$$

Example 1.5: ([11], Example 1.8, [1]) Consider

$$B(n, i) := (B(n, i), \oplus, 0, \odot, 1),$$

where $B(n, i) = \{0, 1, 2, \dots, n-1\}$ and

$a \oplus b = a + b$ if $a + b < n$; otherwise, $a \oplus b = c$ is the unique natural number $i \leq c < n$ satisfying $c \equiv a + b \pmod{n-i}$;

$a \odot b = ab$ if $ab < n$; otherwise, $a \odot b = c$ is the unique natural number $i \leq c < n$ with $c \equiv ab \pmod{n-i}$.

Then $B(n, i)$ is a semiring. Notice that $B(n, 0) = \mathbb{Z}_n$ (a ring) and that $B(2, 1) = \mathbb{B}$ (the Boolean Algebra).

Example 1.6: ([11, page 150, 154]) Let S be a semiring, M be a left S -semimodule and $L \leq_S M$. The **subtractive closure** of L is defined as

$$\bar{L} := \{m \in M \mid m + \ell = \ell' \text{ for some } \ell, \ell' \in L\}. \quad (1)$$

One can easily check that $\bar{L} = \text{Ker}(M \xrightarrow{\pi_L} M/L)$, where π_L is the canonical projection. We say that L is **subtractive** (or a **k -subsemimodule**), iff $L = \bar{L}$. The left S -semimodule M is a **subtractive semimodule**, iff every S -subsemimodule $L \leq_S M$ is subtractive.

Definition 1.7: [11, page 71] Let S be a semiring. A subtractive left (right) ideal of S is called a *left (right) k -ideal* [13]. We say that S is a **left subtractive (right subtractive) semiring**, iff every left (right) ideal of S is subtractive. We say that S is a **subtractive semiring**, iff S is both left and right subtractive.

Remark 1.8: Whether a left subtractive semiring is necessarily right subtractive was an open problem till a counterexample was given in [17, Fact 2.1].

Following [9], we use the following definitions.

1.9. (cf., [7]) The category ${}_S\mathbf{SM}$ of left semimodules over a semiring S is a *variety* in the sense of Universal Algebra (closed under homomorphic images, subobjects and arbitrary products). Whence ${}_S\mathbf{SM}$ is complete, i.e. has all limits (e.g., direct products, equalizers, kernels, pullbacks, inverse limits) and cocomplete, i.e. has all colimits (e.g., direct coproducts, coequalizers, cokernels, pushouts, direct colimits).

1.10. An S -semimodule N is a **direct summand** of an S -semimodule M (i.e. $M = N \oplus N'$ for some S -subsemimodule N' of M) if and only if there exists $\alpha \in \text{Comp}(\text{End}(M_S))$ s.t. $\alpha(M) = N$ where for any semiring T we set

$$\text{Comp}(T) := \{t \in T \mid \exists \tilde{t} \in T \text{ with } t + \tilde{t} = 1_T \text{ and } t\tilde{t} = 0_T = \tilde{t}t\}.$$

Indeed, every direct summand of M is a retract of M ; the converse is not true in general. Golan [11, Proposition 16.6] provided characterizations of direct summands.

Exact Sequences

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S -module is a *left* S -semimodule.

Definition 1.11: A morphism of left S -semimodules $f: L \rightarrow M$ is

k -normal, iff whenever $f(m) = f(m')$ for some $m, m' \in M$, we have $m + k = m' + k'$ for some $k, k' \in \text{Ker}(f)$;

i -normal, iff $\text{im}(f) = \overline{f(L)}$ ($:= \{m \in M \mid m + \ell \in L \text{ for some } \ell \in L\}$).

normal, iff f is k -normal and i -normal.

There are several notions of exactness for sequences of semimodules. In this paper, we use the relatively new notion of exactness introduced by Abuhlail [3, 2.4] which is stronger than that in the sense of [20].

Definition 1.12: ([3, 2.4]) A sequence

$$L \xrightarrow{f} M \xrightarrow{g} N \quad (2)$$

of left S -semimodules is **exact**, iff $f(L) = \text{Ker}(g)$ and g is k -normal.

1.13. We call a (possibly infinite) sequence of S -semimodules

$$\cdots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \rightarrow \cdots \quad (3)$$

a *chain complex*, iff $f_{j+1} \circ f_j = 0$ for every j ;

exact, iff each partial sequence with three terms $M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2}$ is exact.

A **short exact sequence** (or a **Takahashi extension** [21]) of S -semimodules is an exact sequence of the form

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0.$$

2. Noetherian and Artinian Semirings

As before, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S -semimodule is a *left* S -semimodule.

Definition 2.1: A left S -semimodule M is

Noetherian (resp., **k -Noetherian**), iff M satisfies the ACC on its S -subsemimodules (resp., subtractive S -subsemimodules).

Artinian (resp., **k -Artinian**), iff M satisfies the DCC on its S -subsemimodules (resp., subtractive S -subsemimodules).

The corresponding notions for right S -semimodules are defined analogously.

Some authors, e.g. [8], call k -Noetherian (resp. k -Artinian) semirings Noetherian (resp. Artinian). Others, e.g. [15], call them weakly Noetherian (resp. weakly Artinian) semirings. We prefer the terminology in the definition above as it is more informative and avoids confusion.

