

J-projective modules and related concepts

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Abstract

The notion of J -projective modules is presented in this study, (a new type of projectivity relative to the Jacobson radical of the module). Some of the basic properties of this concept are given. In addition, other related concepts like, J -splits, J -direct summand and relationship between them are discussed.

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1. Introduction

The commutative rings R with an identity element are present throughout this work, and a modules be left R -modules which is unitary. For an R -modules, the notation $J(P)$ will stand for the Jacobson radical of P . Projective modules have been studied extensively and many generalizations for the concept of projective modules were given In [1], for example, nearly projective modules. Naoum and AL-Mothafar in [2], introduced the notion of nearly projective modules as follows: An R -modules P is nearly projective modules where $\forall$ epimorphisms $f: M \rightarrow N$ & $\forall$ homomorphisms $g: P \rightarrow N$, $\exists$ homomorphism $h: P \rightarrow M$ s.t $(fh - g)(t) \in J(N), \forall t \in P$. In this paper we introduce another generalization for projective modules, it called J -projective modules (An R -modules P be deemed J -projective in event that every epimorphisms $f: \mathcal{A} \rightarrow \mathcal{B}$ & every homomorphisms $g: P \rightarrow \mathcal{B}$ there is homomorphisms $h: P \rightarrow \mathcal{A}$ s.t $f(h(t)) = g(t), \quad \forall t \in J(P),$ (*i.e.* $fhi = gi$,

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where $i: J(\mathcal{P}) \rightarrow \mathcal{P}$ be inclusion maps), some properties for these modules are discussed. In section 2 some definitions, facts and remarks which are needed in the next sections will be given. In section 3, the concepts of J - $\mathcal{A}$ -projective, J -split (An epimorphism $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ is said to be **J -split**, where $\exists \beta: \mathcal{P} \rightarrow \mathcal{A}$ s.t $\alpha(\beta(t)) = t$, $\forall t \in J(\mathcal{P})$), and the relationship between them are studied. Moreover, the concept J -direct summand and a link between it and J - $\mathcal{A}$ -projective is investigated.

2. Preliminaries

In this section, we will give some basic definitions and properties which are needed later.

Definition 2.1 [3]: $J(M)$ denotes to the radical for modules M that is a intersections for each submodule for M which maximal or is equal to sum for every submodule which small of M . if M not contains maximal submodule we put $J(M) = M$.

Proposition 2.2 [4, 5]: If $f: M \rightarrow N$ be epimorphisms & $\text{Ker}(f) \leq J(M)$, so $J(N) = f(J(M))$.

Definition 2.3 [3]: Let $\mathcal{P}$ & $\mathcal{A}$ are modules. Then $\mathcal{P}$ be $\mathcal{A}$ -projective where $\forall$ epimorphisms $f: \mathcal{A} \rightarrow \mathcal{B}$, & $\forall$ homomorphisms $g: \mathcal{P} \rightarrow \mathcal{B}$, $\exists$ homomorphisms $h: \mathcal{P} \rightarrow \mathcal{A}$ s.t $g = f \circ h$. A module $\mathcal{P}$ be projective if $\mathcal{P}$ is $\mathcal{A}$ -projective for each module $\mathcal{A}$.

3. J -projective modules and related concepts.

Definition 3.1 [6]: Let $\mathcal{A}$ & $\mathcal{B}$ be R -modules. The R -module $\mathcal{P}$ is **J - $\mathcal{A}$ -projective** if $\forall$ epimorphisms $f: \mathcal{A} \rightarrow \mathcal{B}$ and $\forall$ homomorphisms $g: \mathcal{P} \rightarrow \mathcal{B}$ $\exists$ homomorphism $h: \mathcal{P} \rightarrow \mathcal{A}$ s.t $f(h(t)) = g(t)$ $\forall t \in J(\mathcal{P})$ (**i.e.** $fhi = gi$, where $i: J(\mathcal{P}) \rightarrow \mathcal{P}$ be inclusion maps). An R -modules P named **J -projective** if $\mathcal{P}$ be **J - $\mathcal{A}$ -projective** for each R -modules $\mathcal{A}$.

