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Graceful labeling of paths, cycles and caterpillars via inverse transformation

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Abstract

Out of the enormous branches of mathematics, graph theory plays a vital role in the fields of applied mathematics and scientific computing. One of the major fields of graph theory is the study of graph labeling problems. The present work involves the graceful labeling of a cycle of length three by joining isomorphic copies of paths to all of its vertices. The present work also analyzes the design of graceful graphs, which are created by joining caterpillars with either one or two end vertices of a cycle. Further, it establishes graceful labeling by joining two hairy cycle graphs with a common edge. To find all the results in this paper, the inverse transformation technique is used.

Subject Classification: *05C78, 05C10, 05C38.*

Keywords: *Graceful labeling, Path, Cycle, Caterpillar, Inverse transformation, Unicyclic graphs.*

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1. Introduction

By a graph labeling we mean an assignment of integers to the vertices and edges or both, subject to certain conditions. One may refer to the Introduction to Graph Theory textbook [19] for the concepts that are common in graph theory. Most graph labeling methods trace their origin to one introduced by Rosa [14], and the present paper confines itself to graceful labeling of graphs. This kind of graceful labeling was formally referred to as β-labeling. Later, Golomb called it "graceful labeling," which is now the popular term. Over 3600 papers have experimented with more than 350 graph labeling strategies over the years, showing the breadth and depth of this field's research. Lapierre and McGuinness [12] discussed the method of Constructing Trees with Graceful Labeling using Caterpillars. To know more types of graceful caterpillar graphs, one may refer to [15, 17, and 18]. Kumar, Mishra and Kumor [10,11] discussed graceful labeling of cycle graphs as Alpha labeling. Behera, Mishra and Nayak [3] and Behera [4] proved that the union of a cycle of length with a path and a class of corona graphs are graceful. Biatch, Bagga and Arumugam [5] surveyed a new class of graceful unicyclic graphs. Huomani, Bravo, and Pomo [9] presented some new families of graceful spider graphs with particular types of legs. Havior and Kotulovo [8] introduced characterizations of kites as graceful graphs. Basher [2] proved Odd-even gracefulness of splitting of graphs. Boonklurb, Ruamkaew and Singhun [6] presented directed edge-graceful labeling of a digraph consisting of Cycles of the same size. Lau et.al [13] presented some new families of K-super graceful labeling of some standard graphs. Saengsura and Poomsa-ard [16] discussed some graceful spider graphs. Ahmad, Baca, and Siddiqui [1] contributed more explorations on graceful trees. For more results on graceful graphs, one may follow the latest survey by Gallian [7]. Kumar, Anbarasan and Pushparaj [20] proved edge-odd graceful labeling for the Cartesian product of two graph.

The purpose of this paper emphasizes the graceful labeling by the join of a cycle of length m with a caterpillar either at one end or two ends of the cycle. Also, a cycle of length 3 with each of its vertices attached to isomorphic copies of a path is graceful. Further, it is proved in the present study how two Hairy cycle graphs containing a cycle of length 3 having a common edge is graceful. All the results of this paper are found by using graceful labeling and inverse transformation in each step. For a better understanding of inverse transformation, one can follow the Example 2.1, 2.2 in this paper. Till now, no such works on graceful paths, cycles and caterpillars have been done using inverse transformation matching to our results as per the existing literature in graph labeling. Graph labeling solves many practical issues in several domains, such as social networks, communication networks, network security, network addressing, and channel assignment techniques. In computer graphics, force-directed graph drawing algorithms can be benefited from the use of graceful caterpillars when applying graph drawing methods.

2. Preliminaries

Definitions 2.1: Let G be a graph with q edges. The set of labeled vertices is $V(G)=\{0,1,2,3......q\}$ and a set of edges is $\{1,2,3.....q\}$ Then graceful labeling of G is an injection $f:V(G) \rightarrow \{1,2......q\}$ such that $\{|f(u)-f(v)|$ $:\ \{u,v\}$ is an edge of $G\}=\{1,2,....q\}$.

Definitions 2.2: A path is a sequence of vertices followed by edges, with no vertices and no edges.

Definitions 2.3: A cycle is a closed path with three or more vertices.

Definitions 2.4: A caterpillar C is a tree such that there exists a path P called the central path of the tree, from which every vertex of C other than the vertex of a central path is at one distances.

3. Results and discussion

Lemma 2.1: *If h is a graceful labeling of a graph G with q edges, then the labeling* h_q *defined as* $h_q(x)=q-h(x)$, *for all* $x \in V(T)$, *called the inverse transformation or inverse labeling of h is also a graceful labeling of G.*

Example 2.1: Lemma 2.1 is explained by Figure 1 and 2.

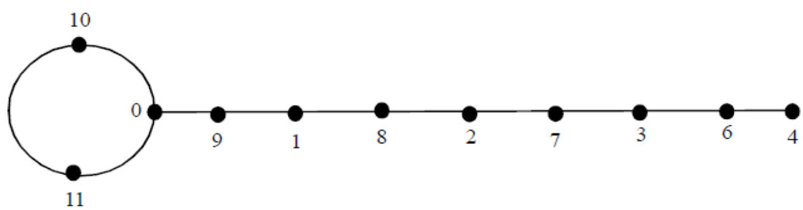

Figure 1

The join of C_3 and P_9 is graceful.