Remark 2.2: Every direct summand of an S -semimodule is subtractive. Let M be an S -semimodule and L a direct summand of M . Then there exists $N \leq_S M$ such that $M = N \oplus L$. Let $m \in M$ and $\ell, \ell' \in L$ be such that $m + \ell = \ell'$. Write $m = \tilde{n} + \tilde{\ell}$ for some $\tilde{n} \in N$ and $\tilde{\ell} \in L$, whence $m + \ell = (\tilde{n} + \tilde{\ell}) + \ell = \tilde{n} + (\tilde{\ell} + \ell) = \ell'$. Since the sum $N + L$ is direct, $\tilde{n} = 0$, and thus $m = \tilde{\ell} \in L$.

The following result is an easy observation; however, we highlight it as it will be used frequently in the proofs of the main results.

Lemma 2.3: Let M be an S -semimodule and N a subtractive S -subsemimodules of M . If $M = L \oplus K$ for some $L \leq_S N$ and $K \leq_S M$, then

$$N = L \oplus (K \cap N).$$

Proof: Clearly, $L + (K \cap N) \subseteq N$. Let $n \in N$. Since $M = L + K$, there exist $k \in K$ and $\ell \in L$ such that $n = \ell + k$. Since $\ell \in N$ and N is subtractive, we have $k \in N$, whence $n \in L + (K \cap N)$. So, $N = L + (K \cap N)$. Suppose now that $\ell + k = \ell' + k'$ for some $\ell, \ell' \in L$ and $k, k' \in K \cap N$. Since the sum $L + K$ is direct, $\ell = \ell'$ and $k = k'$.

Example 2.4: Let $S := M_2(\mathbb{R}^+)$. Consider the left ideals

$$E_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\} \text{ and } E_2 = \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \mid c, d \in \mathbb{R}^+ \right\}$$

and the left ideal

$$N_{\geq 1} := \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} \mid a \leq c, b \leq d, a, b, c, d \in \mathbb{R}^+ \right\}.$$

Then we have $N_{\geq 1} \cap (E_1 \oplus E_2) = N_{\geq 1} \cap S = N_{\geq 1}$, while $N_{\geq 1} \cap E_1 = \{0\}$ and $N_{\geq 1} \cap E_2 = E_2$. So, we have

$$N_{\geq 1} \cap (E_1 \oplus E_2) \neq (N_{\geq 1} \cap E_1) \oplus (N_{\geq 1} \cap E_2).$$

Notice that $N_{\geq 1} \leq_S S$ is not subtractive, whence the condition that N is a subtractive subsemimodule of M in Lemma 2.3 cannot be dropped.

Definition 2.5: Let S be a semiring, M be a left S -semimodule and $N \leq_S M$. A subtractive left S -subsemimodule $L \leq_S M$ is a **maximal subtractive subsemimodule** of N if $L \subsetneq N$ and if L' is a subtractive subsemimodule of M with $L \subseteq L' \subseteq N$, then $L = L'$ or $L' = N$.

Lemma 2.6: If M is a k -Noetherian left S -semimodule, then every non-zero subsemimodule of M contains a maximal subtractive S -subsemimodule.

Proof: Let $N \leq_S S$ be a non-zero subsemimodule and consider

$$\mathcal{I} := \{L \subsetneq_S N \mid L \text{ is a subtractive subsemimodule of } M\}.$$

Notice that $L_0 := \{0_M\} \in \mathcal{I}$. If L_0 is a maximal subtractive subsemimodule of N , then we are done. Otherwise, there exists $L_1 \in \mathcal{I}$ such that $L_0 \subsetneq L_1$. If L_1 is a maximal subtractive subsemimodule of M , we are done. Otherwise, there exists $L_2 \in \mathcal{I}$ such that $L_1 \subsetneq L_2$. If no such maximal subsemimodule of N exists, we obtain a non-terminating strictly ascending chain

$$L_0 \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_k \subsetneq L_{k+1} \subsetneq \cdots$$

of S -subsemimodules of N which are subtractive subsemimodules of M , absurd since M is k -Noetherian.

Definition 2.7: The semiring S is **left Noetherian** (resp., **left k -Noetherian**), iff ${}_S S$ is Noetherian (resp., left k -Noetherian), equivalently every ascending chain condition of left (resp., subtractive left) ideals of S terminates;

left Artinian (resp., left k -Artinian), iff ${}_sS$ is Artinian (resp., left k -Artinian), equivalently every descending chain of left (resp., subtractive left) ideals of S terminates.

The **right $(k-)$ Noetherian** and **right $(k-)$ Artinian semirings** are defined analogously. A semiring which is both left and right $(k-)$ Noetherian is called **$(k-)$ Noetherian**, and a semiring which is both left and right $(k-)$ Artinian is called **$(k-)$ Artinian**.

