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Graded L_e-J_{gr}-prime submodules in commutative graded rings

Shatha Alghueiri [†]
Khaldoun Al-Zoubi *
Department of Mathematics and Statistics
Faculty of Science and Arts
Jordan University of Science and Technology
P. O. Box 3030
Irbid 22110
Jordan

Abstract

In this paper, we introduce the concept of gr-L_e-J_{gr}-PSub in the context of graded commutative rings and graded modules. These submodules extend and unify the properties of gr-L_e-PSub and gr-J_gr-PSub through a fixed gr-Id L=$\bigoplus_{g\in\Omega} L_g$. The research develops foundational definitions, proves equivalences, and examines the structural implications of these submodules under various module-theoretic operations, including localization, direct products, and quotient modules. Examples are provided to illustrate the distinct nature of graded L_e-J_{gr}-prime submodules compared to other graded submodules. Furthermore, conditions under which these submodules exhibit maximal and semisimple properties are characterized, contributing new insights to the theory of graded modules.

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1. Introduction and Preliminaries

The study of graded modules over graded commutative rings has become a cornerstone of modern algebra, driven by their deep connections to topology, geometry, and mathematical physics, as highlighted in various foundational works [1-4].

The concept of graded structures, which extends classical module theory, facilitates the study of algebraic objects with an additional layer of gradation

* *ORCID-ID:* `0000-0001-6082-4480`

[†] *E-mail:* `soalghueiri@just.edu.jo`
* *E-mail:* `kfzoubi@just.edu.jo (Corresponding Author)`

indexed by a group [3, 4]. These structures provide a natural framework for understanding a variety of phenomena, including those arising in the study of graded rings, homogeneous ideals, and their applications in algebraic geometry and representation theory.

Within this context, prime submodules have been a central topic of investigation, with particular attention given to their graded analogs [5]. Graded prime submodules, initially introduced to capture the interaction between module-theoretic properties and the graded structure of their parent modules, have spurred considerable research activity. Notable advancements include the exploration of graded J_{gr}-prime submodules, which incorporate the graded Jacobson radical [6], and graded L_e-prime submodules, which are defined with respect to a fixed graded ideal L [7]. Each of these constructions has proven instrumental in generalizing classical results and uncovering new algebraic phenomena unique to the graded setting.

This paper aims to further advance the theory of gr-Subs by introducing the concept of gr-L_e-J_{gr}-PSubs, a unifying framework that extends the properties of both gr-L_e-PSub and gr-J_{gr}-PSub. A gr-L_e-J_{gr}-PSub is defined as a gr-Sub K of a gr-B-module C with, for any $r_g \in h(B)$ and $c_h \in h(C)$ satisfying $r_g c_h \in K \backslash L_e K$, at least one of $c_h \in K + J_{gr}(C)$ or $r_g \in (K + J_{gr}(C) :_B C)$ holds. This definition encapsulates and extends various prior notions of gr-PSub, providing a robust framework for their analysis.

The study proceeds by exploring the fundamental properties of gr-L_e-J_{gr}-PSub and establishing their relationships with other gr-Subs types. Key results include the behavior of these submodules under localization, direct products, and quotient operations. Examples are presented to highlight the distinctions between graded L_e-J_{gr}-prime submodules and their special cases, illustrating their broader applicability and utility in module theory.

This work not only contributes to the growing body of knowledge on graded algebra but also opens new avenues for further exploration, particularly in understanding the structural properties of graded modules in the context of graded commutative rings. By providing a comprehensive treatment of graded L_e-J_{gr}-prime submodules, this study lays the foundation for future investigations into their applications in both pure and applied mathematics.

We begin by revisiting some key concepts regarding graded rings and modules, which will be useful later on. For a more thorough discussion on these topics, please refer to [3, 4, 8-11].

