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On 2-N-prime ideals over commutative $\mathbb{Z}_2$-graded ring

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Abstract

Let R be a commutative ring with a nonzero unity element. In this article, we introduce and examine new classes of ideals in $\mathbb{Z}_2$-graded rings, expanding upon the earlier concept of N-prime ideals. Using the function $N : R \to R_0$, defined by $N(x) = x_0^2 - x_1^2$ for $x = x_0 + x_1 \in R$, we define and analyze two new types of ideals, 2-N-prime and weakly 2-N-prime ideals. A proper ideal I is called 2-N-prime if $xy \in I$ implies that $N(x^2) \in I$ or $N(y^2) \in I$. Similarly, I is weakly 2-N-prime if $0 \neq xy \in I$ implies that $N(x^2) \in I$ or $N(y^2) \in I$. We explore the fundamental properties of these ideals, highlighting their connections to established structures in graded ring theory. Through examples and counter examples, we illustrate the differences between 2-N-prime and weakly 2-N-prime ideals, underscoring their significance in the theory of

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graded rings. These results not only deepensour understanding of ideal theory in $\mathbb{Z}_2$-graded rings but also open new avenues for futures research.

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1. Introduction

The study of graded rings and their ideal structures has been a central theme in modern algebra, providing a rich framework for exploring connections between ring theory, homological algebra, and algebraic geometry. A graded ring is a ring R that can be written as a direct sum $R = \bigoplus_{g\in G} R_g$, where each component R_g is an additive subgroup of R and satisfies the property $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Here, G is a group, and elements in each R_g are called homogeneous elements. The set of all homogeneous elements, denoted by $h(R)$, is given by $\cup_{g\in G} R_g$. For more terminology and background, see [1, 4, 6, 7].

A $\mathbb{Z}_2$-graded ring is a specific type of graded ring where the group $G = \mathbb{Z}_2 = \{0,1\}$. In this case, each element $x \in R$ can be uniquely written as $x = x_0 + x_1$, where $x_0 \in R_0$ and $x_1 \in R_1$.

Among the most studied examples of graded rings are $\mathbb{Z}_2$-graded rings, which play an important role in representation theory and quantum mechanics. The interaction between grading and ideal theory in these rings has led to many extensions of classical notions, such as prime and semi-prime ideals.

In [2], the concept of N-prime ideals was introduced for $\mathbb{Z}_2$-graded rings using the function $N: R \to R_0$, defined by

$$N(x) = x_0^2 - x_1^2,$$

where $x = x_0 + x_1$ is the homogeneous decomposition of $x \in R$. It is worth mentioning that this function N first appeared in ([3], Theorem 5.8). A proper ideal I is called N-prime if $xy \in I$ implies $N(x) \in I$ or $N(y) \in I$. This approach links the grading structure to the function N, providing deeper insights into graded prime ideals.

In this article, we extend this framework by defining and studying two new classes of ideals: 2-N-prime ideals and weakly 2-N-prime ideals. These concepts generalize the classical idea of 2-prime ideals in a way that respects the grading structure. Specifically, a proper ideal I is called 2-N-prime if $xy \in I$ implies $N(x^2) \in I$ or $N(y^2) \in I$. A proper ideal I is called weakly 2-N-prime if $0 \neq xy \in I$ implies $N(x^2) \in I$ or $N(y^2) \in I$.

These definitions naturally arise from the graded structure of the ring and offer a new perspective on ideal theory in $\mathbb{Z}_2$-graded rings. In this work, we investigate the fundamental properties of these new ideals, explore their relationships, and establish connections to N-prime ideals.

The article is organized as follows. In Section 2, we provide the necessary background and preliminaries on the function N and N-prime ideals. Section 3 develops the theory of 2-N-prime ideals, supported by examples and counterexamples. Section 4 focuses on weakly 2-N-prime ideals, presenting detailed results along with illustrative examples and counterexamples. Section 5 concludes the paper by summarizing the main results and insights. Section 6 highlights directions for future research, while Section 7 poses several open questions for further investigation.