Examples and Remarks (3.2) :

- 1) It is clear that projective modules are J -projective modules.
- 2) if $\mathcal{A}$ is module s.t $J(\mathcal{A}) = \mathcal{A}$ (e.g. $\mathcal{A}$ has no maximal submodule), then $\mathcal{A}$ be projective iff its J -projective.
- 3) If $J(\mathcal{A}) = 0$, then $\mathcal{A}$ be J -projective.
- 4) the simple R -modules be J -projective. For example, $\mathbb{Z}_p$ as a $\mathbb{Z}$ -module is J -projective but not projective.
- 5) The $\mathbb{Z}$ -module $\mathbb{Z}$ is both projective and J -projective, although $J(\mathbb{Z}) = 0$.
- 6) if $\mathcal{P}_1$ is J - $\mathcal{A}$ -projective & $\mathcal{P}_1$ isomorphic to the $\mathcal{P}_2$, so $\mathcal{P}_2$ J - $\mathcal{A}$ -projective.

Proof: Suppose $\mathcal{P}_1$ be J - $\mathcal{A}$ -projective & $\mathcal{P}_1 \cong \mathcal{P}_2$. To proof $\mathcal{P}_2$ be J - $\mathcal{A}$ -projective. We put the diagram where $\mathcal{A}$ & $\mathcal{B}$ any module & $\alpha: \mathcal{A} \rightarrow \mathcal{B}$

epimorphisms & $\beta: \mathcal{P}_2 \rightarrow \mathcal{B}$ homomorphisms. because $\mathcal{P}_1$ be $J\text{-}\mathcal{A}$ -projective, so we get homomorphism $h_1: \mathcal{P}_1 \rightarrow \mathcal{A}$ s.t $\alpha h_1(\mathfrak{t}) = \beta(\theta(\mathfrak{t}))$, $\forall \mathfrak{t} \in J(\mathcal{P}_1)$, where θ is an isomorphism from $\mathcal{P}_1$ to $\mathcal{P}_2$. Define $h_2: \mathcal{P}_2 \rightarrow \mathcal{A}$ by $h_2 = h_1 \theta^{-1}$, and since $\theta(J(\mathcal{P}_1)) = J(\mathcal{P}_2)$. Then $\alpha(h_2(\mathfrak{t})) = \alpha(h_1 \theta^{-1}(\mathfrak{t})) = \beta(\theta \theta^{-1}(\mathfrak{t})) = \beta(\mathfrak{t})$, $\forall \mathfrak{t} \in J(\mathcal{P}_2)$. hence $\mathcal{P}_2$ be $J\text{-}\mathcal{A}$ -projective.

Lemma 3.3: Let $\mathcal{A}$ & $\mathcal{P}$ be R -modules. Then subsequent statements are comparable:

- 1) $\mathcal{P}$ is $J\text{-}\mathcal{A}$ -projective.
- 2) For any submodule K of $\mathcal{A}$ and for any R -homomorphism $g: \mathcal{P} \rightarrow \mathcal{A}/K$, there is an R -homomorphism $h: \mathcal{P} \rightarrow \mathcal{A}$ such that $\pi h i = g i$ where $\pi: \mathcal{A} \rightarrow \mathcal{A}/K$, is the natural map and $i: J(\mathcal{P}) \rightarrow \mathcal{P}$ is the inclusion map.

Proof: (1) $\Rightarrow$ (2) clear (2) $\Rightarrow$ (1) Assume that $\mathcal{A}$ and L are R -modules.

Assume $f: \mathcal{A} \rightarrow L$ is epimorphism & $g: \mathcal{P} \rightarrow L$ homomorphism, where $K = \ker f$. Consider the diagram

By (ii) $\exists h: \mathcal{P} \rightarrow \mathcal{A}$ such that $\pi h i = \theta g i$, where $\theta: L \rightarrow \mathcal{A}/K$ & $\pi = \theta f$, so $\theta f h i = \theta g i$. Thus $f h i = g i$.

Proposition 3.4: Any direct summand for $J\text{-}\mathcal{A}$ -projective be $J\text{-}\mathcal{A}$ -projective.

Proof: If $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ is $J\text{-}\mathcal{A}$ -projective & $\mathcal{A}, \mathcal{B}$ is R -module & $g: \mathcal{P}_1 \rightarrow \mathcal{B}$ a homomorphism, $f: \mathcal{A} \rightarrow \mathcal{B}$ epimorphisms.