Consider a group Ω with an identity element e. A ring B is called a Ω-graded ring if it can be represented as a direct sum of abelian groups, written as:

$$B = \bigoplus_{g \in \Omega} B_g,$$

where the product of elements B_g and B_h belongs to the component B_{gh} for all $g, h \in \Omega$. The non-zero elements in B_h are referred to as homogeneous elements of degree h, and the collection of all homogeneous elements is denoted as $h(B)$, which can be expressed as:

$$h(B) = \bigcup_{h\in\Omega} B_h.$$

Moreover, B_e is a subring of B and $1 \in B_e$.

An ideal L of B is considered a graded ideal (or simply, gr-Id) if it can also be decomposed in a similar fashion:

$$L = \bigoplus_{g\in\Omega} (L \cap B_g) := \bigoplus_{g\in\Omega} L_g.$$

Further details can be found in [3].

Now, let B be a Ω-graded ring and C be a B-module. We say C is a Ω-graded B-module if it can be expressed as a direct sum of additive subgroups:

$$C = \bigoplus_{g\in\Omega} C_g,$$

and it satisfies the condition that for all $g, h \in \Omega$, the product of B_g and C_h lies in C_{gh}. Any element of C that belongs to the union of all C_g's, denoted $h(C)$, is called a homogeneous element.

If B is a Ω-graded ring and C is a graded B-module, then a submodule K of C is called a graded submodule (or simply, gr-Sub) if it can be written as:

$$K = \bigoplus_{g\in\Omega} (K \cap C_g) := \bigoplus_{g\in\Omega} K_g.$$

Here, K_g is referred to as the g-component of the submodule K, see [3].

In this work, we consider Ω to be a group with an identity element e, B to be a commutative Ω-graded ring with an identity element, and C to be a unitary graded B-module.

2. Results

Definition 2.1: Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B, $K = \bigoplus_{g\in\Omega} K_g$ a gr-Sub of C and $\lambda \in \Omega$.

(i) K_λ is called a λ-L_e-J_{gr}-prime submodule (or simply, λ-L_e-J_{gr}-PSub) of a B_e-module C_λ if $K_\lambda \neq C_\lambda$; and whenever $r_e c_\lambda \in K_\lambda \backslash L_e K_\lambda$ where $r_e \in B_e$ and $c_\lambda \in C_\lambda$, then either $c_\lambda \in K_\lambda + J_{gr}(C_\lambda)$ or $r_e \in (K_\lambda + J_{gr}(C_\lambda) :_{B_e} C_\lambda)$.

(ii) K is called a graded L_e-J_{gr}-prime submodule (or simply, gr -L_e-J_{gr}-PSub) of C if $K \neq C$; and whenever $r_g c_h \in K \backslash L_e K$ where $r_g \in h(B)$ and $c_h \in h(C)$, then $c_h \in K + J_{gr}(C)$ or $r_g \in (K + J_{gr}(C) :_B C)$.

A proper gr-Sub K of C is called a gr-PSup of C if whenever $w_g c_h \in C$ where $w_g \in h(B)$ and $c_h \in h(C)$, then $c_h \in K$ or $w_g \in (K :_B C)$, see [5].

Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B and K a proper gr-Sub of C. Then K is called a gr-L_e-PSub of C if whenever $w_g c_h \in K \backslash L_e K$ where $w_g \in h(B)$ and $c_h \in h(C)$, then $c_h \in K$ or $w_g \in (K :_B C)$, see [12].

Clearly, every gr-PSup and every gr-L_e-PSub is a gr-L_e-J_{gr}-PSub.