This study not only enriches the theory of $\mathbb{Z}_2$-graded rings but also sets the stage for further exploration of functional mappings in algebraic structures.

2. Preliminaries on the function N and N-prime ideals

In this article, we focus on $\mathbb{Z}_2$-graded rings $R = R_0 \oplus R_1$, where for all $i, j \in \mathbb{Z}_2$, the multiplication satisfies $R_i R_j \subseteq R_{i+j}$. Here, R_0 and R_1 are additive subgroups of R, with R_0 being a subring of R, and $1 \in R_0$. For each element $x \in R$, we can uniquely express $x = x_0 + x_1$, where $x_0 \in R_0$ and $x_1 \in R_1$. We then define a function $N(x) = x_0^2 - x_1^2$ for every $x \in R$. This function N first appeared in Theorem 5.8 of [3].

Theorem 2.1: [2] *Let S be a $\mathbb{Z}_2$-graded ring. Then*

(1) $N(\alpha) \in S_0$, for every $\alpha \in S$. Hence, N is a function from S to S_0.

(2) $N(0) = 0$.

(3) $N(1) = 1$.

(4) $N(\alpha\beta) = N(\alpha)N(\beta)$, for every $\alpha, \beta \in S$.

Theorem 2.2: [2] *Let S be a $\mathbb{Z}_2$-graded ring and $\alpha, \beta \in S$. Then*

$$N(\alpha + \beta) = N(\alpha) + N(\beta) + 2(\alpha_0\beta_0 - \alpha_1\beta_1).$$

Theorem 2.3: [2] *Let R be a $\mathbb{Z}_2$-graded ring and $x \in h(R)$. Then*

$$N(x) = \begin{cases} x^2, & x \in R_0 \\ -x^2, & x \in R_1 \end{cases}$$

Lemma 2.4: [2] *Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R. Then $N(I) \subseteq I \cap R_0 \subseteq I$.*

If I is a graded ideal of a graded ring R, then R/I is graded by $(R/I)_g = (R_g + I)/I$, for all $g \in G$.

Lemma 2.5: [2] *Let R be a $\mathbb{Z}_2$-graded ring and A be a graded ideal of R. Then*

$$N_{R/A}(\alpha + J) = N_R(\alpha) + A, \text{ for every } \alpha \in R.$$

Definition 2.6: [2] Let S be a $\mathbb{Z}_2$-graded ring and A be a proper ideal of S. Then A is said to be an N-prime ideal of S if whenever $r, s \in S$ such that $rs \in A$, then either $N(r) \in A$ or $N(s) \in A$.

Theorem 2.7: [2] *Let S be a $\mathbb{Z}_2$-graded ring and A be an ideal of S. If A is a prime ideal of S, then A is an N-prime ideal of S.*

Indeed, ([2], Example 5) introduces an N-prime ideal that is not a prime ideal. So, the converse of Theorem 2.7 is not necessarily true.

3. 2-N-Prime Ideals

Recall that from [8], a proper ideal I of a ring R is said to be 2-prime if whenever $xy \in I$, we have $x^2 \in I$ or $y^2 \in I$. In this section, we establish and examine 2-N-prime ideals.

Definition 3.1: Let S be a $\mathbb{Z}_2$-graded ring and A be a proper ideal of S. Then A is said to be a 2-N-prime ideal of S if whenever $r, s \in S$ such that $rs \in A$, then either $N(r^2) \in A$ or $N(s^2) \in A$.

Theorem 3.2: *Let S be a $\mathbb{Z}_2$-graded ring and A be an ideal of S. If A is a prime ideal of S, then A is a 2-N-prime ideal of S.*

Proof: Let $r, s \in S$ such that $rs \in A$. Then either $r \in A$ or $s \in A$ and so either $r^2 \in A$ or $s^2 \in A$ and then by Lemma 2.4, $N(r^2) \in N(A) \subseteq A$ or $N(s^2) \in N(A) \subseteq A$. Hence, A is a 2-N-prime ideal of S.