Example 2.8: ([4]) The semiring $\mathbb{Z}^+$ is Noetherian but not Artinian. Setting $I_k := \{0, k, k+1, k+2, \dots\}$ yields the strictly descending *non-terminating* chain of ideals of $\mathbb{Z}^+$:

$$I_1 \supsetneq I_2 \supsetneq \dots \supsetneq I_k \supsetneq I_{k+1} \supsetneq \dots,$$

i.e. $\mathbb{Z}^+$ is not Artinian.

Lemma 2.9: *The only non-trivial proper subtractive left ideals of $S := M_2(\mathbb{R}^+)$ are*

$$\begin{aligned} E_1 &= \text{Span}\left(\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right) = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+\right\} \\ E_2 &= \text{Span}\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}^+\right\} \\ N_r &= \left\{\begin{bmatrix} ra & a \\ rb & b \end{bmatrix} \mid a, b \in \mathbb{R}^+, r \in \mathbb{R}^+ \setminus \{0\}\right\}. \end{aligned}$$

We prove this technical lemma in three steps.

Step I: E_1 , E_2 and N_r ($r \in \mathbb{R}^+ \setminus \{0\}$) are subtractive left ideals of S .

$E_1 \leq S$ is a left ideal: for every $a, b, c, d, p, q, r, s \in \mathbb{R}^+$ we have

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} pa + qb + c & 0 \\ ra + sb + d & 0 \end{bmatrix} \in E_1.$$

Moreover, E_1 is subtractive since

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$$

implies $q = 0 = s$ and $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in E_1$. Similarly, E_2 is a subtractive left ideal of S .

For any nonzero $r \in \mathbb{R}^+$, N_r is a left ideal since (for all $a, b, c, d, k, \ell, m, n \in \mathbb{R}^+$) we have

$$\begin{bmatrix} k & \ell \\ m & n \end{bmatrix} \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} + \begin{bmatrix} rc & c \\ rd & d \end{bmatrix} = \begin{bmatrix} r(ka + \ell b + c) & ka + \ell b + c \\ r(ma + nb + d) & ma + nb + d \end{bmatrix} \in N_r.$$

Moreover, $N_r \leq S$ is subtractive since

$$\begin{bmatrix} k & \ell \\ m & n \end{bmatrix} + \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} = \begin{bmatrix} rc & c \\ rd & d \end{bmatrix},$$

whence $c = a + k/r = a + \ell, d = b + m/r = b + n$. So, $k = r\ell, m = rn$, and

$$\begin{bmatrix} k & \ell \\ m & n \end{bmatrix} \in N_r.$$

Step II: E_1 , E_2 and N_r ($r \in \mathbb{R}^+ \setminus \{0\}$) are maximal subtractive left ideals of S .

Let I be a subtractive left ideal of $M_2(\mathbb{R}^+)$ such that $E_1 \subsetneq I$. Then there exists $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in I$ such that $q \neq 0$ or $s \neq 0$, whence $\begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} \in I$ as $\begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} \in I$ and

$$\begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} + \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in I$$

If $q \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/q & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix}.$$

If $s \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/s \end{bmatrix} \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix}.$$

Either way $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$, which implies $E_2 \subseteq I$ and $I = S$. Similarly, if I is a subtractive left ideal of $M_2(\mathbb{R}^+)$ such that $E_2 \subsetneq I$, then $I = S$.

Let $r \in \mathbb{R}^+ \setminus \{0\}$ and I be a subtractive left ideal of $M_2(\mathbb{R}^+)$ such that $N_r \subsetneq I$. Then there exists $\begin{bmatrix} k & \ell \\ m & n \end{bmatrix} \in I$ such that $k \neq r\ell$ or $m \neq rn$. Consider the case $k < r\ell$. Then $k + p = r\ell$ for some $p \in \mathbb{R}^+ \setminus \{0\}$. Thus $\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} \in I$ or $\begin{bmatrix} p & 0 \\ u & q \end{bmatrix} \in I$ for some $u, q \in \mathbb{R}^+$ as

$$\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} + \begin{bmatrix} k & \ell \\ m & n \end{bmatrix} = \begin{bmatrix} r\ell & \ell \\ rn & n \end{bmatrix} \in I$$

or

$$\begin{bmatrix} p & 0 \\ u & q \end{bmatrix} + \begin{bmatrix} k & \ell \\ m & n \end{bmatrix} = \begin{bmatrix} r\ell & \ell \\ r(q+n) & q+n \end{bmatrix} \in I.$$

Thus

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ u & q \end{bmatrix}.$$

Either way we have $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$, whence $E_1 \subsetneq I$ and $I = S$. The other cases are handled similarly.

Step III: E_1 , E_2 and N_r ($r \in \mathbb{R}^+ \setminus \{0\}$) are the only non-trivial proper subtractive left ideals of S .