Example 2.2: Let $\Omega = \mathbb{Z}_2$ and $B = \mathbb{Z}$. In this setup, B becomes a Ω-graded ring with $B_0 = \mathbb{Z}$ and $B_1 = 0$. Now, consider $C = \mathbb{Z}_{12}$. Here, C forms a graded B-module, where $C_0 = \mathbb{Z}_{12}$ and $C_1 = 0$. Next, take $K = \langle \bar{4} \rangle$. Then K is a gr-Sub of C. To analyze further, start with the gr-Id $L = 4\mathbb{Z}$ in B. Under these conditions, K qualifies as a gr-L_e-J_{gr}-PSub of C because $K \backslash L_0 K = \emptyset$. However, K is not a gr-J_{gr}-PSub of C. This is evident because neither $\bar{2} \in K$ nor $2 \in (K :_B C) = 4\mathbb{Z}$, yet $2 \cdot \bar{2} = \bar{4} \in K$. Now, consider another gr-Id $P = 0$ in B. Basic calculations confirm that K is a gr-P_e-J_{gr}-PSub of C. Nevertheless, K does not qualify as a gr-P_e-PSub of C, since $2 \cdot \bar{2} = \bar{4} \in K \backslash P_0 K$ and neither $\bar{2} \in K$ nor $2 \in (\langle \bar{4} \rangle :_{\mathbb{Z}} \mathbb{Z}_{12})$.

A proper gr-Sub K of C is called a gr-J_{gr}-PSub of C if whenever $r_g c_h \in K$ where $r_g \in h(B)$ and $c_h \in h(C)$, then either $c_h \in K + J_{gr}(C)$ or $r_g \in (K + J_{gr}(C) :_B C)$, see [6].

It is recognizable that a gr-J_{gr}-PSub is a gr-L_e-J_{gr}-PSub. The opposite doesn't hold true, as demonstrated in the next example.

Example 2.3: Consider a group $\Omega = \mathbb{Z}_2$ and a ring $B = \mathbb{Z}$, where B is a Ω-graded ring. The grading on B is such that $B_0 = \mathbb{Z}$ and $B_1 = 0$. Now, let $C = \mathbb{Z}_6$, which is a graded B-module. The grading on C is such that $C_0 = \mathbb{Z}_6$ and $C_1 = 0$. Next, we look at the gr-Id $L = 0$ in B and the gr-Sub $K = \langle \bar{0} \rangle$ in C. We can conclude that K is not a gr-J_{gr}-PSub of C because the element $2 \cdot \bar{3}$ is in K, but neither $\bar{3}$ is in $K + J_{gr}(C) = \{0\}$, nor is 2 in $(K + J_{gr}(C) :_B C) = \{0\}$. However, since the difference between K and $L_e K$ is empty (i.e., $K \backslash L_e K = \emptyset$), it follows that K is a gr-L_e-J_{gr}-PSub of C. Let $L_1 = \bigoplus_{g \in \Omega} L_{1_g}$ and $L_2 = \bigoplus_{g \in \Omega} L_{2_g}$ be two gr-Ids of B with $L_1 \subseteq L_2$ and K and U two gr-Subs of C with $K \emptyset U$.

(i) If K is a gr-L_{1_e}-J_{gr}-PSub of C and $J_{gr}(C) \subseteq J_{gr}(U)$, then K is a gr-L_{1_e}-J_{gr}-PSub of U.

(ii) If K is a gr-L_{1_e}-J_{gr}-PSub of C, then K is a gr- L_{2_e}-J_{gr}-PSub of C.

Proof:

(i) Let $r_g u_h \in K \backslash L_{1_e} K$ where $r_g \in h(B)$ and $u_h \in h(C) \cap U$, then we conclude that either $u_h \in K + J_{gr}(C)$ or $r_g C \subseteq K + J_{gr}(C)$ as K is a gr-L_{1_e}-J_{gr}-PSub of C. As $J_{gr}(C) \subseteq J_{gr}(U)$, either $u_h \in K + J_{gr}(C) \subseteq K + J_{gr}(U)$ or $r_g U \subseteq r_g C \subseteq K + J_{gr}(C) \subseteq K + J_{gr}(U)$. This follows that either $u_h \in K + J_{gr}(U)$ or $r_g U \subseteq K + J_{gr}(U)$. Therefore, K is a gr-L_{1_e}-J_{gr}-PSub of U.