The next example shows that the converse of Theorem 3.2 is not necessarily true:

Example 3.3: Let S = $\mathbb{Z}[i]$ and G = $\mathbb{Z}_2$. Then S is graded by $S_0 = \mathbb{Z}$ and $S_1 = i\mathbb{Z}$. For $r = \alpha + i\beta \in S$, $N(r) = \alpha^2 + \beta^2$. Consider the ideal $A = 9S$ of S. A is not a prime ideal of S since $3.6 \in A$ but $3,6 \notin A$. Let $r, s \in S$ such that $rs \in A$. Then $rs = 9(\alpha + i\beta)$, for some $\alpha, \beta \in \mathbb{Z}$, and then $N(r)N(s) = N(rs) = 9^2(\alpha^2 + \beta^2)$. So, 3 divides $N(r)N(s)$ in $\mathbb{Z}$, and then 3 divides $N(r)$ or 3 divides $N(s)$, and hence $N(r^2) \in A$ or $N(s^2) \in A$. Thus, A is a 2-N-prime ideal of S.

Theorem 3.4: *Let S be a $\mathbb{Z}_2$-graded ring and A be a graded ideal of S such that A_0 is a 2-prime ideal of S_0. Then A is a 2-N-prime ideal of S.*

Proof: Let $r, s \in S$ such that $rs \in A$. Then $N(r), N(s) \in S_0$ such that $N(r)N(s) = N(rs) \in N(A) \subseteq A_0$ by Lemma 2.4. Since A_0 is 2-prime, $N(r^2) = (N(r))^2 \in A_0 \subseteq A$ or $N(s^2) = (N(s))^2 \in A_0 \subseteq A$, and hence A is a 2-N-prime ideal of S.

Theorem 3.5: *Let S be a $\mathbb{Z}_2$-graded ring. If A is a 2-N-prime ideal of S, then $\sqrt{A}$ is an N-prime ideal of S.*

Proof: Let $r, s \in S$ such that $rs \in \sqrt{A}$. Then there exists a positive integer k such that $r^k s^k = (rs)^k \in A$, and then either $(N(r))^{2k} = N(r^{2k}) \in A$ or

$(N(s))^{2k} = N(s^{2k}) \in A$ which implies that $N(r) \in \sqrt{A}$ or $N(s) \in \sqrt{A}$. Hence, $\sqrt{A}$ is an N-prime ideal of S.

Theorem 3.6: *Let T and S be two $\mathbb{Z}_2$-graded rings. Suppose that $f: T \to S$ is a ring homomorphism such that $f \circ N_T = N_S \circ f$. If A is a 2-N_S-prime ideal of S, then $f^{-1}(A)$ is a 2-N_T-prime ideal of T.*

Proof: Let $t, y \in T$ such that $ty \in f^{-1}(A)$. Then $f(t)f(y) = f(ty) \in A$, and then $N_S(f(t^2)) \in A$ or $N_S(f(y^2)) \in A$, which gives that $f(N_T(t^2)) \in A$ or $f(N_T(y^2)) \in A$, so $N_T(t^2) \in f^{-1}(A)$ or $N_T(y^2) \in f^{-1}(A)$. Hence, $f^{-1}(A)$ is a 2-N_T-prime ideal of T.

Theorem 3.7: *Let T and S be two $\mathbb{Z}_2$-graded rings. Suppose that $f: T \to S$ is a ring epimorphism such that $f \circ N_T = N_S \circ f$. If A is a 2-N_T-prime ideal of T with $Ker(f) \subseteq A$, then $f(A)$ is a 2-N_S-prime ideal of S.*

Proof: Let $s, y \in S$ such that $sy \in f(A)$. Then there exist $t, b \in T$ such that $f(t) = s$ and $f(b) = y$. Now, $f(tb) = f(t)f(b) = sy \in f(A)$, so $tb \in A$ as $Ker(f) \subseteq A$, and then $N_T(t^2) \in A$ or $N_T(b^2) \in A$, which implies that $f(N_T(t^2)) \in f(A)$ or $f(N_T(b^2)) \in f(A)$, and then $N_S(f(t^2)) \in f(A)$ or $N_S(f(b^2)) \in f(A)$, so $N_S(s^2) \in f(A)$ or $N_S(y^2) \in f(A)$. Hence, $f(A)$ is a 2-N_S-prime ideal of S.