Let I be a proper non-trivial subtractive left ideal of S . Then $\begin{bmatrix} k & \ell \\ m & n \end{bmatrix} \in I \setminus \{0\}$ for some $k, \ell, m, n \in \mathbb{R}^+$. If $k \neq 0$, then

$$\begin{bmatrix} 1/k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k & \ell \\ m & n \end{bmatrix} = \begin{bmatrix} 1 & \ell/k \\ 0 & 0 \end{bmatrix}$$

whence $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$, or $\begin{bmatrix} k/\ell & 1 \\ 0 & 0 \end{bmatrix} \in I$, and it follows that $I \in \{E_1, N_{k/\ell}, S\}$ as I contains E_1 or $N_{k/\ell}$. If $\ell \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 1/\ell & 0 \end{bmatrix} \begin{bmatrix} k & \ell \\ m & n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k/\ell & 1 \end{bmatrix},$$

whence $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$ or $\begin{bmatrix} 0 & 0 \\ k/\ell & \ell \end{bmatrix} \in I$, and so $I \in \{E_2, N_{k/\ell}, S\}$ as I contains E_2 or $N_{k/\ell}$. If $m \neq 0$, then

$$\begin{bmatrix} 0 & 1/m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k & \ell \\ m & n \end{bmatrix} = \begin{bmatrix} 1 & n/m \\ 0 & 0 \end{bmatrix}$$

whence $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$ or $\begin{bmatrix} m/n & 1 \\ 0 & 0 \end{bmatrix} \in I$, and it follows that $I \in \{E_1, N_{m/n}, S\}$ as I contains E_1 or $N_{m/n}$. If $n \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} k & \ell \\ m & n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ m/n & 1 \end{bmatrix}$$

whence $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$ or $\begin{bmatrix} 0 & 0 \\ m/n & \ell \end{bmatrix} \in I$. So, $I \in \{E_2, N_{m/n}, S\}$ as I contains E_2 or $N_{m/n}$.

We provide an example of a semiring which is left k -Artinian and left k -Noetherian but neither left Artinian nor left Noetherian:

Example 2.10: Let $S = M_2(\mathbb{R}^+)$. By Lemma 2.9, the only subtractive left ideals of S are 0 , S , E_1 , E_2 and N_r ($r \in \mathbb{R}^+ \setminus \{0\}$). Notice that for $r \neq s$, the left ideals N_r, N_s are not comparable. Thus, the longest ascending (descending) chain of subtractive left ideals of S is $0 \subsetneq N \subsetneq S$ ($S \supsetneq N \supsetneq 0$) with $N = E_2$ or $N = N_r$ for some $r \in \mathbb{R}^+$. Whence, S is left k -Artinian and left k -Noetherian.

On the other hand, for every $r \in \mathbb{R}^+$ we have a left ideal of S given by

$$N_{\geq r} = \left\{ \begin{bmatrix} a & p \\ b & q \end{bmatrix} : p \geq ra, q \geq rb, a, b, p, q \in \mathbb{R}^+ \right\}.$$

Thus, we have an infinite strictly descending chain of left ideal that does not terminate

$$N_1 \supsetneq N_{\geq 2} \supsetneq N_{\geq 3} \supsetneq \cdots \supsetneq N_{\geq m} \supsetneq N_{\geq m+1} \supsetneq \cdots,$$

i.e. S is not left Artinian. On the other hand, we have an infinite ascending chain of left ideals that does not terminate

$$N_{\geq 1} \subsetneq N_{\geq \frac{1}{2}} \subsetneq N_{\geq \frac{1}{3}} \subsetneq \cdots \subsetneq N_{\geq \frac{1}{m}} \subsetneq N_{\geq \frac{1}{m+1}} \subsetneq \cdots,$$

i.e. S is not left Noetherian.

An additional example of a commutative k -Noetherian semiring that is not Noetherian was communicated to Abuhlail by T. Nam:

Example 2.11: The semiring $\mathbb{R}^+[x]$ is k -Noetherian but not Noetherian.

Proof: The semiring $\mathbf{B}[x]$, where $\mathbf{B}$ is the Boolean semiring, is not Noetherian. The surjective morphism of semirings

$$f: \mathbb{R}^+ \rightarrow \mathbf{B}, r \mapsto \begin{cases} 1, & r \neq 0 \\ 0, & r = 0 \end{cases}$$

induces a surjective morphism of semirings $\mathbb{R}^+[x] \rightarrow \mathbf{B}[x]$, whence $\mathbb{R}^+[x]$ is not Noetherian.

We do not know whether k -Artinian semirings are k -Noetherian. However, we have the following interesting result.

Lemma 2.12: *A left S -semimodule M satisfies the ACC on direct summands if and only if M satisfies the DCC on direct summands.*

Proof: ($\Rightarrow$) Assume that M satisfies the ACC on direct summands. Let

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots \supseteq N_i \supseteq N_{i+1} \supseteq \cdots \quad (4)$$

be a descending chain of direct summands of M . For every $i \in \mathbb{N}$, there exists a direct summand $L_i \leq_S M$ such that $M = N_i \oplus L_i$. Since $M = N_2 \oplus L_2$ and $N_1 \supseteq N_2$, we have (taking into consideration Remark 2.2):

$$N_1 \stackrel{\text{Lemma 2.3}}{=} N_2 \oplus (N_1 \cap L_2) \text{ and } M = N_1 \oplus L_1 = N_2 \oplus (N_1 \cap L_2) \oplus L_1.$$

Set $K_1 := L_1$ and $K_2 := (N_1 \cap L_2) \oplus L_1$, so that $N_1 \oplus K_1 = M = N_2 \oplus K_2$ and $K_1 \subseteq K_2$.