(ii) Let $r_g c_h \in K \backslash L_{2_e} K$ where $r_g \in h(B)$ and $c_h \in h(C)$, then $r_g c_h \in K \backslash L_{2_e} K \subseteq K \backslash L_{1_e} K$ as $L_1 \subseteq L_2$. Since K is a gr-L_{1_e}-J_{gr}-PSub of C, we get either $c_h \in K + J_{gr}(C)$ or $r_g C \subseteq K + J_{gr}(C)$. Therefore, K is a gr-L_{2_e}-J_{gr} -PSub of C.

Theorem 2.5: *Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B and K a gr-L_e-J_{gr}-PSub of C.*

(i) *If $J_{gr}(C) \subseteq K$, then K is a gr-L_e-PSub of C.*

(ii) *If K is a gr-maximal Sub and C is a gr-local B-module, then K is a gr-L_e-PSub of C.*

(iii) *If C is a gr-semisimple B-module, then K is a gr-L_e-PSub of C.*

Proof:

(i) The proof is trivial.

(ii) Since C is a gr-local B-module and K is a gr-maximal Sub of C, $J_{gr}(C) = K$.

(iii) Since C is a gr-semisimple B-module, $J_{gr}(C) = \{0\}$.

Theorem 2.6: *Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B and K a proper gr-Sub of C. The following statements are equivalent:*

(a) K is a gr-L_e-J_{gr}-PSub of C.

(b) *If for any gr-Id I of B and gr-Sub U of C with $IU \subseteq K\backslash L_eK$, then either $U \subseteq K + J_{gr}(C)$ or $I \subseteq (K + J_{gr}(C):_B C)$.*

Proof: $(a) \Rightarrow (b)$ Let I be a gr-Id of B and U a gr-Sub of C such that $IU \subseteq K\backslash L_eK$. Assume that neither $U \subseteq K + J_{gr}(C)$ nor $I \subseteq (K + J_{gr}(C):_B C)$. Hence, there exist $i_g \in (I\backslash(K + J_{gr}(C):_B C)) \cap h(B)$ and $u_h \in (U\backslash(K + J_{gr}(C))) \cap h(C)$. Now, since $i_g u_h \in K\backslash L_eK$ and K is a gr-L_e-J_{gr}-PSub of C, either $u_h \in K + J_{gr}(C)$ or $i_g \in (K + J_{gr}(C):_B C)$, a contradiction. Therefore, either $U \subseteq K + J_{gr}(C)$ or $I \subseteq (K + J_{gr}(C):_B C)$.

$(b) \Rightarrow (a)$ Suppose that (b) holds. Let $r_g c_h \in K\backslash L_eK$ where $r_g \in h(B)$ and $c_h \in h(C)$. Let $I = r_gB$ be a gr-Id of B generated by r_g and $U = Bc_h$ be a gr-Sub of C generated by c_h. Now, by our assumption $IU \subseteq K\backslash L_eK$. Hence, we obtain either $U \subseteq K + J_{gr}(C)$ or $I \subseteq (K + J_{gr}(C):_B C)$ and then either $c_h \in K + J_{gr}(C)$ or $r_g \in (K + J_{gr}(C):_B C)$. Thus, K is a gr-L_e-J_{gr}-PSub of C.