Example 2.2: If the join of C_3 and P_9 is graceful, then by applying the inverse transformation to the graceful graph in Example 2.1.

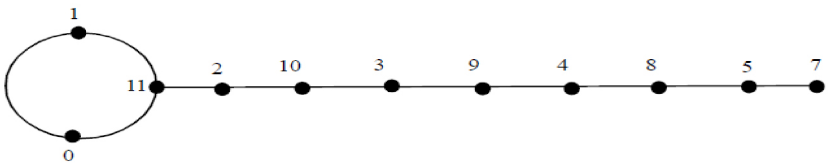

Figure 2

The graceful graph obtained by applying the inverse transformation to the graceful graph in Figure 1.

Theorem 2.1: *The graph obtained by joining one vertex of a graceful cycle C_m with either one end vertex of the central path or a leaf adjacent to one end vertex of the central path of a caterpillar has a graceful labeling.*

Proof: Consider a graceful cycle C_m. Suppose that the vertex x of C_m possess the label 0. Take a caterpillar with $p = x_1x_2....x_n$ as its central path. Let x_i be adjacent to r_i leaves for $i = 1,2,3.....n.$ Let c_m has m edges. Let us join the vertex x_1 of P_n to and assign a label $m + 1$ to it. Clearly, the resultant graph is graceful. Next, apply the inverse transform to the above graph and get a new graceful labeling of it in which x_1 gets the label 0. Now add the $n_1 + 1$ vertices to x_1 assign the labels $m+2,\ m+3.,\ ......\ m+n_1+2$. Clearly, the resultant trees thus obtained are graceful. Now designate the vertex whose label is $m+n_1+2$ as x_2 and apply inverse labeling so that the label of x_2 becomes 0. Continue this process until n vertices, namely $x_1x_2....x_{n-1}$ along with the leaves adjacent to them are added one by one. Now join x_n to x_{n-1} and assign the label $m + \sum_{i=1}^{n-1} r_i + n.$ Apply the inverse labeling to the above tree. So the resultant tree becomes graceful, and the vertex x_n gets the label 0. Finally attach r_n leaves to x_n and give labels as $t+1,\ t+2,\ .....\ t+r_n$ to get the desired graph with a graceful labeling.

Corollary 2.1: *The graph obtained by joining one vertex of a graceful cycle C_3 with either one end vertex of the central path or a leaf adjacent to one end vertex of the central path of a caterpillar has a graceful labeling.*

Example 2.2: Consider the cycle, with the graceful labeling given in Figure 3 and the caterpillar with the central path $x_1x_2x_3x_4$ in Figure 4. We first join x_1 with x and assign the label 4 to x_1 and get the resultant graph shown in Figure 5. Figure 6 is the graceful graph obtained by applying inverse labeling to the graceful graph in Figure 5 and subsequently attaching four new vertices to x_1 with labels 5, 6, 7 8. Figure 7 is the graceful graph obtained by applying inverse labeling to the graceful graph in Figure 6 and subsequently attaching three new vertices to x_2 with labels 9, 10, 11. Figure 8 is the graceful graph obtained by applying inverse labeling to the graceful graph in Figure 7 and subsequently attaching one new vertex to x_3 with labels 12. Figure 9 is the graceful graph obtained from the graceful graph in Figure 8 by attaching five new vertices to x_4 with labels 13, 14, 15, 16, 17.

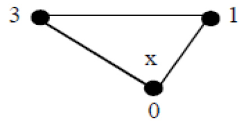

Figure 3

The cycle C_3 with a graceful labeling

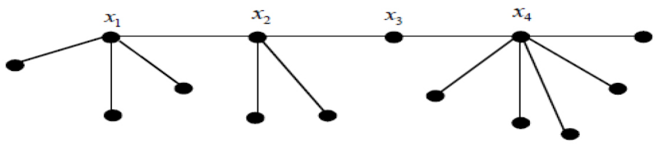

Figure 4

The caterpillar with the central path $x_1x_2x_3x_4$

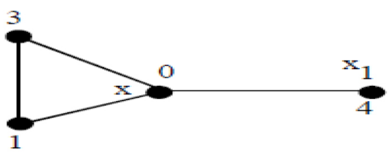

Figure 5

The graceful graph obtained by joining x to x_1

Figure 6

The graceful graph obtained by applying inverse labeling to the graceful graph in Figure 5. And attaching new vertices to and attaching x_1 with the labels 5, 6, 7 and 8.

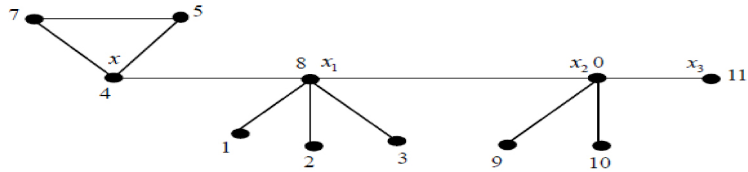

Figure 7

The graceful graph obtained by applying inverse labeling to the graceful graph in Figure 6 and attaching three new vertices with x_2 label 9, 10, 11.