Now, $N_2 \supseteq N_3$ and $M = N_3 \oplus L_3$, whence $N_2 \stackrel{\text{Lemma 2.3}}{=} N_3 \oplus (N_2 \cap L_3)$ and so

$$M = N_2 \oplus K_2 = N_3 \oplus (N_2 \cap L_3) \oplus K_2.$$

Set $K_3 := (N_2 \cap L_3) \oplus K_2$, so that $M = N_3 \oplus K_3$ and $K_2 \subseteq K_3$. Continuing this way, we obtain an ascending chain

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_i \subseteq K_{i+1} \subseteq \cdots \quad (5)$$

of direct summands of ${}_S M$. By our assumption, the ascending chain (5) terminates, whence there exists $t \in \mathbb{N}$ such that $K_i = K_t$ for any $i \geq t$. For any $i \geq t$, we have $N_t \supseteq N_i$, $M = N_i \oplus K_i$ and $N_t \cap K_t = 0$ and so

$$N_t \stackrel{\text{Lemma 2.3}}{=} N_i \oplus (N_t \cap K_i) = N_i \oplus (N_t \cap K_t) = N_i,$$

thus the descending chain (4) terminates.

($\Leftarrow$) Assume that M satisfies the DCC on direct summands. Let

$$L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L_i \subseteq L_{i+1} \quad (6)$$

be an ascending chain of direct summands of M . For every $i \in \mathbb{N}$, there exists an S -subsemimodule $N_i \leq_S M$ such that $M = L_i \oplus N_i$; in particular $M = L_1 \oplus N_1$. Since $L_1 \subseteq L_2$ it follows (taking into consideration Remark 2.2) that $L_2 \stackrel{\text{Lemma 2.3}}{=} L_1 \oplus (L_2 \cap N_1)$, whence

$$M = L_2 \oplus N_2 = L_1 \oplus (L_2 \cap N_1) \oplus N_2.$$

Since $L_2 \cap N_1 \subseteq N_1$ it follows that

$$N_1 \stackrel{\text{Lemma 2.3}}{=} L_2 \cap N_1) \oplus (N_1 \cap (L_1 \oplus N_2)),$$

whence

$$M = L_1 \oplus N_1 = L_1 \oplus (L_2 \cap N_1) \oplus (N_1 \cap (L_1 \oplus N_2)).$$

Setting $N'_1 := N_1$ and $N'_2 := N_1 \cap (L_1 \oplus N_2)$, we have $L_1 \oplus N'_1 = M = L_2 \oplus N'_2$ where $N'_1 \supseteq N'_2$. Since $M = L_2 \oplus N'_2$ and $L_2 \subseteq L_3$, it follows that $L_3 \stackrel{\text{Lemma 2.3}}{=} L_2 \oplus (L_3 \cap N'_2)$, whence

$$M = L_3 \oplus N_3 = L_2 \oplus (L_3 \cap N'_2) \oplus N_3.$$

Since $L_3 \cap N'_2 \subseteq N'_2$, we have

$$N'_2 \stackrel{\text{Lemma 2.3}}{=} (L_3 \cap N'_2) \oplus (N'_2 \cap (L_2 \oplus N_3)).$$

Setting $N'_3 := N'_2 \cap (L_2 \oplus N_3)$, we have $N'_2 \supseteq N'_3$ and

$$M = L_2 \oplus N'_2 = L_2 \oplus (L_3 \cap N'_2) \oplus N'_3 = L_3 \oplus N'_3.$$

Continuing this process, we obtain a descending chain

$$N'_1 \supseteq N'_2 \supseteq \cdots \supseteq N'_i \supseteq N'_{i+1} \supseteq \cdots \quad (7)$$

of *direct summands* of M such that $M = L_i \oplus N'_i$ for every $i \in \mathbb{N}$. By our assumption, the descending chain (7) terminates, *i.e.* there exists some $k \in \mathbb{N}$ such that $N'_i = N'_k$ for every $i \geq k$.

Now, for every $i \geq k$, we have $L_k \subseteq L_i$, $M = L_k \oplus N'_k$ and $L_i \cap N'_i = 0$ and so

$$L_i \stackrel{\text{Lemma 2.3}}{=} L_k \oplus (L_i \cap N'_k) = L_k \oplus (L_i \cap N'_i) = L_k.$$

Thus the ascending chain (6) terminates.

A ring in which every left ideal is a direct summand is left Artinian and left Noetherian [22, 3.4 and 4.1] (in fact, left semisimple). The following result extends this fact to semirings.

Theorem 2.13: *If every subtractive left ideal of S is a direct summand, then S is left k -Artinian and left k -Noetherian.*

Proof: Assume that every subtractive left ideal of S is a direct summand.

Claim I: S is left k -Artinian.

Suppose that

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots \supsetneq I_i \supsetneq I_{i+1} \supsetneq \cdots \quad (9)$$

is a strictly descending chain of left subtractive ideals of S that does not terminate. For every $k \in \mathbb{N}$, there exists, by our assumption, some left ideal $N_k \leq_S S$ such that $S = I_k \oplus N_k$. The left ideals I_k, N_k are non-zero as the chain does not terminate, and are subtractive by Remark 2.2.