Theorem 2.7: *Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B and $S \subseteq h(B)$ a multiplicatively closed subset of B. If K is a gr-L_e-J_{gr}-PSub of C with $(K:_B C) \cap S = \emptyset$ and $S^{-1}(L_eK) \subseteq (S^{-1}L)_e(S^{-1}K)$, then $S^{-1}K$ is a gr-$(S^{-1}L)_e$-J_{gr}-PSub of the gr-$S^{-1}B$-module $S^{-1}C$.*

Proof: Let $(K:_B C) \cap S = \emptyset$. Thus, $S^{-1}K$ is a proper gr-Sub of $S^{-1}C$. Let $\frac{r_g}{x_1}\frac{c_h}{x_2} = \frac{r_g c_h}{x_1 x_2} \in (S^{-1}K)\backslash(S^{-1}L)_e(S^{-1}K)$ where $\frac{r_g}{x_1} \in h(S^{-1}B)$ and $\frac{c_h}{x_2} \in h(S^{-1}C)$. Hence, there exists $v \in S$ such that $vr_g c_h \in K\backslash L_eK$ and then either $vc_h \in K + J_{gr}(C)$ or $r_g \in (K + J_{gr}(C):_B C)$ as K is a gr-L_e-J_{gr}-PSub of C. This yields that either $\frac{c_h}{x_2} = \frac{vc_h}{vx_2} \in S^{-1}K + J_{gr}(S^{-1}C)$ or $\frac{r_g}{x_1} \in S^{-1}(K +$

$J_{gr}(C):_B C) = (S^{-1}K + J_{gr}(S^{-1}C):_{S^{-1}B} S^{-1}C)$. As a result, $S^{-1}K$ is a gr-$(S^{-1}L)_e$-J_{gr}-PSub of $S^{-1}C$.

Let $K = \bigoplus_{g\in\Omega} K_g$ be a proper gr-Sub of C and $\lambda \in \Omega$. Then K_λ is called a λ-J_{gr}-PSub of the B_e-module C_λ if whenever $r_e \in B_e$ and $c_\lambda \in C_\lambda$ with $r_e c_\lambda \in K_\lambda$, then either $c_\lambda \in K_\lambda + J_{gr}(C_\lambda)$ or $r_e \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$, see [6].

Theorem 2.8: *Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B, $K = \bigoplus_{g\in\Omega} K_g$ a gr-Sub of C and $\lambda \in \Omega$. If K_λ is a λ-L_e-J_{gr}-PSub of the B_e-module C_λ with $(K_\lambda:_{B_e} C_\lambda)K_\lambda \nsubseteq L_e K_\lambda$, then K_λ is a λ-J_{gr}-PSub of C_λ.*

Proof: Suppose that K_λ is a λ-L_e-J_{gr}-PSub of the B_e-module C_λ and $(K_\lambda:_{B_e} C_\lambda)K_\lambda \nsubseteq L_e K_\lambda$. Let $r_e c_\lambda \in K_\lambda$ where $r_e \in B_e$ and $c_\lambda \in C_\lambda$. If $r_e c_\lambda \notin L_e K_\lambda$, then we get either $c_\lambda \in K_\lambda + J_{gr}(C_\lambda)$ or $r_e \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$. So, we assume $r_e c_\lambda \in L_e K_\lambda$. Now, let $r_e K_\lambda \nsubseteq L_e K_\lambda$, then there exists $k_{1_\lambda} \in K_\lambda$ with $r_e k_{1_\lambda} \notin L_e K_\lambda$. Thus, $r_e(c_\lambda + k_{1_\lambda}) \in K_\lambda \backslash L_e K_\lambda$ and then we get either $c_\lambda + k_{1_\lambda} \in K_\lambda + J_{gr}(C_\lambda)$ or $r_e \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$ and then either $c_\lambda \in K_\lambda + J_{gr}(C_\lambda)$ or $r_e \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$. Now, let $(K_\lambda:_{B_e} C_\lambda)c_\lambda \nsubseteq L_e K_\lambda$, then there exists $s_{1_e} \in (K_\lambda:_{B_e} C_\lambda)$ such that $s_{1_e} c_\lambda \notin L_e K_\lambda$. Hence, $(r_e + s_{1_e})c_\lambda \in K_\lambda \backslash L_e K_\lambda$ which follows that either $r_e + s_{1_e} \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$ or $c_\lambda \in K_\lambda + J_{gr}(C_\lambda)$ and then either $r_e \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$ or $c_\lambda \in K_\lambda + J_{gr}(C_\lambda)$. So, assume that $r_e K_\lambda \subseteq L_e K_\lambda$ and $(K_\lambda:_{B_e} C_\lambda)c_\lambda \subseteq L_e K_\lambda$. Since $(K_\lambda:_{B_e} C_\lambda)K_\lambda \nsubseteq L_e K_\lambda$, there exist $s_{2_e} \in (K_\lambda:_{B_e} C_\lambda)$ and $k_{2_\lambda} \in K_\lambda$ with $s_{2_e} k_{2_\lambda} \notin L_e K_\lambda$. Now, $(r_e + s_{2_e})(c_\lambda + k_{2_\lambda}) = r_e c_\lambda + r_e k_{2_\lambda} + s_{2_e} c_\lambda + s_{2_e} k_{2_\lambda} \in K_\lambda \backslash L_e K_\lambda$, then either $r_e + s_{2_e} \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$ or $c_\lambda + k_{2_\lambda} \in K_\lambda + J_{gr}(C_\lambda)$. This follows that either $r_e \in (K_\lambda + J_{gr}(C_\lambda):_{B_e} C_\lambda)$ or $c_\lambda \in K_\lambda + J_{gr}(C_\lambda)$. As a result, K_λ is a λ-J_{gr}-PSub of C_λ.