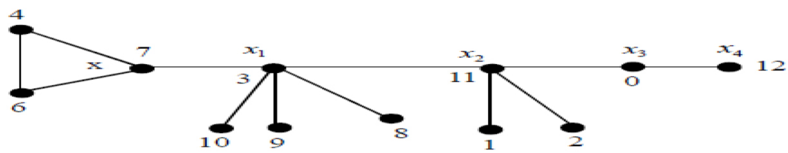

Figure 8

The graceful graph obtained by applying inverse labeling to the graceful graph in Figure 7and subsequently attaching x_4 to x_3 with label 12.

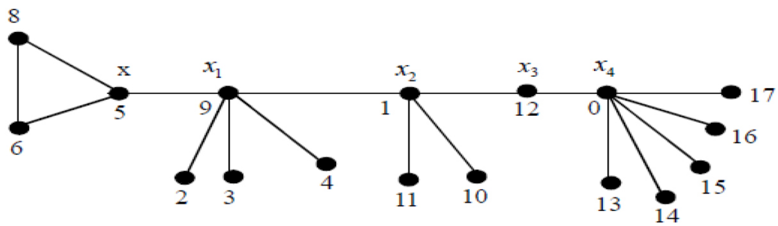

Figure 9

Graceful graph obtained from the graceful graph in Figure 8 by attaching five new vertices to x_4 with the labels 13, 14, 15, 16, 17.

Theorem 3.1: *A cycle C_3 with two of its vertices attached to caterpillars is graceful.*

Proof: Any caterpillar has a graceful labeling in which one of the end vertices of the central path gets label 0. Let that C and C_1 be two caterpillars with central paths $p_n = x_1, x_2, \ldots\ldots x_n$ and $Q_m = y_1 y_2 \ldots\ldots y_m$ respectively. Construct a caterpillar from c by adding one extra edge incident on x_n. We know that this caterpillar possesses a graceful labeling, with one among x_1 and x_n assumes the label 0. Without loss of generality, let x_1 assumes the label 0. Then the edge has a label 1 must be incident on x_n. Let us attach a new vertex to x_1 say y and assign the label $q + 2$ to it, where q is the number of edges of C. This new tree thus obtained is graceful. Apply the inverse labeling to this new tree, as a result of which x_1 gets label q + 2, y gets label 0. Now attach the vertex y_1 of Q_m to y and assign the label q + 3. Now join the vertex y_1 and x_1 so that we get a unicyclic graph on the cycle C_3 formed by the vertices x_1, y and y_1 Here, we observe that the edges of the cycle get labels $q + 3$, $q + 2$, and 1. We remove the edge incident on x_n having the label 1. So the new graph thus formed is graceful. Now we can add the remaining portion of the second caterpillar to the central path Q_m by following the procedure in the proof involving the Theorem 2.1.

Example 3.1: Figures given below represent the construction of a unicyclic graceful graph in which two vertices of a cycle of length 3 are attached to two distinct caterpillars given in Figure 10 to Figure 17.

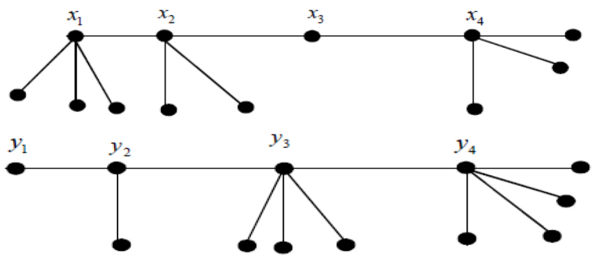

Figure 10

Two distinct caterpillars.

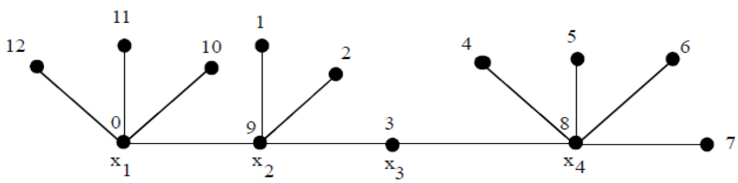

Figure 11

A graceful caterpillar

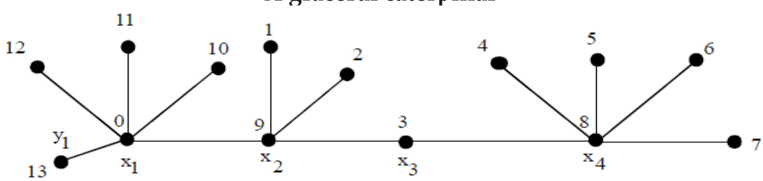

Figure 12

New graceful tree obtained by attaching one edge labeled with 13 to x_1 of Figure 11.

