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An approach to coding theory and the k-Narayana sequence

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Abstract

In this paper, we initially investigate k-Narayana sequence and then define negative indexed k-Narayana sequence. Moreover, we obtain some interesting properties for them. Furthermore, we give a new approach to the coding theory with k-Narayana sequence. Finally, we illustrate the coding steps with two examples and verify the results via software.

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1. Introduction

The Narayana sequence is a third order sequence that is defined by the following recurrence relation [1].

$$N_n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1 \text{ or } n = 2, \\ N_{n-1} + N_{n-3}, & \text{if } n \geq 3. \end{cases} \tag{1}$$

The negative indexed Narayana sequence [2, 3] is defined as below:

$$N_{-n} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n = 2, \\ -N_{-(n-2)} + N_{-(n-3)}, & \text{if } n \geq 3. \end{cases} \tag{2}$$

The Narayana sequence is named after the Indian mathematician Narayana Pandit from the fourteenth century. It is defined as the yearly number of cows

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born, similar to the Fibonacci sequence. For more information, see [2], [3], [4], [5], [6], [7], [8]. The Narayana sequence has many applications in various fields including cryptography, combinatorics, etc., please see the references [9], [10], [14], [15] and [17] therein.

In [2], the explicit formula for the Narayana sequence, often referred to as the Binet representation, is derived based on the roots of the characteristic equation $x^3 - x^2 - 1 = 0$. The sequence is expressed as follows:

$$N_p = \frac{\alpha^{p+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{p+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{p+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{3}$$

where α, β and γ are roots of the equation, given below:

$$\alpha = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \tag{4}$$

$$\beta = \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \tag{5}$$

$$\gamma = \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \tag{6}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp\left(\frac{2\pi i}{3}\right).$$

Note that $\alpha \approx 1.4655$ and $\gamma = \bar{\beta}$ and $|\beta| = |\gamma| < 1$. In [12, 7], the authors show that the following inequality holds for the Narayana sequence;

$$\alpha^{p-2} \leq N_p \leq \alpha^{p-1} \tag{7}$$

where for $p \geq 1$. The properties and characteristics of the Narayana sequence allow us to use it in various fields of science. For example, the sequence has very useful autocorrelation and cross-correlation properties that can be used in cryptographic and key generation applications [6],[7]. Moreover, this sequence has a close connection with some famous numbers, such as Catalan numbers [11]. The Narayana sequence also plays an important role in coding theory that is used to create universal codes [10], [12], [13]. Recently, many researcher is interested in the generalization of the sequence, please see. For instance, the k-Narayana sequence, denoted by $\{N_{k,p}\}_{k=0}^{\infty}$, is defined by the following recurrence relation

$$N_{k,p} = kN_{k,p-1} + N_{k,p-3} \tag{8}$$

with initial conditions $N_{k,0} = 0, N_{k,1} = 1, N_{k,2} = k$, where $p \geq 0$. The first few terms of the k-Narayana sequence are given below: [5]

$$N_{k,0} = 0, \quad N_{k,1} = 1, \quad N_{k,2} = k, N_{k,3} = k^2, \quad N_{k,4} = k^3 + 1,$$
$$N_{k,5} = k^4 + 2k, \quad N_{k,6} = k^5 + 3k^2, \quad N_{k,7} = k^6 + 4k^3 + 1,$$
$$N_{k,8} = k^7 + 5k^4 + 3k, \quad N_{k,9} = k^8 + 6k^5 + 6k^2,$$
$$N_{k,10} = k^9 + 7k^6 + 10k^3 + 1$$

Some special cases for k values are defined in OEIS[16].

$\{N_{2,p}\}_{p=0}^{\infty} = \{0,1,2,4,9,20,44,97,214,472,1041,2296, \ldots\}$, OEIS A008998.

$\{N_{3,p}\}_{p=0}^{\infty} = \{0,1,3,9,28,87,270,838,2601,8073,25057, \ldots\}$, OEIS A052541.

For $p \geq 0$, the k-Narayana sequence can be expressed by Binet's formula, as roots of the characteristic equation $x^3 - kx^2 - 1 = 0$; i.e.;

$$N_{k,p} = \frac{\alpha_k^{p+1}}{(\alpha_k - \beta_k)(\alpha_k - \gamma_k)} + \frac{\beta_k^{p+1}}{(\beta_k - \alpha_k)(\beta_k - \gamma_k)} + \frac{\gamma_k^{p+1}}{(\gamma_k - \beta_k)(\gamma_k - \alpha_k)} \tag{9}$$

In [3, 4], the generating matrix of the k-Narayana sequence is given as below;

$$Q_k = \begin{pmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{10}$$

and, the pth power of the Q_k matrix is,

$$(Q_k)^p = \begin{pmatrix} N_{k,p+1} & N_{k,p-1} & N_{k,p} \\ N_{k,p} & N_{k,p-2} & N_{k,p-1} \\ N_{k,p-1} & N_{k,p-3} & N_{k,p-2} \end{pmatrix} \tag{11}$$

where k is any integer, $k \neq 0$ and $p \geq 3$.