Theorem 3.8: *Let R be a $\mathbb{Z}_2$-graded ring. If I is a 2-N-prime ideal of R and J, K are two subsets of R such that $JK \subseteq I$, then $\{N(x^2): x \in J\} \subseteq I$ or $\{N(x^2): x \in K\} \subseteq I$.*

Proof: If there exist $x \in J$ and $y \in K$ such that $N(x^2) \notin I$ and $N(y^2) \notin I$, we get $xy \notin I$ because I is a 2-N-prime ideal. This is a contradiction with the fact that $xy \in JK \subseteq I$.

Theorem 3.9: *Let R be a $\mathbb{Z}_2$-graded ring, B be a graded ideal of R and A be an ideal of R with $B \subseteq A$. Then A is a 2-N_R-prime ideal of R if and only if A/B is a 2-$N_{R/B}$-prime ideal of R/B.*

Proof: Suppose that A is a 2-N_R-prime ideal of R. Let $x + B, y + B \in R/B$ such that $(x + B)(y + B) \in A/B$. Then $xy \in A$, and then $N_R(x^2) \in A$ or $N_R(y^2) \in A$, which implies that $N_{R/B}((x + B)^2) = N_R(x^2) + B \in A/B$ or $N_{R/B}((y + B)^2) = N_R(y^2) + B \in A/B$. Hence, A/B is a 2-$N_{R/B}$-prime ideal of R/B. Conversely, let $x, y \in R$ such that $xy \in A$. Then $(x + B)(y + B) \in A/B$, and then $N_{R/B}((x + B)^2) \in A/B$ or $N_{R/B}((y + B)^2) \in A/B$, which implies that $N_R(x^2) + B \in A/B$ or $N_R(y^2) + B \in A/B$, so $N_R(x^2) \in A$ or $N_R(y^2) \in A$. Thus, A is a 2-N_R-prime ideal of R.

Theorem 3.10: *Let R be a $\mathbb{Z}_2$-graded ring. If I is an N-prime ideal of R, then I^2 is a 2-N-prime ideal of R.*

Proof: Let $x, y \in R$ such that $xy \in I^2 \subseteq I$. Then $N(x) \in I$ or $N(y) \in I$, and then $N(x^2) \in I^2$ or $N(y^2) \in I^2$. Hence, I^2 is a 2-N-prime ideal of R.

Let R and S be two G-graded rings. Then $R \times S$ is G-graded by $(R \times S)_g = R_g \times S_g$, for all $g \in G$ [7].

Lemma 3.11: *Let T and S be two $\mathbb{Z}_2$-graded rings. Then*

$$N_{T\times S}(t, s) = (N_T(t), N_S(s)), \text{ for every } (t, s) \in T \times S.$$

Proof: Let $(t, s) \in T \times S$. Then $(t, s) = (t, s)_0 + (t, s)_1 = (t_0, s_0) + (t_1, s_1)$, and then $N(t, s) = (t_0, s_0)^2 - (t_1, s_1)^2 = (t_0^2, s_0^2) - (t_1^2, s_1^2) = (t_0^2 - t_1^2, s_0^2 - s_1^2) = (N_T(t), N_S(s))$.

Theorem 3.12: *Let R and S be two $\mathbb{Z}_2$-graded rings. Then I is a 2-N_R-prime ideal of R if and only if $I \times S$ is a 2-$N_{R\times S}$-prime ideal of $R \times S$.*

Proof: Suppose that I is a 2-N_R-prime ideal of R. Let $(a, b)(c, d) \in I \times S$, for some $(a, b), (c, d) \in R \times S$. Then $ac \in I$. Since I is 2-N-prime, either $N_R(a^2) \in I$ or $N_R(c^2) \in I$. Hence either $N_{R\times S}((a, b)^2) \in I \times S$ or $N_{R\times S}((c, d)^2) \in I \times S$. Thus $I \times S$ is a 2-$N_{R\times S}$-prime ideal of $R \times S$. Conversely, let $ab \in I$, for some $a, b \in R$. Then $(a, 1)(b, 1) \in I \times S$. Hence $N_{R\times S}((a, 1)^2) \in I \times S$ or $N_{R\times S}((b, 1)^2) \in I \times S$. Therefore $N_R(a^2) \in I$ or $N_R(b^2) \in I$. Thus I is a 2-N_R-prime ideal of R.