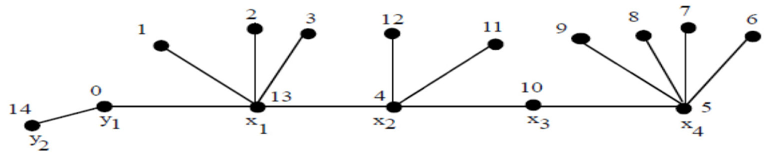

Figure 13

Graceful tree obtained by applying inverse labeling to Figure 12 and attaching one more edge to y_1.

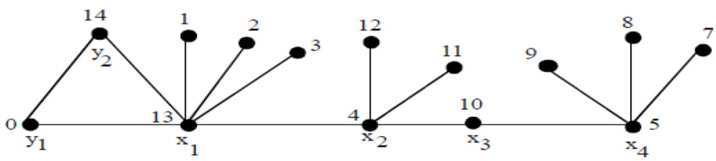

Figure 14

Graceful tree obtained by joining x_1 with y_2 in Figure 13.

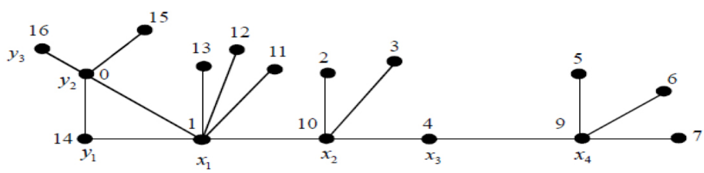

Figure 15

Graceful tree obtained by applying inverse labeling to Figure 14 and attaching two more edges to y_2.

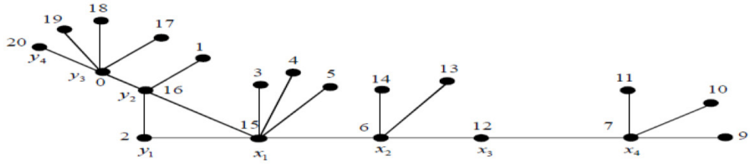

Figure 16

Graceful tree obtained by applying inverse labeling to Figure 15and attaching four more edges to y_3.

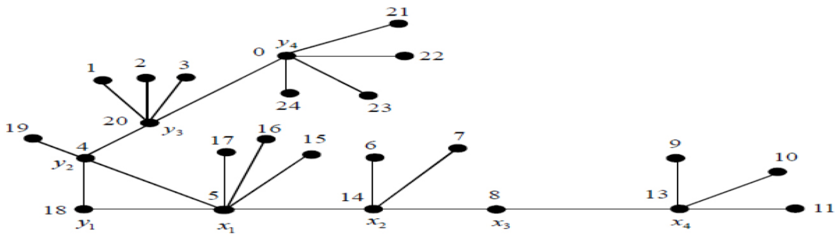

Figure 17

Graceful tree obtained by applying inverse labeling to Figure 16 and attaching four more edges to y_4.

Theorem 4.1: *A cycle C_3 whose each vertex is attached to isomorphic copies of a path P_n is graceful.*

Proof: Consider a path $p_n = x_0\, x_1\, x_2 \,........ x_n$ whose vertex labeling *f* is given by

$$f(x_i) = \begin{cases} \dfrac{i}{2} & if\ \ i = 0, 2, 4, 6, \ldots \\ n - \dfrac{i-1}{2} & if\ \ i = 1, 3, 5, \ldots \end{cases}$$

Take three isomorphic copies of p_n say

$$p_{n1} = (x_{01}\, x_{11}\, x_{21} \,.......... x_{n1})$$

$$p_{n2} = (x_{02} x_{12} x_{22} .... x_{n2})$$

$$p_{n3} = (x_{03} x_{13} x_{23} .... x_{n3})$$

Where x_{ij} is the i^{th} vertex of j^{th} copy of p_{ni}. construct the tree $p_n^{\ 3}$ from p_{n1}, P_{n2} P_{n3} by identifying the vertices x_{01}, x_{02} and x_{03}. Denote this new identified vertex as x_0. The tree $p_n^{\ 3}$ looks as in the Figure 18 given below.

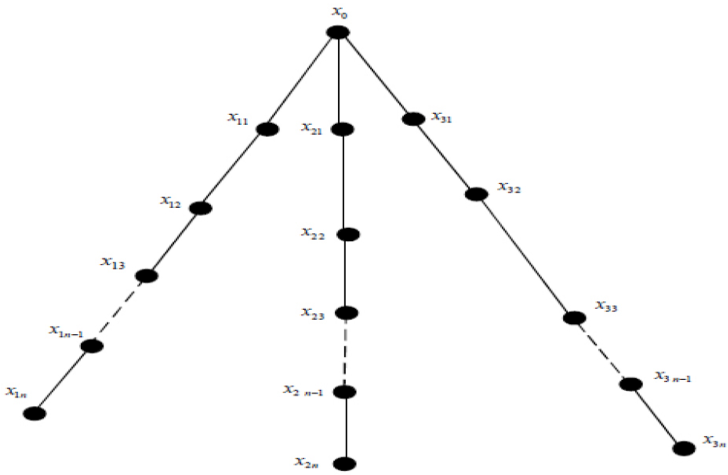

Figure 18

The tree $p_n^{\ 3}$ **Consider the labeling g of** $p_n^{\ 3}$ **given by** $g(x_0)=0$

$$g(x_{ij})=\begin{cases} in-\dfrac{j-1}{2} & if\ \ j\ is\ odd \\ \dfrac{j}{2}+(3-j)n & if\ j\ is\ \ even \end{cases}$$

for $i=1,2,3....,n$ and $j=1,2,3$

The labels of the edges of p_{n1} are $n,n+1,n+2,..........2n-1$. The labels of the edges of p_{n2} are $2n,n-1,n-2,..........3,2,1$. The labels of the edges of p_{n3} are $3n,3n-1,3n-2,........2n+1$.

