

Burning number of path and cycle related graphs

M. S. Mary Jenifer *

A. S. Shanthi [†]

Department of Mathematics

Stella Maris College

(Affiliated to University of Madras)

Cathedral Road 600086

Chennai

India

Abstract

In network theory the circulation of data is modelled into a graph and the idea of graph burning is applied. Graph burning is the concept of finding the minimum number of moves that takes for the propagation of data in a network. Even in the cases of memes passed in social media platforms like Facebook and Instagram how quickly an information reaches its followers and the impact of it which affects the entire network can be determined by finding the burning number of the network. In this paper we have determined the burning number of a graph which has path attached with a graph which has diameter 2, flower network and sunlet graph.

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Keywords: Burning, Cycle, Diameter, Flower network, Sunlet.

1. Introduction

The recent trend which is flourishing across the globe is the study of data which is obtained through different sources. When data is propagated, specifically through social media, the impact person makes relies on the information that is heard or seen and then passed on to others. Therefore, the data is shared with many individuals if the influencer has greater connections. The relationship between the influencer and the connections they have with the individuals can be modelled into a graph. Graph burning allows us to analyze this relationship between them in a network.

* ORCID-ID: 0009-0000-7751-9431

[†] ORCID-ID: 0009-0005-0133-1535

* E-mail: 23dmtf001@stellamariscollege.edu.in (Corresponding Author)

[†] E-mail: shanthi.a.s@stellamariscollege.edu.in

Finding a network's burning number shows how soon the network is impacted or in the case of data spread, how rapidly the information spread has occurred can be seen. In case of an encounter of an infected cell with a host cell, burning number in a network relates the transmission that has spread among the individuals. Thus, a network can be modelled if there are conceptual spread, interactional spread, emotional spread, behavior spread, in computer networks, communication networks, social networks and pathogens spread can be linked with burning number.

Graph burning is a process in which all the nodes are unburned in zero state, and when the process begins, when a node is burnt, it remains in the burned state itself. In step 1 a node is selected, and it is allowed to burn, in step 2 the neighbors of the of the burned nodes are allowed to burn. The process is repeated if there is an existence of vertices in unburned state, one among them is chosen and the procedure is iterated.

Bonato, Janssen, and Roshanbin, [4] gave the definition of burning sequence as for a simple graph G with V number of vertices, E number of edges a burning sequence of G is defined as $(x_1, x_2, x_3 \dots x_k)$ a sequence of vertices of G such that for $1 \leq i \leq k$, the fire spread from x_i will burn only all the nodes within distance $k - i$ from x_i by the end of the k^{th} step. On the other hand, every node $v \in V(G)$ must be either a source of fire or burned from at least one of the sources of fire by the end of the k^{th} step. Initially, Bonato, Janssen, and Roshanbin, [4] determined the burning number for paths, cycles, and certain graph classes. In a similar way, burning number can be extended to cycle related graphs [3] so that the structural properties can be known for real world networks. The burning number of graph G is denoted by $b(G)$. The minimum $b(G)$ demonstrates the rapid spread in G .

2. Literature Review

The first person to initiate the process of Burning a graph was Bonato, Janssen, and Roshanbin, [4]. The bootstrap percolation [1] of graphs and fire-fighter problems [2] served as the model for graph burning visualizations. Song, Qi, and Cao, [7], found burning number by using a new model. Upper and lower bounds were also discussed. To evaluate the performance of non-exact algorithms optimal solutions for graphs burning were given Garcia et al. [6]. A new parameter for the generalized burning number of graphs was given by Yinkui et al. [15]. Rahul et al. [11] proposed three heuristics based on the centrality measure of eigen vector. Since for disconnected graphs ordering the component was crucial an attempt was made to use heuristic measures. These heuristics were tested for vertices more than fifty thousand and advantages of algorithms were implemented. Yasuaki et al. [14] studied the parameterized complexity and they used a different technique to show the parameterization by distance to split graphs. Sumathi et al. [12] proposed an algorithm to calculate the burning number by selecting activators. And by this method they found that there was an improvement made in the process of finding the burning number. Omar and Rohilla, [9] explained how to solve conjecture for several classes of

graphs by developing systematic machinery to be used as test beds. Nilay et al. [10] determined the burning number for Jahangir graphs and thus the burning number for graphs with single chord was determined. Mondal et al. [5] gave the approximation for the k-burning problem and showed the NP-hardness while computing the burning sequence. Burning number for graphs that are equal to three were characterized by Wen li [16].