Similarly, one can prove the following:

Theorem 3.13: *Let R and S be two $\mathbb{Z}_2$-graded rings. Then J is a 2-N_S-prime ideal of S if and only if $R \times J$ is a 2-$N_{R\times S}$-prime ideal of $R \times S$.*

Theorem 3.14: *Let R be a $\mathbb{Z}_2$-graded ring. If I is a 2-N-prime ideal of R and $a \in R$ such that $N(a) \notin \sqrt{I}$, then $(I: N(a^4))$ is a 2-N-prime ideal of R and $N((I: N(a^4))) \subseteq \sqrt{I} \subseteq \sqrt{(I: N(a^4))}$.*

Proof: If $(I: N(a^4)) = R$, then $1 \in (I: N(a^4))$, and then $N(a^4) \in I$ and so $N(a) \in \sqrt{I}$, a contradiction. Thus $(I: N(a^4))$ is a proper ideal of R. Let $x, y \in R$ such that $xy \in (I: N(a^4))$. Then $(xN(a^2))(yN(a^2)) \in I$, and then $N(x^2)N(a^4) \in I$ or $N(y^2)N(a^4) \in I$ which gives that $N(x^2) \in (I: N(a^4))$ or $N(y^2) \in (I: N(a^4))$. Hence, $(I: N(a^4))$ is a 2-N-prime ideal of R. Let $b \in N((I: N(a^4)))$. Then $b = N(t)$, for some $t \in (I: N(a^4))$. So, $tN(a^4) \in I$ and then as I is 2-N-prime and $N(a) \notin \sqrt{I}$, we have $N(t^2) \in I$ that is

$b = N(t) \in \sqrt{I}$. Thus $N((I{:}N(a^4))) \subseteq \sqrt{I}$. On the other hand, $I \subseteq (I{:}N(a^4))$, so $\sqrt{I} \subseteq \sqrt{(I{:}N(a^4))}$. Hence, $N((I{:}N(a^4))) \subseteq \sqrt{I} \subseteq \sqrt{(I{:}N(a^4))}$.

The next example shows that the intersection of two N-prime (2-N-prime) ideals is not necessarily N-prime (2-N-prime):

Example 3.15: Consider the graded ring given in Example 3.3. By ([2], Example 5), $I = 2R$ and $J = 5R$ are N-prime ideals. But $I \cap J$ is not N-prime (not 2-N-prime) as $2.5 \in I \cap J$ with $N(2) = 4 \notin I \cap J$ and $N(5) = 25 \notin I \cap J$ ($N(2^2) = 16 \notin I \cap J$ and $N(5^2) = 625 \notin I \cap J$).

Theorem 3.16: *Let (R, M) be a local $\mathbb{Z}_2$-graded ring and I be an N-prime ideal of R. Then IM is a 2-N-prime ideal of R.*

Proof: Let $x, y \in R$ such that $xy \in IM \subseteq I$. Then $N(x) \in I$ or $N(y) \in I$. Assume that $N(x) \in I$. Then $N(x)$ is not a unit, so $N(x) \in M$, and then $N(x^2) \in IM$. Hence, IM is a 2-N-prime ideal of R.

Theorem 3.17: Let R be a $\mathbb{Z}_2$-graded ring. If every 2-N-prime ideal of R is prime, then $I^2 = I$, for every N-prime ideal I of R.

Proof: Let I be an N-prime ideal of R. Then by Theorem 3.10, I^2 is a 2-N-prime ideal of R and hence I^2 is a prime ideal of R which gives that $I^2 = I$.