So the labels of the edges of $p_n^{\ 3}$ is the set $\{1,2,3,.........., 3n\}$ and such g is a graceful labeling of $p_n^{\ 3}$. Next, take the path $p_4=a_0\,a_1\,a_2a_3$ with a graceful labeling h given by $h(a_0)=0$, $h(a_1)=3$, $h(a_2)=1$, $h(a_3)=2$. Multiply each label of p_4 by n so that the labels of a_0, a_1, a_2, a_3 become 0, $3n$, n and $2n$ respectively. Now construct a graceful tree, say T by merging a_0 with x_0, a_1 with x_{11}, a_2 with x_{31} and a_3 with x_{21}. The graceful tree looks as in the Figure 19 given below.

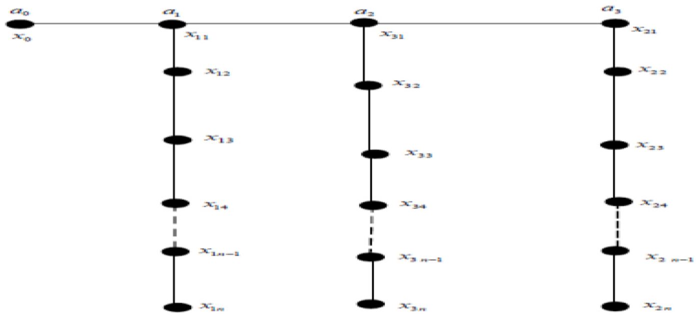

Figure 19

Graceful tree T obtained by merging a_0 with x_0, a_1 with x_{11}, a_2 with x_{31} and a_3 with x_{21}. Apply the inverse labeling to ψ_{3n} so that the label of a_0 becomes $3n$ and that of a_1 becomes 0. Delete the vertex a_0 so that the resultant tree T_1 looks as shown in the figure 20 given below.

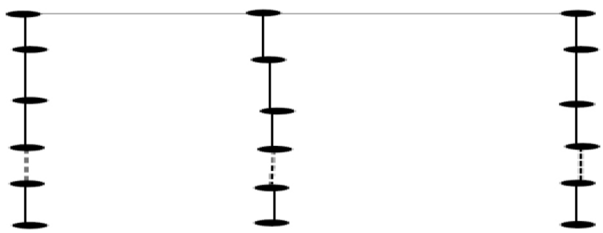

Figure 20

Graceful tree T_1 obtained by deleting the vertex a_0.

Now, the desired graph by joining a_1 and a_2 gives a labeling ϕ as follows

$$\phi(x) = \begin{cases} \psi_{3n}(x) & if\ \psi_{3n}(x) < 2n \\ \psi_{3n}(x)+1 & if\ \psi_{3n}(x) \geq 2n \end{cases}$$

$\Rightarrow$ Label of the vertices. Obviously, ϕ is a graceful labeling of the graph.

Example 4.1: Figures 21 to Figure 27 given below represent the construction of a unicyclic graceful graph in which each of the three vertices of a cycle is attached isomorphic copies of a path.

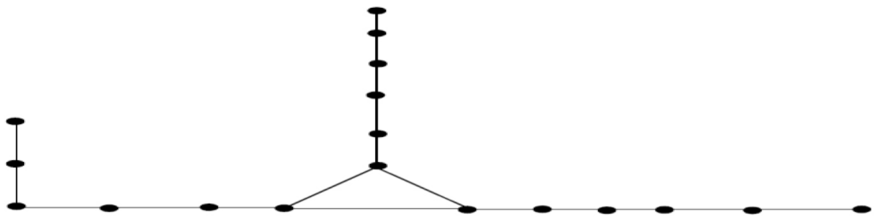

Figure 21

Unicyclic graph with a cycle that's each vertex is attached to an isomorphic copys of p_n.