Corollary 2.9: *Let $K = \bigoplus_{g\in\Omega} K_g$ be a gr-Sub of C and $\lambda \in \Omega$. If K_λ is a λ-0_e-J_{gr} -PSub of the B_e-module C_λ with $(K_\lambda:_{B_e} C_\lambda)K_\lambda \neq \{0\}$, then K_λ is a λ-J_{gr}-PSub of C_λ.*

Proof: In Theorem 2.8, take $L = \{0\}$ and then we obtain the result.

Corollary 2.10: *Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B, $K = \bigoplus_{g\in\Omega} K_g$ a gr-Sub of C and $\lambda \in \Omega$. If C_λ is a multiplication B_e-module and K_λ is a λ-L_e-J_{gr}-PSub of C_λ, then either K_λ is a λ-J_{gr}-PSub of C_λ or $K_\lambda^2 \subseteq L_e K_\lambda$.*

Proof: Suppose that K_λ is not a λ-J_{gr}-PSub of C_λ. So, by Theorem 2.8 we get $(K_\lambda:_{B_e} C_\lambda)K_\lambda \subseteq L_e K_\lambda$ as K_λ is a λ-L_e-J_{gr}-PSub of C_λ. Now, as C_λ is a multiplication B_e-module we get $K_\lambda^2 = (K_\lambda:_{B_e} C_\lambda)^2 C_\lambda \subseteq (K_\lambda:_{B_e} C_\lambda)K_\lambda \subseteq L_e K_\lambda$. Therefore, $K_\lambda^2 \subseteq L_e K_\lambda$.

Theorem 2.11: *Let B_i be a Ω-graded ring, C_i a graded B_i-module and $L_i = \bigoplus_{g\in\Omega} L_{i_g}$ a gr-Id of B_i, for $i = 1,2$. Let $B = B_1 \times B_2$, $C = C_1 \times C_2$ and $L = L_1 \times L_2$. Then:*

(i) *If K_1 is a gr-L_{1_e}-J_{gr}-PSub of C_1 with $L_{1_e}K_1 \times C_2 \subseteq L_e(K_1 \times C_2)$ and $J_{gr}(C_1) \times C_2 \subseteq J_{gr}(C_1 \times C_2)$, then $K_1 \times C_2$ is a gr-L_e-J_{gr}-PSub of C.*