Let π & i be projection & injection between $\mathcal{P}, \mathcal{P}_1$. but $\mathcal{P}$ be $J\text{-}\mathcal{A}$ -projective, so $\exists$ homomorphism

$h: \mathcal{P} \rightarrow \mathcal{A}$ s.t $f h(\mathfrak{t}) = g(\pi(\mathfrak{t})) \forall \mathfrak{t} \in J(\mathcal{P}) = J(\mathcal{P}_1) \oplus J(\mathcal{P}_2)$, $\pi(\mathfrak{t}) \in J(\mathcal{P}_1)$.

Now, define $h_1: \mathcal{P}_1 \rightarrow \mathcal{A}$ by $h_1 = h i$. Then for all $\mathfrak{t}$ in $J(\mathcal{P}_1)$,

$$f(h_1(\mathfrak{t})) = f h(i(\mathfrak{t})) = g(\pi i(\mathfrak{t})) = g(\mathfrak{t}).$$

Proposition 3.5: Let $\mathcal{P} = \bigoplus_{i \in I} \mathcal{P}_i$, then is $J\text{-}\mathcal{A}$ -projective iff $\mathcal{P}_i$ is $J\text{-}\mathcal{A}$ -projective module, for all $i \in I$.

Proof: ($\Rightarrow$) follows by Proposition (3.4).

($\Leftarrow$) Assume that $\mathcal{P} = \bigoplus_{i \in I} \mathcal{P}_i$, and $f: \mathcal{A} \rightarrow \mathcal{B}$ be R -epimorphisms & $g: \mathcal{P} \rightarrow \mathcal{B}$ is R -homomorphisms, where $\mathcal{A}$ & $\mathcal{B}$ any modules.

Since $\mathcal{P}_i$ be $J\text{-}\mathcal{A}$ -projective so $\forall i$, $\exists$ homomorphisms

$$h_i: \mathcal{P}_i \rightarrow \mathcal{A} \text{ s.t } f(h_i(\mathfrak{t}_i)) = g(j_i(\mathfrak{t}_i)), \forall \mathfrak{t}_i \in J(\mathcal{P}_i).$$

Note that for any $\mathfrak{t} \in \mathcal{P}$, $\mathfrak{t} = (\pi_i(\mathfrak{t}))$, also if $\mathfrak{t} \in J(\mathcal{P})$, $\pi_i(\mathfrak{t}) \in J(\mathcal{P}_i)$.

Define $h: \mathcal{P} \rightarrow \mathcal{A}$ by $h(\mathfrak{t}) = \sum h_i(\pi_i(\mathfrak{t}))$. It is clear that h is well defined.

Now, for any $\mathfrak{t} \in J(\mathcal{P})$,

$$(h(\mathfrak{t})) = f(\sum h_i(\pi_i(\mathfrak{t}))) = \sum fh_i(\pi_i(\mathfrak{t})) = \sum g(j_i(\pi_i(\mathfrak{t}))) = g(\sum j_i\pi_i(\mathfrak{t})) = g(\mathfrak{t}).$$

Proposition 3.6: Let $\mathcal{A}$ & $\mathcal{P}$ be R -modules & $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$.

If $\mathcal{P}$ be J - $\mathcal{A}$ -projective, then $\mathcal{P}$ be J - $\mathcal{A}_1$ -projective.

Proof: Assume $\mathcal{P}$ be J - $\mathcal{A}$ -projective & $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, to claim $\mathcal{P}$ be J - $\mathcal{A}_1$ -projective. let K any R -module, $f: \mathcal{A}_1 \rightarrow K$ epimorphisms, $g: \mathcal{P} \rightarrow K$ homomorphisms & $\pi: \mathcal{A} \rightarrow \mathcal{A}_1$ be projection maps.

Now, $\mathcal{P}$ be J - $\mathcal{A}$ -projective, so $\exists$ homomorphism $h: \mathcal{P} \rightarrow \mathcal{A}$ s.t

$f(\pi(h(\mathfrak{t}))) = g(\mathfrak{t}), \forall \mathfrak{t} \in J(\mathcal{P})$. Define $h_1: \mathcal{P} \rightarrow \mathcal{A}_1$ by $h_1 = \pi h$, then

$fh_1(\mathfrak{t}) = f\pi(h(\mathfrak{t})) = g(\mathfrak{t}) \forall \mathfrak{t} \in J(\mathcal{P})$. hence $\mathcal{P}$ be J - $\mathcal{A}_1$ -projective.