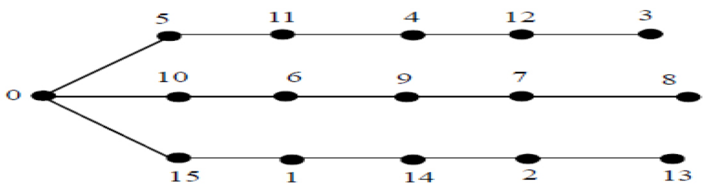

Figure 22

p_6^3 **is graceful**

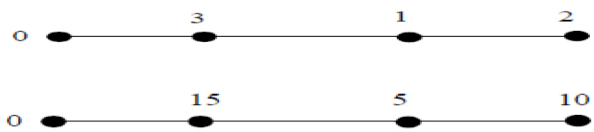

Figure 23

Two Graceful paths P_4

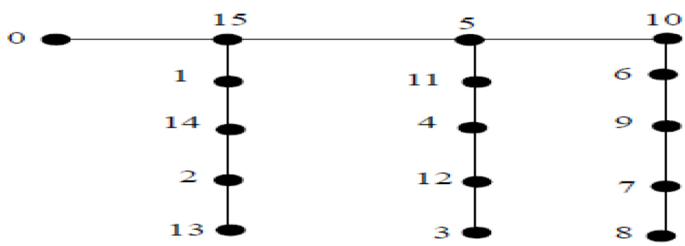

Figure 24

Graceful tree T **obtained by merging** a_0 **with** x_0, a_1 **with** x_{11}, a_2 **with** x_{31} **and** a_3 **with** x_{21} **of** p_6^3.

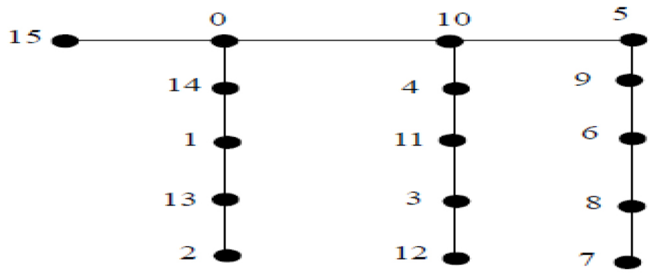

Figure 25

Graceful tree obtained by applying inverse labeling to Figure 24

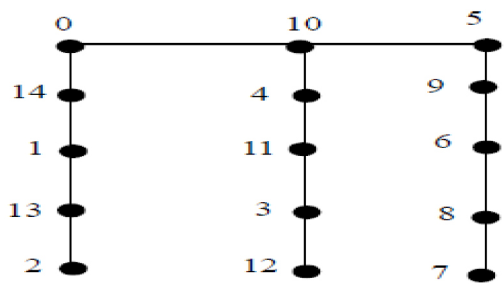

Figure 26

Graceful tree obtained by deleting the edge with the vertex label 15

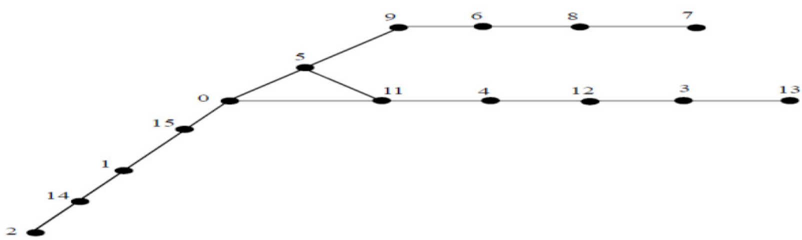

Figure 27

C_3 whose each vertex is attached to isomorphic copies of P_5 is graceful.

Theorem 4.2: $C_3 \oplus 3K_{1,n} - C_3 \oplus 3K_{1,n}$ *is a graceful graph.*

Proof: Consider the graph $C_3 \oplus 3K_{1,n}$. Let three vertices of C_3 be a, b, c and each of them is attached to n edges. Consider the labeling f of $C_3 \oplus 3K_{1,n}$ given by

$$f(a)=0, f(b)=1,\ f(c)=2n+3$$

$$f(x)=\begin{cases} i\ ,\ 2n+4\le i\le 3n+3 & \text{if } x \text{ is adjacent to } a \\ 2j\ ,\ 2\le j\le n+l & \text{if } x \text{ is adjacent to } b \\ 2k+l\ ,\ 1\le l\le n & \text{if } x \text{ is adjacent to } c \end{cases}$$

Figure 29 given below represents the graph $C_3 \oplus 3K_{1,n}$ with the labeling *f*.

We observe that

$$\{|f(u)-f(v)|:(u,v) \text{ is an edge of } C_3 \oplus K_{1,n}\}=\{1,\ 2,3,........,\ 3n+3\}$$

So $C_3 \oplus 3K_{1,\,n}$ is graceful. Next we join the tree given below to the vertex of $C_3 \oplus 3K_{1,\,n}$. So as to construct a new graceful graph. Add the vertex *d* to *a* and assign the label $3n+4$ to it. Figure 30 gives the resultant graceful graph. Apply inverse labelling to the graph in Figure 30 so that the label of d becomes 0. Join $n+1$ new vertices to d and give them the labels $3n+5,\ 3n+6,\ ......\ 4n+5.$ The vertex labeled $4n+5$ is taken as Figure 31 is the resultant graceful tree. Next apply inverse labeling to the graph in Figure 31 so that the vertex e gets the label 0. Next attach $n+1$ vertices to *e* and assign them label $4n+6,\ 4n+7,...,\ \ 5n+6.$ The vertex labeled $5n+6$ is taken as *f*. The resultant graceful graph is represented by Figure 32. Next apply inverse labeling to the graph in Figure 32 so that the vertexes *f* get the label 0. Attach the *n* new vertices to *f* and give them labels $5n+7,\ 5n+8,\ .....\ 6n+6.$ Figure 33 represents the resultant graceful graph. Let this graceful labeling is represented by *g*. Next construct the desired graph $C_3 \oplus 3K_{1,n} - C_3 \oplus 3K_{1,n}$ by joining d and *f*.