(ii) *If K_2 is a gr-L_{2_e}-J_{gr}-PSub of C_2 with $C_1 \times L_{2_e}K_2 \subseteq L_e(C_1 \times K_2)$ and $C_1 \times J_{gr}(C_2) \subseteq J_{gr}(C_1 \times C_2)$, then $C_1 \times K_2$ is a gr-L_e-J_{gr}-PSub of C.*

Proof:

(i) Let $(r_{1_g}, r_{2_g})(c_{1_h}, c_{2_h}) = (r_{1_g}c_{1_h}, r_{2_g}c_{2_h}) \in (K_1 \times C_2)\backslash L_e(K_1 \times C_2)$ where $(r_{1_g}, r_{2_g}) \in h(B)$ and $(c_{1_h}, c_{2_h}) \in h(C)$. So, $(K_1 \times C_2)\backslash L_e(K_1 \times C_2) \subseteq (K_1 \times C_2)\backslash(L_{1_e}K_1 \times C_2) = (K_1\backslash L_{1_e}K_1) \times C_2$ as $L_{1_e}K_1 \times C_2 \subseteq L_e(K_1 \times C_2)$. Then, $r_{1_g}c_{1_h} \in K_1\backslash L_{1_e}K_1$ which follows that either $r_{1_g}C_1 \subseteq K_1 + J_{gr}(C_1)$ or $c_{1_h} \in K_1 + J_{gr}(C_1)$ as K_1 is a gr-L_{1_e}-J_{gr}-PSub of C_1. So, either $(r_{1_g}, r_{2_g})(C_1 \times C_2) \subseteq (K_1 \times C_2) + (J_{gr}(C_1) \times C_2) \subseteq (K_1 \times C_2) + J_{gr}(C_1 \times C_2)$ or $(c_{1_h}, c_{2_h}) \in K_1 \times C_2 + (J_{gr}(C_1) \times C_2) \subseteq (K_1 \times C_2) + J_{gr}(C_1 \times C_2)$. As a result, $K_1 \times C_2$ is a gr -L_e-J_{gr}-PSub of C.

(ii) The proof is similar to part (i).

Theorem 2.12: *Let B_i be a Ω-graded ring, C_i a graded B_i-module, K_i a gr-Sub of C_i and $L_i = \bigoplus_{g\in\Omega} L_{i_g}$ a gr-Id of B_i, for $i = 1,2$. Let $B = B_1 \times B_2$, $C = C_1 \times C_2$ and $L = L_1 \times L_2$. Then:*

(a) If $L_{i_e}K_i = K_i$ for $i = 1,2$, then $K_1 \times K_2$ is a gr-L_e-J_{gr}-PSub of C.

(b) If K_1 is a gr-PSup of C_1, then $K_1 \times C_2$ is a gr-L_e-J_{gr}-PSub of C.

(c) If K_1 is a gr-L_{1_e}-J_{gr}-PSub of C_1 with $L_{2_e}C_2 = C_2$ and $J_{gr}(C_1) \times C_2 \subseteq J_{gr}(C_1 \times C_2)$, then $K_1 \times C_2$ is a gr-L_e-J_{gr}-PSub of C.

(c) If K_2 is a gr-PSup of C_2, then $C_1 \times K_2$ is a gr-L_e-J_{gr}-PSub of C.

(c) If K_2 is a gr-L_{2_e}-J_{gr}-PSub of C_2 with $L_{1_e}C_1 = C_1$ and $C_1 \times J_{gr}(C_2) \subseteq J_{gr}(C_1 \times C_2)$, then $C_1 \times K_2$ is a gr-L_e-J_{gr}-PSub of C.

Proof:

(a) Since $L_{i_e}K_i = K_i$ for $i = 1,2$, $L_e(K_1 \times K_2) = L_{1_e}K_1 \times L_{2_e}K_2 = K_1 \times K_2$. This yields that $(K_1 \times K_2)\backslash L_e(K_1 \times K_2) = \emptyset$, so we get the result.