4. Weakly 2-*N*-Prime Ideals

Recall that from [5], a proper ideal I of a ring R is weakly 2-prime if whenever $0 \neq xy \in I$, we have $x^2 \in I$ or $y^2 \in I$. In this section, we introduce and study the concept of weakly 2-N-prime ideals.

Definition 4.1: Let R be a $\mathbb{Z}_2$-graded ring and I be a proper ideal of R. Then I is said to be a weakly 2-N-prime ideal of R if whenever $x, y \in R$ such that $0 \neq xy \in I$, then either $N(x^2) \in I$ or $N(y^2) \in I$.

Clearly, every 2-N-prime ideal is weakly 2-N-prime, but the converse is not necessarily true, see the following example:

Example 4.2: Consider $R = \mathbb{Z}_6[i]$, and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_6$ and $R_1 = i\mathbb{Z}_6$. Clearly, $I = \{0\}$ is a weakly 2-N-prime ideal of R, but I is not a 2-N-prime ideal of R since $2.3 \in I$, $N(2^2) = 4 \notin I$ and $N(3^2) = 3 \notin I$.

By Theorem 3.5, if I is a 2-N-prime ideal of R, then $\sqrt{I}$ is an N-prime ideal of R. The next example shows that if I is a weakly 2-N-prime ideal of R, then $\sqrt{I}$ is not necessarily N-prime ideal of R:

Example 4.3: Consider $R = \mathbb{Z}_{12}[i]$, and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_{12}$ and $R_1 = i\mathbb{Z}_{12}$. Now, $I = \{0\}$ is a weakly 2-N-prime ideal of R, but $\sqrt{I} = 6R$ is not an N-prime ideal of R since $2.3 \in \sqrt{I}$, $N(2) = 4 \notin \sqrt{I}$ and $N(3) = 9 \notin \sqrt{I}$.

Theorem 4.4: *Let R and S be two $\mathbb{Z}_2$-graded rings.*

- *Suppose that $f: R \to S$ is a ring epimorphism. If I is a weakly 2-N_R-prime ideal of R containing $Ker(f)$ and $N_S \circ f = f \circ N_R$, then $f(I)$ is a weakly 2-N_S-prime ideal of S provided that $f(I)$ is a proper ideal of S.*
- *Suppose that $f: R \to S$ is a ring monomorphism. If K is a weakly 2-N_S-prime ideal of S and $N_S \circ f = f \circ N_R$, then $f^{-1}(K)$ is a weakly 2-N_R-prime ideal of R provided that $f^{-1}(K)$ is a proper ideal of R.*

Proof:

- Let $0 \neq ab \in f(I)$ for some $a, b \in S$. Since f is epimorphism, we can write $f(x) = a$ and $f(y) = b$ for some $x, y \in R$. Then we have $ab = f(x)f(y) = f(xy) \in f(I)$. As $Ker(f) \subseteq I$, we have $0 \neq xy \in I$. Since I is weakly 2-N_R-prime, we get either $N_R(x^2) \in I$ or $N_R(y^2) \in I$ and this yields $N_S(a^2) = N_S((f(x))^2) = f(N_R(x^2)) \in f(I)$ or $N_S(b^2) = N_S(f(y^2)) \in f(I)$. Hence $f(I)$ is a weakly 2-N_S-prime ideal of S.
- Let $0 \neq xy \in f^{-1}(K)$ for some $x, y \in R$. As f is monomorphism, $0 \neq f(xy) = f(x)f(y) \in K$. Since K is a weakly 2-N_S-prime ideal, we get either $N_S((f(x))^2) = f(N_R(x^2)) \in K$ or $N_S((f(y))^2) = f(N_R(y^2)) \in K$, and thus $N_R(x^2) \in f^{-1}(K)$ or $N_R(y^2) \in f^{-1}(K)$. Hence, $f^{-1}(K)$ is a weakly 2-N_R-prime ideal of R.