Case I: suppose that *n* is even, then $3n+3$ is odd and it is adjacent to *c* labeled $4n+4$.

Give a labeling *h* to $C_3 \oplus 3K_{1,n} - C_3 \oplus 3K_{1,n}$ as given below

$$h(x)=\begin{cases} g(x) & \text{if } x\le 4n+4 \\ g(x)+1 & \text{if } x>4n+4 \end{cases}$$

Next replace the label $3n+3$ adjacent to *b* labeled $4n+4$ with $4n+3$ and the label $2n+4$ adjacent to *c* labeled $2n+2$ with $4n+5.$ The resultant graph thus obtained has a graceful labeling and this fact is represented by Figure 34.

Case II : Suppose that n is odd, then $3n+3$ is even and it is adjacent to b labeled $2n+2$.

Give a labeling h to $C_3 \oplus 3K_{1,n} - C_3 \oplus 3K_{1,n}$ as given below

$$h(x) = \begin{cases} g(x) & \text{if } x \le 3n+3 \\ g(x)+1 & \text{if } x > 3n+4 \end{cases}$$

The resultant graph thus obtained has a graceful labeling and this fact is represented by Figure 35.

A tree that's each vertex of the central path P_3 attached with n leaves is shown in Figure 28.

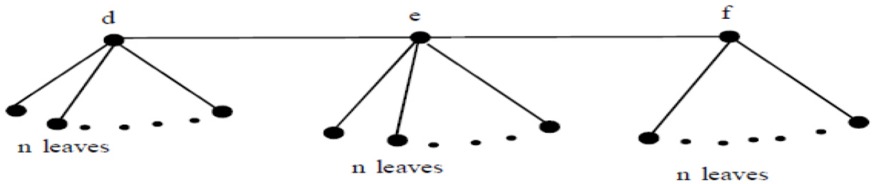

Figure 28

A tree that's each vertex of the central path P_3 attached with n leaves

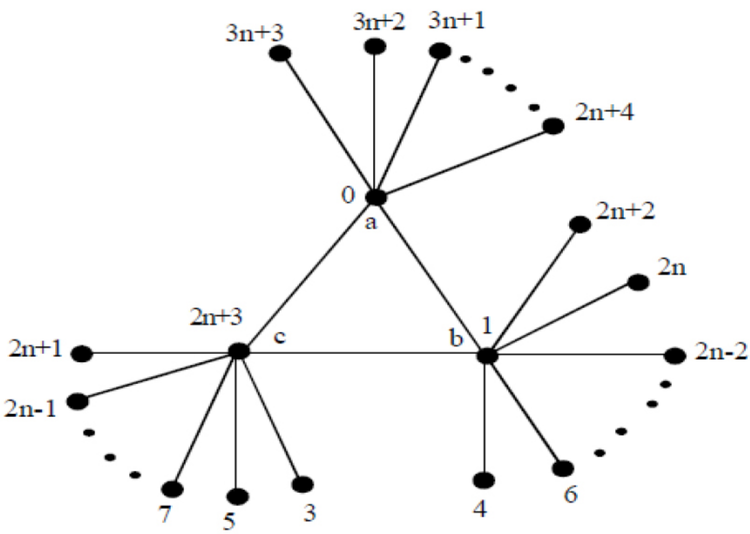

Figure 29

$C_3 \oplus 3K_{1,n}$ **with graceful labeling**

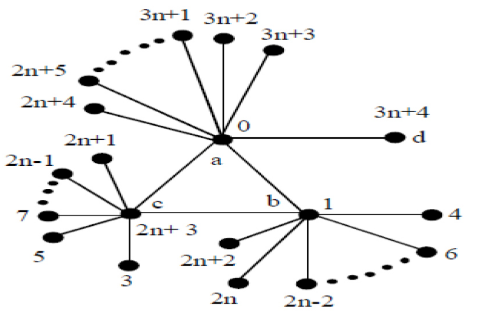

Figure 30

Graceful graph obtained from the graceful graph in Figure 29 by joining the vertex d to a and assigning the label $3n+3$ to it.