Definition 3.7: Let $\mathcal{P}$ and $\mathcal{A}$ be R -modules. An epimorphism $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ is named **J -split**, if $\exists \beta: \mathcal{P} \rightarrow \mathcal{A}$ s.t $\alpha(\beta(z)) = z$, for each $z \in J(\mathcal{P})$.

Remark 3.8: Let $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ be an epimorphism, then

(1) If α is J -split then $\ker \alpha \leq^{\oplus} \alpha^{-1}(J(\mathcal{P}))$.

(2) If $\ker \alpha \leq^{\oplus} \alpha^{-1}(J(\mathcal{P}))$, then $\exists \beta: J(\mathcal{P}) \rightarrow \mathcal{A}$ s.t

$$\alpha(\beta(z)) = z, \text{ for each } z \in J(\mathcal{P}).$$

Proof 1: Assume that $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ is J -split, then $\exists \beta: \mathcal{P} \rightarrow \mathcal{A}$ s.t $\alpha(\beta(z)) = z$ for all $z \in J(\mathcal{P})$. So, $J(\mathcal{P}) = \alpha(\beta(J(\mathcal{P})))$, hence $\alpha^{-1}(J(\mathcal{P})) = \beta(J(\mathcal{P})) + \ker \alpha$. Let $\mathfrak{t} \in \beta(J(\mathcal{P})) \cap \ker \alpha$, then $\alpha(\mathfrak{t}) = 0$ & $\mathfrak{t} = \beta(z)$, for some $z \in J(\mathcal{P})$. But $\alpha(\mathfrak{t}) = \alpha(\beta(J(\mathcal{P})))$ that is, $z = 0$, so $\mathfrak{t} = \beta(0) = 0$. Therefore $\alpha^{-1}(J(\mathcal{P})) = \beta(J(\mathcal{P})) \oplus \ker \alpha$.

Proof 2: Assume that $\ker \alpha \leq^{\oplus} \alpha^{-1}(J(\mathcal{P}))$, that is $\alpha^{-1}(J(\mathcal{P})) = \mathcal{B} \oplus \ker \alpha$ where $\mathcal{B} \leq \alpha^{-1}(J(\mathcal{P})) \leq \mathcal{A}$.

Hence $\alpha(\mathcal{B}) = J(\mathcal{P})$. Let $z \in J(\mathcal{P})$, then $z = \alpha(b)$ for some b in $\mathcal{B}$.

Define $\beta: J(\mathcal{P}) \rightarrow \mathcal{A}$ by $\beta(z) = b$ where $(z = \alpha(b))$. Now, if $z = \alpha(b) = \alpha(\hat{b})$, then $b - \hat{b} \in \ker \alpha \cap \mathcal{B}$ so $b - \hat{b} = 0$ that is $b = \hat{b}$. Hence β is well define. Clearly, $\alpha(\beta(z)) = \alpha(b) = z$.

Proposition 3.9: An R -module $\mathcal{P}$ is J -projective iff any epimorphism $\alpha: \mathcal{A} \rightarrow \mathcal{P}$, J -splits.

Proof: ($\Rightarrow$) Assume $\mathcal{P}$ be J -projective & $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ be epimorphisms, so from diagram $\exists \beta: \mathcal{P} \rightarrow \mathcal{A}$ s.t $\alpha\beta(\mathfrak{t}) = \mathfrak{t}, \forall \mathfrak{t} \in J(\mathcal{P})$. Therefore α be J -split.

($\Leftarrow$) suppose any epimorphism $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ is J -split & assume F is free modules with $\alpha: F \rightarrow \mathcal{P}$ epimorphisms. Then α J -splits, and $\exists \beta: \mathcal{P} \rightarrow F$ such that $\alpha(\beta(z)) = z$, for all $z \in J(\mathcal{P})$. Now, consider the diagram

$\rightarrow \mathcal{A}$ s.t $\beta(\alpha(z)) = z \quad \forall z \in J(\mathcal{P})$. Where g epimorphism, $\mathcal{A}$ & $\mathcal{B}$ any two modules. Since F is projective $\exists \theta: F \rightarrow \mathcal{A}$ with $g\theta = f\alpha$. Take $h = \theta\beta: \mathcal{P} \rightarrow \mathcal{A}$, then for $z \in J(\mathcal{P})$, $g(h(z)) = g(\theta(\beta(z))) = f(\alpha(\beta(z))) = f(z)$.