Since $I_1 \supseteq I_2$ and $S = I_2 \oplus N_2$, we have

$$I_1 \stackrel{\text{Lemma 2.3}}{=} I_2 \oplus (I_1 \cap N_2).$$

Then $J_1 := I_1 \cap N_2$ is a subtractive left ideal of S , which is non-zero as $I_1 \not\supseteq I_2$, and $I_1 = I_2 \oplus J_1$. Since $I_2 \supseteq I_3$ and $S = I_3 \oplus N_3$, we have

$$I_2 \stackrel{\text{Lemma 2.3}}{=} I_3 \oplus (I_2 \cap N_3).$$

Then $I_2 \cap N_3$ is a subtractive left ideal of S , which is non-zero as $I_2 \not\supseteq I_3$, and $I_1 = I_2 \oplus J_1 = I_3 \oplus J_2 \oplus J_1$. Continuing this process, we obtain at the k th step, a non-zero subtractive left ideal $J_k \leq_S S$ such that

$$I_k = I_{k+1} \oplus J_k \text{ and } I_1 = I_{k+1} \oplus J_k \oplus \cdots \oplus J_1.$$

Setting $J'_i := J_1 \oplus \cdots \oplus J_i$ for each $i \in \mathbb{N}$, we have $S = J'_i \oplus I_{i+1} \oplus N_1$ whence J'_i is subtractive (by Remark 2.2). One can easily show that $J := \bigcup_{i \in \mathbb{N}} J'_i$ is subtractive.

By our assumption, $S = J \oplus N$ for some left ideal of $N \leq_S S$. Thus $1_S = j + n$ for some $j \in J$ and $n \in N$. Since $j \in J'_i$ for some $i \in \mathbb{N}$, it can be written in a unique way as $j = j_1 + j_2 + \cdots + j_i$ for some uniquely determined $j_k \in J_k$, $k = 1, 2, \dots, i$. Since $J'_{i+1} \subseteq J$, the sum $J'_{i+1} + N$ is direct, whence the sum $J_1 + J_2 + \cdots + J_i + J_{i+1} + N$ is direct. Setting

$$K := J_1 \oplus \cdots \oplus J_i \oplus N,$$

this means that the sum $J_{i+1} + K$ is direct. For any $s_{i+1} \in J_{i+1} \setminus \{0\}$, we have

$$\begin{aligned} s_{i+1} &= s_{i+1} 1_S = s_{i+1} (j_1 + j_2 + \cdots + j_i + n) \\ &= s_{i+1} j_1 + s_{i+1} j_2 + \cdots + s_{i+1} j_i + s_{i+1} n \end{aligned}$$

where $s_{i+1} j_k \in J_k$ for $k = 1, 2, \dots, i$ and $s_{i+1} n \in N$. It follows that $s_{i+1} \in J_{i+1} \cap K = 0$, absurd since $s_{i+1} \neq 0$. So, the descending chain (8) terminates.

Claim II: S is left k -Noetherian.

Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_i \subseteq I_{i+1} \subseteq \cdots \quad (9)$$

be an ascending chain of subtractive left ideals of S . Since every direct summand of ${}_S S$ is subtractive (by Remark 2.2), it follows from the proof of **Claim I** that ${}_S S$ satisfies DCC on direct summands, whence ${}_S S$ satisfies ACC on direct summands by Lemma 2.12. Since (9) is an ascending chain of subtractive left ideals of S , whence of direct summands of ${}_S S$ (by our assumption), the chain terminates.

Example 2.14: Let p be a prime number. Every subtractive ideal of the semiring $S = B(p + 1, p)$ is a direct summand, and S is k -Artinian and k -Noetherian.

Proof: S has no non-trivial proper subtractive ideals, thus every subtractive left ideal of S is a direct summand. Notice that S is k -Artinian and k -Noetherian since it has finitely many elements.

Example 2.15: Let $S = \mathbb{B}^{\mathbb{N}}$ with the canonical structure of a semiring induced by that on $\mathbb{B}$. Then S has a subtractive left ideal which is not a direct summand and S is neither k -Artinian nor k -Noetherian.

Proof: The subtractive left ideal $\bigoplus_{n \in \mathbb{N}} \mathbb{B}$ is not a direct summand. Notice that neither the ascending chain

$$\mathbb{B} \times \prod_{n \geq 2} \{0\} \subsetneq \mathbb{B}^2 \times \prod_{n \geq 3} \{0\} \subsetneq \dots \subsetneq \mathbb{B}^i \times \prod_{n \geq i+1} \{0\} \subsetneq \mathbb{B}^{i+1} \times \prod_{n \geq i+2} \{0\} \subsetneq \dots$$

nor the descending chain

$$\{0\} \times \prod_{n \geq 2} \mathbb{B} \supsetneq \{0\}^2 \times \prod_{n \geq 3} \mathbb{B} \supsetneq \dots \supsetneq \{0\}^i \times \prod_{n \geq i+1} \mathbb{B} \supsetneq \{0\}^{i+1} \times \prod_{n \geq i+2} \mathbb{B} \supsetneq \dots$$

terminates, thus S is neither k -Noetherian nor k -Artinian.