3. Burning number of vertices in Graph G

Theorem 3.1: *The graph G is a path P_m attached with H at one end with $b(H) = 2$, when $\text{diam}(H) = 2$, $b(G) = \lceil \sqrt{m+2} \rceil$, $\text{diam}(H) = 1$, $b(G) = \lceil \sqrt{m+1} \rceil$.*

Proof: Let one end of the path be a and the other end of the path be b . H is attached to the end b . Let the vertices of the path be labelled as p_i , $1 \leq i \leq m$, and the vertices of H be labelled as h_i , $1 \leq i \leq n-1$, $|V(G)| = m+n-1$. It is shown in Figure 1. Let the vertex that is chosen as source 1 such that $b(H) = 2$ be labelled as h_1 and h_i , $i = 2, \dots, n-1$ be the neighbourhood of h_1 . The first source is chosen at a distance of $\lceil \sqrt{m+2} \rceil - 1$ from the end where H is attached. The neighbouring vertices $\lceil \sqrt{m+2} \rceil + 1$ and $\lceil \sqrt{m+2} \rceil - 1$ get burnt in step 2. The second activator is selected as $p_{\lceil \sqrt{m+2} \rceil + \lceil \sqrt{m+2} \rceil - 1 + \lceil \sqrt{m+2} \rceil - 2 + 1}$ from the end b , the neighbouring nodes of the burnt vertices get burnt in the next step. The process is pursued until all the vertices are burned. We find that h_1 and the vertices of H that are neighbours of P_m get burnt in the $(\lceil \sqrt{m+2} \rceil - 1)^{\text{th}}$ step for all source 1 selected at a distance of 2 from the end b . The other vertices of H get burnt in $\lceil \sqrt{m+2} \rceil^{\text{th}}$ step. Else in the second step the vertices of H get burnt. Thus, we find in $\lceil \sqrt{m+2} \rceil$ steps the entire graph is burnt. $b(P_m) = \lceil \sqrt{m} \rceil$ [4] and $b(H) = 2$. Hence $\lceil \sqrt{m} \rceil \leq b(G) \leq \lceil \sqrt{m} \rceil + 2$. The graph H has vertices of maximum distance 2. As n becomes greater $\lceil \sqrt{m+2} \rceil$ is closer to $\lceil \sqrt{m} \rceil$, hence the proof. Similarly for diameter = 1, $b(G) = \lceil \sqrt{m+1} \rceil$.

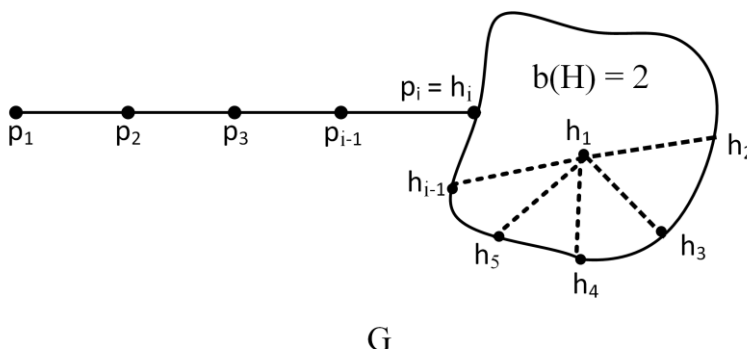


Figure 1
Path attached with H at one end

Theorem 3.2: The graph G is a path P_k attached with H at both ends with $b(H) = 2$, $b(G) = \lceil \sqrt{k+4} \rceil$.

Proof: Let H^1 be attached to one end of the path and H^2 be attached to the other end of the path. Let the vertices of H^1 be labelled as h_i^1 , $2 \leq i \leq n-1$ and let the vertices of H^2 be labelled as h_i^2 , $2 \leq i \leq n-1$. The vertices of the path are labelled as p_i , $1 \leq i \leq k$. h_1^1 and h_2^2 are labelled for the nodes of H^i , $i = 1, 2$ on selection of such a node will have their burning number as two as shown in Figure 2. Let us choose the first initiator from the end where H^1 is attached. The first initiator is chosen at a distance of $\lceil \sqrt{k+4} \rceil - 1$. The neighbouring nodes which are at a distance of $\lceil \sqrt{k+4} \rceil$ and $\lceil \sqrt{k+4} \rceil - 2$ gets burned in step 2. Step 2 ends by choosing $p_{\lceil \sqrt{k+4} \rceil + \lceil \sqrt{k+4} \rceil - 1 + \lceil \sqrt{k+4} \rceil - 2 + 1}$. In the next step the adjacent vertices of the nodes that are already burnt are burned. The process continues until all the vertices are burned. The node h_1^2 is selected as the source in $(\lceil \sqrt{k+4} \rceil - 1)^{\text{st}}$ step such that in step $\lceil \sqrt{k+4} \rceil$ all the unburnt vertices of H^2 are burnt thus obtaining an optimum burning number. $b(P_m) = \lceil \sqrt{m} \rceil$ [4] and $b(H) = 2$. In this graph G , H is attached to both the ends of the path. Hence $\lceil \sqrt{m} \rceil \leq b(G) \leq \lceil \sqrt{m} \rceil + 4$. The graph H has vertices of maximum distance 2. As n tends to a larger number, $\lceil \sqrt{m+4} \rceil$ is closer to $\lceil \sqrt{m} \rceil$.