Therefore $\mathcal{P}$ be J -projective.

Definition 3.10: An R -module P is J -direct summand (in short J -d.s) of $\mathcal{A}$ if for each epimorphism $\alpha: \mathcal{A} \rightarrow \mathcal{P} \exists \beta: \mathcal{P} \rightarrow \mathcal{A}$ such that $\alpha(\beta(t)) = t, \forall t \in J(\mathcal{P})$.

Definition 3.11 [7, 8] : An R -module $\mathcal{A}$ is said to have (D_2) condition, if $\mathcal{A} / \mathcal{P}$ isomorphic to direct summand for $\mathcal{A}$ for every submodules $\mathcal{P}$ of $\mathcal{A}$, then $\mathcal{P}$ itself is direct summand for $\mathcal{A}$.

Lemma 3.12: Assume P be direct summand for $\mathcal{A}$ and $\mathcal{A}$ has (D_2) then every epimorphism $\alpha: \mathcal{A} \rightarrow \mathcal{P}$, splits.

Proof: Let $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ be epimorphism, then $\mathcal{P} \cong \mathcal{A} / \ker \alpha$. But $\mathcal{P} \leq^{\oplus} \mathcal{A}$ and $\mathcal{A} \text{ ht}s (D_2)$

Hence $\ker \alpha \leq^{\oplus} \mathcal{A}$. So, α is split.

Remarks 3.13:

- 1) If $\mathcal{P}$ is a direct summand of $\mathcal{A}$ and $\mathcal{A}$ has (D_2) , then $\mathcal{P}$ be J -direct summand for $\mathcal{A}$. The proof is directly from Lemma (3.12).
- 2) If $\mathcal{P}$ be J -direct summand for $\mathcal{A}$ & $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ an epimorphism then $\ker \alpha \leq^{\oplus} \alpha^{-1}(J(\mathcal{P}))$.
- 3) If $\mathcal{P}$ be J -direct summand and a homomorphic image of $\mathcal{A}$ with $J(\mathcal{A}) = \mathcal{A}$, then $\mathcal{P}$ isomorphic to directsummand for $\mathcal{A}$.

Proof: Suppose $\alpha: \mathcal{A} \rightarrow \mathcal{P}$ epimorphism. Since $\mathcal{P}$ be J -direct summand for $\mathcal{A}$, then $\exists \beta: \mathcal{P} \rightarrow \mathcal{A}$ s.t $\alpha(\beta(t)) = t, \forall t \in J(\mathcal{P})$ & $\ker \alpha \leq^{\oplus} \alpha^{-1}J(\mathcal{P})$.

But $J(\mathcal{P}) = \mathcal{P}$, so $\ker \alpha \leq^{\oplus} \alpha^{-1}(\mathcal{P}) = \mathcal{A}$. Hence α is split, this implies that $\mathcal{A} = \beta(\mathcal{P}) \oplus \ker \alpha$. But β is a monomorphism, then $\beta(\mathcal{P}) \cong \mathcal{P}$.

- 4) If $\mathcal{P}$ is homomorphic image of a free R -module F and $\mathcal{P}$ is J -direct summand of F , then $\mathcal{P}$ is J -projective.

Proof: Assume $\mathcal{P}$ homomorphic image for free R -modules F , so $\exists$ epimorphisms $\alpha: F \rightarrow \mathcal{P}$. Since $\mathcal{P}$ is J -direct summand for F , so $\exists \beta: \mathcal{P} \rightarrow F$ such that $\alpha(\beta(z)) = z, \forall z \in J(\mathcal{P})$. Now, by consider the diagram below, and since F is projective, so $\exists h: F \rightarrow \mathcal{A}$ s.t $fh = g\alpha$, (where $\mathcal{A}$ & $\mathcal{B}$ are R -modules, $f: \mathcal{A} \rightarrow \mathcal{B}$ epimorphisms & $g: \mathcal{P} \rightarrow \mathcal{B}$ homomorphisms). Define $\theta: \mathcal{P} \rightarrow \mathcal{A}$ by $\theta = h\beta$. Now, $\forall t \in J(\mathcal{P})$ we get $f(\theta(t)) = f(h(\beta(t))) = fh(\beta(t)) = g(\alpha(\beta(t))) = g(t)$.

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