2.16. Let I be a left S -semimodule.

For a left S -semimodule M , we say that I is

M -injective [11, page 197], iff for every *injective* S -linear map $f: L \rightarrow M$ and any S -linear map $g: L \rightarrow I$, there exists an S -linear map $h: M \rightarrow I$ such that $h \circ f = g$;

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M \\ & & \downarrow g & \searrow h & \\ & & I & & \end{array}$$

M - i -injective [6], iff for every *normal monomorphism* $f: L \rightarrow M$ and any S -linear map $g: L \rightarrow I$, there exists an S -linear map $h: M \rightarrow I$ such that $h \circ f = g$;

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{f \text{ (normal)}} & M \\ & & \downarrow g & \searrow h & \\ & & I & & \end{array}$$

M - e -injective [5], iff for every short exact sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ of left S -semimodules, the following sequence $0 \rightarrow \text{Hom}_S(N, I) \xrightarrow{(g, I)} \text{Hom}_S(M, I) \xrightarrow{(f, I)} \text{Hom}_S(L, I) \rightarrow 0$ of commutative monoids is exact.

We say that I is **injective** (resp. **i -injective**, **e -injective**), iff I is M -injective (resp. M - i -injective, M - e -injective) for every left S -semimodule M .

A Bass-Papp Theorem for Semirings

The celebrated *Bass-Papp Theorem* states that a ring R is left (right) Noetherian if and only if every direct sum of left (right) injective R -modules is $(R$ -)injective (e.g., [19, 3.39], [12, page 407]).

An e -injective version of this theorem was obtained by Abuhlail et al.:

Proposition 2.17: ([16, Theorem 3.6], [5, Theorem 5.4]) The following are equivalent for a semiring S :

- (1) S is a left Noetherian ring;
- (2) every direct sum of injective left S -semimodules is injective and the left S -semimodule $S/V(S)$ can be embedded in an injective left S -semimodule.
- (3) every direct sum of e -injective left S -semimodules is e -injective and the left S -semimodule $S/V(S)$ can be embedded in an e -injective left S -semimodule.

Notice that the assumptions in Proposition 2.17 force the semiring S to be a ring. In what follows we provide a partial version of the Bass-Papp characterization for left k -Noetherian semirings.

2.18. We say that a semiring S **has enough S - i -injective left semimodules**, iff every left S -semimodule can be embedded into an S - i -injective left S -semimodule.

Theorem 2.19: Let S be a semiring with enough left S - i -injective left semimodules. If every direct sum of left S - i -injective S -semimodules is S - i -injective, then S is left k -Noetherian.

Proof: Let

$$L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L_i \subseteq L_{i+1} \subseteq \cdots \quad (10)$$

be a chain of subtractive left ideals of S and consider the left ideal $L := \bigcup_{n \in \mathbb{N}} L_n$. It is clear that L is subtractive. By our assumption, there exists for every $n \in \mathbb{N}$ an S - i -injective left S -semimodule J_n and an embedding $S/L_n \xrightarrow{\iota_n} J_n$. Set $J := \bigoplus_{n \in \mathbb{N}} J_n$ and consider the S -linear map

$$\varphi_n: S \xrightarrow{\pi_i} S/L_n \xrightarrow{\iota_n} J_n \text{ and } \varphi: L \rightarrow J, x \mapsto \sum_{k=1}^{\infty} \varphi_k(x).$$

Notice that φ is well defined as each $x \in L$ belongs to L_n for some $n \in \mathbb{N}$ and so $\varphi_k(x) = 0$ for all $k \geq n$, i.e. $\varphi(x) = \sum_{k=1}^{\infty} \varphi_k(x) = \sum_{k=1}^{n-1} \varphi_k(x)$.

By our assumption, J is S - i -injective and so there exists an S -linear map $\psi: S \rightarrow J$ such that $\psi \circ \iota = \varphi$.

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & S \\ & & \downarrow \varphi & \nearrow \psi & \\ & & J & & \end{array}$$

Let $\psi(1_S) = \sum t_k \in J$. Then $\psi(1_S) \in \bigoplus_{k=1}^{m-1} J_k$ for some m , whence $\psi(x) = \psi(x \cdot 1_S) = x\psi(1_S) \in \bigoplus_{k=1}^{m-1} J_k$ for every $x \in L$. In particular, $\varphi_m(x) = (\pi_m \circ \phi)(x) = 0$ where π_m is the projection on J_m . Thus $x \in L_n$ and $L = L_n$, whence $L_k = L_n$ for all $k \geq n$, i.e. the chain terminates. Consequently, S is k -Noetherian.

If S is an additively idempotent semiring such that every direct sum of left S - i -injective S -semimodules is S - i -injective, then S is left k -Noetherian. This results from Theorem 2.19 and the fact that every semimodule over an additively idempotent semiring can be embedded into an e -injective (whence an i -injective) left S -semimodule [5, 4.5].