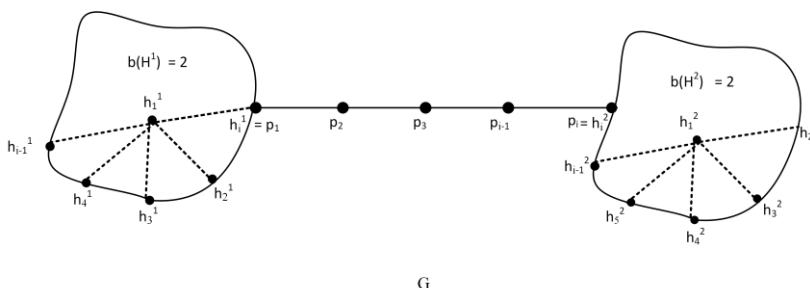


Figure 2

Path attached with H at both the ends

Definition 3.1: When i^{th} copy of K_n is grafted at the i^{th} edge of C_m , flower (C_m, K_n) is obtained, where C_m is the cycle on m vertices and K_n is the complete graph on n vertices. It is denoted by FC_m, K_n [13] is shown in Figure 3.

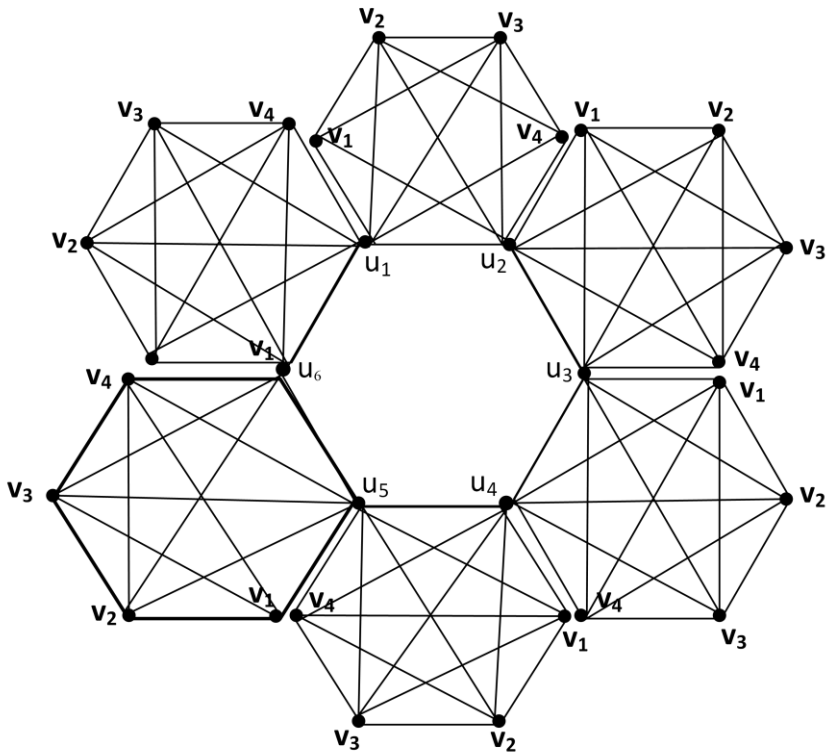


Figure 3
Flower network (C6, K6)

Theorem 3.3: The burning number of flower network F_{C_m, K_n} $b(G) = \lfloor 1 \cdot 5 + \sqrt{n} \rfloor$.