(b) $K_1 \times C_2$ is a gr-PSup of C as K_1 is a gr-PSup of C_1. Thus, we conclude that $K_1 \times C_2$ is a gr-L_e-J_{gr}-PSub of C.

(c) Since $L_{2_e}C_2 = C_2$, $L_{1_e}K_1 \times C_2 \subseteq L_e(K_1 \times C_2)$. Hence, $K_1 \times C_2$ is a gr-L_e-J_{gr}-PSub of C by Theorem 2.11.

(d) The proof is similar to part (b).

(e) The proof is similar to part (c).

Theorem 2.13: *Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B and K a proper gr-Sub of C. Then the following statements are equivalent:*

(i) K is a gr-L_e-J_{gr}-PSub of C.

(ii) K/L_eK is a gr-0_e-J_{gr}-PSub of C/L_eK.

Proof: $(i) \Rightarrow (ii)$ Let $r_g(c_h + L_eK) = r_gc_h + L_eK \in (K/L_eK)\backslash 0_e(K/L_eK)$ where $r_g \in h(B)$ and $(c_h + L_eK) \in h(C/L_eK)$. So, $r_gc_h \in K\backslash L_eK$ which yeilds that either $c_h \in K + J_{gr}(C)$ or $r_gC \subseteq K + J_{gr}(C)$ as K is a gr-L_e-J_{gr}-PSub of C. Then we conclude that either $c_h + L_eK \in K/L_eK + J_{gr}(C/L_eK)$ or $r_g(C/L_eK) \subseteq K/L_eK + J_{gr}(C/L_eK)$. As a result, K/L_eK is a gr-0_e-J_{gr}-PSub of C/L_eK.

$(ii) \Rightarrow (i)$ Let $r_gc_h \in K\backslash L_eK$ where $r_g \in h(B)$ and $c_h \in h(C)$. So, $r_gc_h + L_eK \in (K/L_eK)\backslash 0_e(K/L_eK)$ and then we get either $c_h + L_eK \in K/L_eK + J_{gr}(C/L_eK)$ or $r_g(C/L_eK) \subseteq K/L_eK + J_{gr}(C/L_eK)$. Thus, either $c_h \in K + J_{gr}(C)$ or $r_gC \subseteq K + J_{gr}(C)$. As a result, K is a gr-L_e-J_{gr} -PSub of C.

Theorem 2.14: *Let $L = \bigoplus_{g\in\Omega} L_g$ be a gr-Id of B and K, U be two gr-Subs of C such that $U \subseteq K$. If K is a gr-L_e-J_{gr}-PSub of C with $K/U + J_{gr}(C/U) = (K + J_{gr}(C))/U$, then K/U is a gr-L_e-J_{gr}-PSub of C/U.*

Proof: Let $r_g(c_h + U) = r_gc_h + U \in (K/U)\backslash L_e(K/U)$ where $r_g \in h(B)$ and $c_h + U \in h(C/U)$. So, $r_gc_h \in K\backslash L_eK$ and then we get either $c_h \in K + J_{gr}(C)$ or $r_gC \subseteq K + J_{gr}(C)$. Since $K/U + J_{gr}(C/U) = (K + J_{gr}(C))/U$, either $c_h + U \in K/U + J_{gr}(C/U)$ or $r_g(C/U) \subseteq K/U + J_{gr}(C/U)$. Thus, K/U is a gr-L_e-J_{gr}-PSub of C/U.

Conclusion

This paper introduces and systematically analyzes graded L_e-J_{gr}-prime submodules, demonstrating their significance as a generalization of previously studied graded submodule types. Through rigorous proofs and illustrative examples, we establish their distinct theoretical properties and practical implications in graded module theory. Notable results include their behavior under localization and direct product operations, as well as criteria for identifying maximal and semisimple graded submodules. This work provides a robust framework for further exploration of graded structures and primes, paving the way for applications in both abstract algebra and graded ring theory.

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