Theorem 4.5: *Let R be a $\mathbb{Z}_2$-graded ring, J be an ideal of R and I be a graded ideal of R such that $I \subseteq J$.*

(1) *If J is a weakly 2-N_R-prime ideal of R, then J/I is a weakly 2-$N_{R/I}$-prime ideal of R/I.*

(2) *If J/I is a weakly 2-$N_{R/I}$-prime ideal of R/I and I is a 2-N_R-prime ideal of R, then J is a 2-N_R-prime ideal of R.*

(3) *If J/I is a weakly 2-$N_{R/I}$-prime ideal of R/I and I is a weakly 2-N_R-prime ideal of R, then J is a weakly 2-N_R-prime ideal of R.*

Proof:

(1) Define $f: R \to R/I$ by $f(x) = x + I$ for each $x \in R$. Then f is a ring epimorphism with $Ker(f) = I \subseteq J$ and $f(N_R(x)) = N_R(x) + I = N_{R/I}(x + I) = N_{R/I}(f(x))$. So, by Theorem 4.4, J/I is a weakly 2-$N_{R/I}$-prime ideal of R/I.

(2) Let $ab \in J$ for some $a, b \in R$. If $ab \in I$, then since I is 2-N_R-prime, either $N_R(a^2) \in I \subseteq J$ or $N_R(b^2) \in I \subseteq J$, and hence J is a 2-N_R-prime ideal of R. Suppose that $ab \notin I$. Then $0 + I \neq (a + I)(b + I) \in J/I$, and

then since J/I is weakly 2-$N_{R/I}$-prime, either $N_{R/I}((a+I)^2) = N_R(a^2) + I \in J/I$ or $N_{R/I}((b+I)^2) = N_R(b^2) + I \in J/I$, which implies that either $N_R(a^2) \in J$ or $N_R(b^2) \in J$. Therefore, J is a 2-N_R-prime ideal of R.

(3) Similar to the part (2).

Theorem 4.6: *Let R be a $\mathbb{Z}_2$-graded ring and I be a graded ideal of R such that R_0ÚI. If I is a weakly 2-N-prime ideal of R, then I_0 is a weakly 2-N-prime ideal of R_0.*

Proof: Let $a, b \in R_0$ such that $0 \neq ab \in I_0 \subseteq I$. Since I is a weakly 2-N-prime, we have $N(a^2) \in I$ or $N(b^2) \in I$. Since $N(a^2), N(b^2) \in R_0 R_0 \subseteq R_0$, we get either $N(a^2) \in I \cap R_0 = I_0$ or $N(b^2) \in I \cap R_0 = I_0$. Thus, I_0 is a weakly 2-N-prime ideal of R_0.

Theorem 4.7: *Let R and S be two $\mathbb{Z}_2$-graded rings, I be an ideal of R and J be an ideal of S. If $I \times J$ is a nonzero weakly 2-$N_{R\times S}$-prime ideal of $R \times S$, then I is a 2-N_R-prime ideal of R and $J = S$ or $I = R$ and J is a 2-N_S-prime ideal of S.*

Proof: Without loss of generality, we may assume that $I \neq 0$. Then there exists a nonzero element $x \in I$. This gives $(0_R, 0_S) \neq (x, 1_S)(1_R, 0_S) = (x, 0_S) \in I \times J$. As $I \times J$ is a weakly 2-$N_{R\times S}$-prime ideal of $R \times S$, we have that $N_{R\times S}((x,1)^2) = (N_R(x^2), 1_S) \in I \times J$ or $N_{R\times S}((1_R, 0_S)^2) = (1_R, 0_S) \in I \times J$. We have $I = R$ or $J = S$. Without loss of generality, we may assume that $I = R$. Now, we will show that J is a 2-N_S-prime ideal of S. Let $ab \in J$ for some $a, b \in S$. Then, we obtain $(0,0) \neq (1_R, a)(1_R, b) = (1_R, ab) \in R \times J$. As $R \times J$ is a weakly 2-$N_{R\times S}$-prime ideal of $R \times S$, we have $N_{R\times S}((1_R, a)^2) \in R \times J$ or $N_{R\times S}((1_R, b)^2) \in R \times J$ which implies that $N_S(a^2) \in J$ or $N_S(b^2) \in J$. Hence, J is a 2-N_S-prime ideal of S.