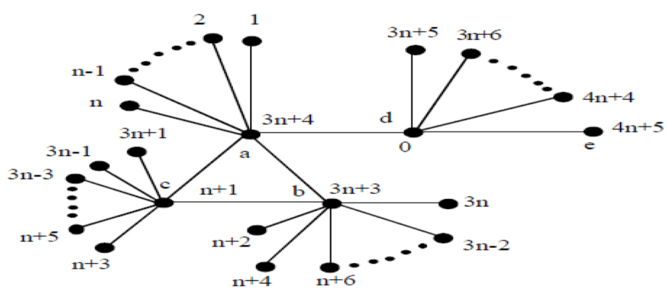

Figure 31

Graceful graph obtained from the graceful graph in Figure 30 by applying inverse transformation and adding $n+1$ new vertices to d and assigning them with labels $3n+5, 3n+6, \ldots\ldots 4n+5$

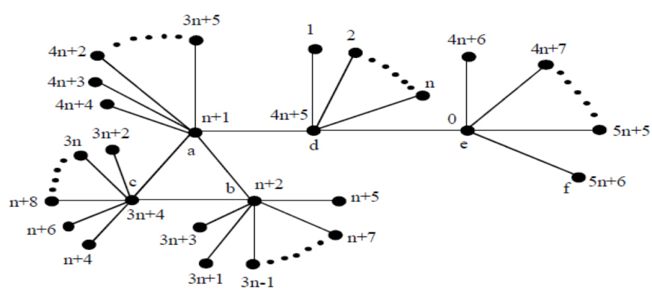

Figure 32

Graceful graph obtained from the graceful graph in Figure 31 by applying inverse transformation and $n+1$ new vertices to e and assigning the labels $4n+6, 4n+7, \ldots\ldots 5n+6$ to them.

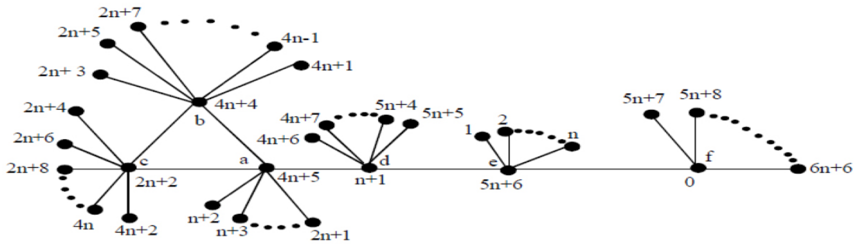

Figure 33

The graceful graph obtained from the graceful graph in figure 32 by applying inverse labeling and attaching n new vertices to f and assigning them with the labels $5n+7, 5n+8, \ldots.. 6n+6$ to them.

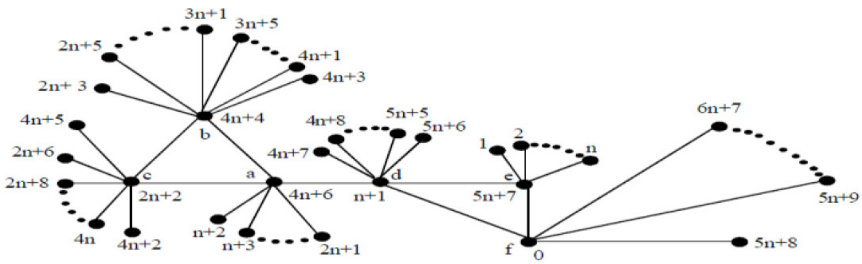

Figure 34

Desired graph with graceful labeling when n is even.

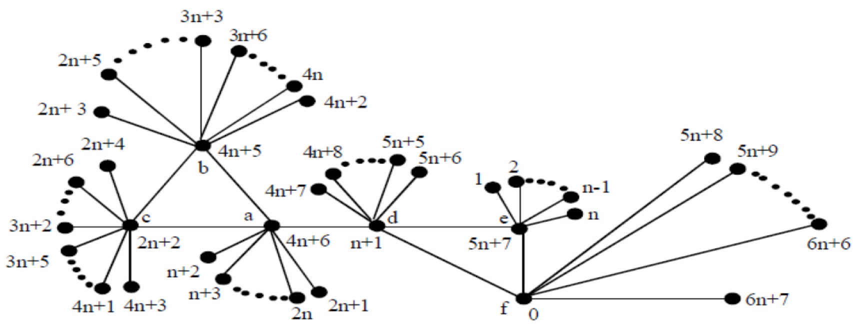

Figure 35

Desired graph with graceful labeling when n is odd.

Example 4.2: Figure 36 given below represents the gracefulness of $C_3 \oplus 3K_{1,n} - C_3 \oplus 3K_{1,n}$.

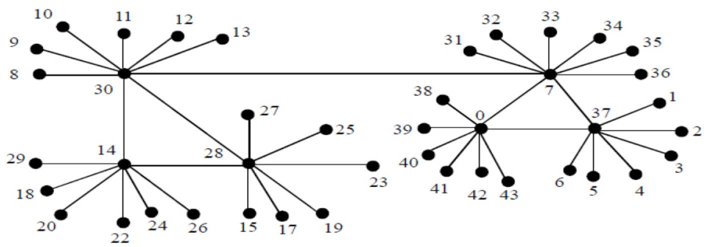

Figure 36

The graph $C_3 \oplus 3K_{1,n} - C_3 \oplus 3K_{1,n}$ **with graceful labeling**

5. Discussion and Conclusion

By following the above paper, it is concluded that paths, cycles and caterpillars using various joining and inverse transformation are graceful. One may further extend this paper by attaching caterpillars to more than two end vertices of a cycle or joining two unicyclic graphs with more than one common edge is graceful. Also one may prove the graceful labeling by joining isomorphic copies of a path with all vertices of a cycle of length more than 3.

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