Proof: Let the vertices of the cycle be labelled as u_i , $1 \leq i \leq m$ and the vertices of the complete graph connected to every u_i $1 \leq i, j \leq n-1$ be labelled as v_i , $1 \leq i \leq n-2$. The first source is selected from u_i . Let it be $u_{\lfloor \sqrt{n} \rfloor}$. The vertices that are adjacent to the first source namely $u_{\lfloor \sqrt{n} \rfloor+1}$, $u_{\lfloor \sqrt{n} \rfloor-1}$, v_i $1 \leq i \leq n-2$, vertices of $u_{\lfloor \sqrt{n} \rfloor+1}$ $u_{\lfloor \sqrt{n} \rfloor}$, $u_{\lfloor \sqrt{n} \rfloor-1}$ $u_{\lfloor \sqrt{n} \rfloor}$ gets burned in step 2. Step 2 ends by choosing $u_{(\lfloor \sqrt{n} \rfloor + \lfloor \sqrt{n} \rfloor - 1 + \lfloor \sqrt{n} \rfloor - 2) + 1}$ (for $m \geq 9$) as the activator. The process gets repeated until all the vertices of the graph are burnt. It is noted that certain nodes of the complete graph get burned only in the next step that is $(\lfloor \sqrt{n} \rfloor + 1)^{\text{st}}$ step. Hence the burning number of flower network F_{C_m, K_n} , $b(G) = \lfloor 1 \cdot 5 + \sqrt{n} \rfloor$.

Definition 3.2: The graph obtained by adding pendant edge to each vertex of the cycle C_n is sunlet graph denoted by Gallian [8]. It is shown in Figure 4.

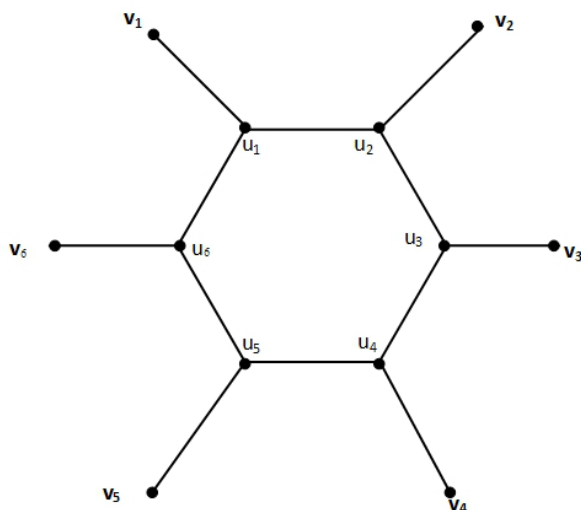


Figure 4
Sunlet Graph L_6

Theorem 3.4: *The burning number of the sunlet graph is*

$$\begin{cases} \lceil \sqrt{n} \rceil & \text{if } n = k^2 + 1 \text{ for some integer } k \geq 2, \\ \lceil \sqrt{n} \rceil + 1 & \text{otherwise} \end{cases}$$

Proof: Let the vertices of the cycle be labelled as u_i $1 \leq i \leq n$ and the pendant vertices be labelled as v_i $1 \leq i \leq n$. Without loss of generality let us choose $n = 5$, the first initiator that is selected be u_3 . In the second step u_2, u_4, v_3 gets burned. By selecting u_5 as source, step 2 ends. In the 3rd step u_1, v_4, v_5 gets burned. Step 3 ends by choosing v_1 as the activator. Hence for all $n \geq 5$ and for $n = k^2 + 1$ for some integer $k \geq 2$, we conclude that, let the first initiator be chosen from u_i , let it be $u_{\lceil \sqrt{n} \rceil}$. In the proceeding steps, by burning the neighbours and by selecting the sources, all u_i, v_i $1 \leq i \leq n$ gets burned in $\lceil \sqrt{n} \rceil$ steps.

Let us choose $n = 6$, In step 1, we choose u_3 as the source. In the second step, u_2, u_4, v_3 gets burned and u_6 is selected to complete step 2. In step 3 $u_1, u_5, v_1, v_2, v_4, v_6$ gets burned. Step 3 ends by choosing v_1 as the source. We find v_4 as the only unburnt vertex that can be burned in step 4. Hence for all $n \geq 6$ and for $n \neq k^2 + 1$, in $\lceil \sqrt{n} \rceil$ steps some vertices of v_i remain unburned. Hence, they are burnt only in $\lceil \sqrt{n} \rceil + 1$ step.

4. Conclusion

Data transmission in social networks affects a larger number of individuals; in certain situations, people's decisions are even influenced by the information posted on social media. Knowing how a particular piece of data spreads across a network in the shortest amount of time is made possible by this paper. The

“burning number” notion will allow us to determine how rapidly a network can be impacted, since it aids in determining the minimum steps in which the spread has occurred. The choice of finding burning number for path and cycle related graphs is effective more specifically in real world network. A researcher will be able to determine the bare minimum of actions required for a disease to spread throughout the network in situations such as these.

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