Theorem 4.8: *Let R be a $\mathbb{Z}_2$-graded ring, I be a weakly 2-N-prime ideal of R and $x \in R$ such that $N(x^2)$ is a nonzero divisor element. If $a, b \in R$ such that $0 \neq ab \in (I: N(x^2))$, then either $N(a) \in \sqrt{(I: N(x^4))}$ or $N(b) \in \sqrt{(I: N(x^4))}$.*

Proof: Now, $0 \neq abN(x^2) = \big(aN(x)\big)\big(bN(x)\big) \in I$, and then either $N(a^2)N(x^4) \in I$ or $N(b^2)N(x^4) \in I$, which implies that either $N(a^2) \in (I: N(x^4))$ or $N(b^2) \in \big(I: N(x^4)\big)$. Therefore, $N(a) \in \sqrt{(I: N(x^4))}$ or $N(b) \in \sqrt{(I: N(x^4))}$.

5. Conclusion

In this article, we introduced and studied two new classes of ideals, 2-N-prime and weakly 2-N-prime ideals in the context of $\mathbb{Z}_2$-graded rings.

These concepts were motivated by the earlier notion of N-prime ideals and extend the classical theory of prime and 2-prime ideals in a way that respects the underlying grading structure.

We established the definitions of 2-N-prime and weakly 2-N-prime ideals using the functional mapping $N: R \to R_0$, defined by $N(x) = x_0^2 - x_1^2$. Key properties of these ideals were explored, along with examples and counterexamples that demonstrated their distinct behaviors and relationships.

Our results highlight the significance of these new ideals in extending the ideal theory of $\mathbb{Z}_2$-graded rings. They also shed light on connections between grading, functional mappings, and ideal structures, providing a richer framework for further investigation.

We believe that the results presented here open new directions for future research, including deeper studies of related algebraic structures, variations of the N-function, and potential applications in areas such as representation theory and noncommutative algebra.

This work not only builds upon the existing theory of graded rings but also encourages further exploration of functional mappings and their impact on ideal theory.

6. Future Work

Several directions for future research arise naturally from this study. One potential avenue is to investigate whether the concepts of 2-N-prime and weakly 2-N-prime ideals can be extended to more general graded ring structures, such as G-graded or noncommutative graded rings.

Another interesting line of inquiry involves exploring modifications or generalizations of the function N to define new classes of ideals and examine their algebraic properties. Studying how these new ideals behave under homomorphisms, extensions, and localizations could also provide valuable insights.

Additionally, connections between the newly introduced ideals and other areas, such as module theory, homological algebra, and mathematical physics, merit further exploration. These connections could reveal broader applications and deepen our understanding of graded algebraic systems.

Finally, identifying practical examples and applications of these ideals in mathematical models and theoretical frameworks will help bridge the gap between abstract algebra and applied mathematics, offering exciting opportunities for future work.

7. Open Questions

This study raises several intriguing questions that can guide future research:

(1) Can the concepts of 2-N-prime and weakly 2-N-prime ideals be extended to more general graded structures, such as G-graded rings with arbitrary groups?

(2) How do these new ideals behave under tensor products, and localizations?

(3) Are there alternative definitions of the function N that lead to other meaningful classes of ideals?

(4) What connections can be established between 2-N-prime ideals and classical algebraic structures, such as primary decompositions?

(5) Can these concepts be applied to noncommutative graded rings, and if so, what modifications are necessary?

(6) Are there applications of 2-N-prime ideals in fields like representation theory, coding theory, or mathematical physics?

(7) How do the new ideals interact with module theory, and can analogous definitions be developed for submodules?

(8) What combinatorial or geometric interpretations, if any, can be associated with these ideals in graded settings?

Addressing these questions could lead to significant advancements in the theory of graded rings and their applications.

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