Theorem 2.21: *If S is a semiring such that every short exact sequence of left S -semimodules $0 \rightarrow L \rightarrow S \rightarrow M \rightarrow 0$ is left splitting, then S is a left k -Noetherian.*

Proof: Let

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_k \subsetneq N_{k+1} \subsetneq \cdots$$

be a non-terminating ascending chain of subtractive left ideals of S . Notice that $N := \bigcup_{i \in \mathbb{N}} N_i$ is a subtractive ideal of S , whence (by assumption) the following short exact sequence

$$0 \rightarrow N \xrightarrow{\iota} S \xrightarrow{\pi} S/N \rightarrow 0$$

of left S -semimodules is left splitting. Let $h: S \rightarrow N$ be an S -linear map such that $h \circ \iota = id_N$. Then $h(1_S) \in N$, that is $h(1_S) \in N_i$ for some $i \in \mathbb{N}$. If $x \in N_{i+1} \setminus N_i$, then

$$x = (h \circ \iota)(x) = h(x) = h(x1_S) = xh(1_S) \in N_i$$

a contradiction. Hence S is k -Noetherian.

Corollary 2.22: *If S is a semiring such that every subtractive left ideal is S - i -injective, then S is a left k -Noetherian semiring.*

Compliance with Ethical Standards

Funding: This study was funded by the Deanship of Scientific Research at King Fahd University of Petroleum & Minerals (grant number: RG1304-1 & RG1304-2).

Conflict of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

References

- [1] F. Alarcon and D. Anderson, "Commutative semirings and their lattices of ideals", *Houston J. Math.*, vol. 20, pp. 571-590 (1994).
- [2] J. Y. Abuhlail and R. G. Noegraha, "Injective Semimodules - Revisited", *J. Algebra Appl.*, vol. 21, no. 1, Article ID 2250242, 28 pp. (2022).
- [3] J. Abuhlail, "Exact sequence of commutative monoids and semimodules", *Homology Homotopy Appl.*, vol. 16, no. 1, pp. 199-214 (2014).
- [4] P. J. Allen and L. Dale, "Ideal theory in Z^+ ", *Publ. Math. Debrecen*, vol. 22, no. 3-4, pp. 219-224 (1975).
- [5] J. Abuhlail, S. Il'in, Y. Katsov and T. Nam, "Toward homological characterization of semirings by e-injective semimodules", *J. Algebra Appl.*, vol. 17, no. 4, 1850059 (24 pages) (2018).
- [6] H. Al-Thani, "Injective semimodules", *J. Inst. Math. Comput. Sci.*, vol. 16, no. 3, pp. 143-152 (2003).
- [7] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories; The Joy of Cats*. Dover: Dover Publications Edition (2009).
- [8] R. Ebrahimi Atani and S. Ebrahimi Atani, "Ideal theory in commutative semirings", *Bul. Acad. Stiinte Repub. Mold. Mat.*, vol. 57, no. 2, pp. 14-23 (2008).
- [9] R. El Bashir, J. Hurt, A. Jancarik and T. Kepka, "Simple commutative semirings", *J. Algebra*, vol. 236, no. 1, pp. 277-306 (2001).
- [10] K. Głazek, *A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences. With Complete Bibliography*. Dordrecht: Kluwer Academic Publishers (2002).
- [11] J. Golan, *Semirings and their Applications*. Dordrecht: Kluwer Academic Publishers (1999).
- [12] P. A. Grillet, *Abstract Algebra*. Second Edition, New York: Springer (2007).
- [13] M. Henriksen, "Ideals in semirings with commutative addition", *Notices Amer. Math. Soc.*, vol. 5, no. 3, pp. 321 (1958).
- [14] U. Hebisch and H. J. Weinert, *Semirings: Algebraic Theory and Applications in Computer Science*. River Edge, NJ: World Scientific Publishing Co., Inc. (1998).
- [15] P. Lescot, "Prime and primary ideals in semirings", *Osaka J. Math.*, vol. 52, no. 3, pp. 721-736 (2015).

- [16] S. N. Il'in, " Direct sums of injective semimodules and direct products of projective semimodules over semirings" , *Russ. Math.*, vol. 54, no. 10, pp. 27-37 (2010).
- [17] Y. Katsov, T. G. Nam and N. X. Tuyen, " More on subtractive semirings: simpleness, perfectness, and related problems" , *Comm. Algebra*, vol. 39, no. 11, pp. 4342-4356 (2011).
- [18] G. L. Litvinov and V. P. Maslov, *Idempotent Mathematics and Mathematical Physics*, In Contemporary Mathematics, vol. 377, American Mathematical Society, Providence, RI, 2005. Papers from the International Workshop held in Vienna, February 3-10 (2003).
- [19] J. Rotman, *An Introduction to Homological Algebra*, Second edition, New York: Springer (2009).
- [20] M. Takahashi, "On the bordism categories. III. Functors Hom and for semimodules", *Math. Sem. Notes Kobe Univ.*, vol. 10, no. 2, pp. 551-562 (1982).
- [21] M. Takahashi, "Extensions of Semimodules I", *Math. Sem. Notes Kobe Univ.*, vol. 10, pp. 563-592 (1982).
- [22] R. Wisbauer, *Foundations of Module and Ring Theory: A Handbook for Study and Research*. Reading: Gordon and Breach (1991).

Received July, 2025