This paper aims to give a new approach to coding theory by exploiting some properties of the generalized k-Narayana numbers, given by ([4]). This approach allows us to establish some linear and combinatorial properties of k-Narayana sequence. The content of this study is organized as follows. Section 2 concerns some properties of the k-Narayana sequence. In Section 3 presents the k-Narayana coding method.

2. Properties of k-Narayana Sequence

In this section, we initially define the negative indexed k-Narayana sequence. Then, we investigate its matrix representation and obtain some spectacular properties of the k-Narayana sequence with negative indexed.

Definition 2.1: For $p \geq 3$

$$N_{k,-p-3} = N_{k,-p} - kN_{k,-p-1} \tag{12}$$

with initial conditions $N_{k,0} = 0$.

The first few terms are given in Table 1.

Table 1
k-Narayana negative indexed.

$N_{k,-1}$	$N_{k,-2}$	$N_{k,-3}$	$N_{k,-4}$	$N_{k,-5}$	$N_{k,-6}$	$N_{k,-7}$	$N_{k,-8}$	$N_{k,-9}$	$N_{k,-10}$
0	1	0	$-k$	1	k^2	$-2k$	$-k^3+1$	$3k^2$	k^4-3k

Theorem 1: *For* $p \geq 1$*;*

$$(Q_k)^{-p} = \begin{pmatrix} N_{k,-p+1} & N_{k,-p-1} & N_{k,-p} \\ N_{k,-p} & N_{k,-p-2} & N_{k,-p-1} \\ N_{k,-p-1} & N_{k,-p-3} & N_{k,-p-2} \end{pmatrix} \tag{13}$$

where k *is any integer,* $k \neq 0$.

Theorem 2: *The following equality holds:*

$$Q_k^p = Q_k^s Q_k^{p-s} \tag{14}$$

where s, p are integers such that $0 < s < p$.

In the following theorem, we give a new property for the k-Narayana sequence. It can be considered as a general form of [4].

Theorem 3: *For* $p \geq 1$*, the following inequality holds:*

$$\alpha_k^{p-2} \leq N_{k,p} \leq \alpha_k^{p-1} \tag{15}$$

where $N_{k,p}$ *is the* k*-Narayana number.*

Proof: Let's prove it by induction. For $p = 1$

$$\alpha_k^{-1} \leq N_{k,1} \leq \alpha_k^0.$$

Suppose that it is true for $p = r$

$$\alpha_k^{r-2} \leq N_{k,r} \leq \alpha_k^{r-1}.$$

Lets prove that it is true for $p = r + 1$. Initially, multiply each side of the inequality by k.

$$\mathrm{k.}\ \alpha_k^{r-2} \leq \mathrm{k.}\, N_{k,r} \leq \mathrm{k.}\, \alpha_k^{r-1} \tag{16}$$

and if we consider $n = r - 2$ at the theorem, we have the following inequality;

$$\alpha_k^{r-4} \leq N_{k,r-2} \leq \alpha_k^{r-3}. \tag{17}$$

Then, by adding the inequalities (16) and (17), we get the equation (18)

$$k\alpha_k^{r-2} + \alpha_k^{r-4} \leq \mathrm{k}N_{k,r} + N_{k,r-2} \leq \mathrm{k}\alpha_k^{r-1} + \alpha_k^{r-3}$$

$$\alpha_k^{r-4}(k \cdot \alpha_k^2 + 1) \leq N_{k,r+1} \leq \alpha_k^{r-3}(\mathrm{k} \cdot \alpha_k^2 + 1)$$

$$\alpha_k^{r-4} \cdot \alpha_k^3 \leq N_{k,r+1} \leq \alpha_k^{r-3} \cdot \alpha_k^3$$

$$\alpha_k^{r-1} \leq N_{k,r+1} \leq \alpha_k^r. \tag{18}$$

Therefore, the proof is completed.

The matrix formulation of these expressions for k-Narayana sequence will also allow us to provide some new identities.

Theorem 4: *The ratio of two consecutive* k*-Narayana numbers converges*

$$\lim_{r\to\infty} \frac{N_{k,r}}{N_{k,r-1}} = \alpha_k \tag{19}$$

where α_k *is the real root of the characteristic equation* $x^3 - kx^2 - 1 = 0$.

Proof: Let's divide each side of the equation $N_{k,r} = kN_{k,r-1} + N_{k,r-3}$ by $N_{k,r-1}$, i.e.

$$\lim_{r\to\infty}\left(\frac{N_{k,r}}{N_{k,r-1}}\right) = k + \lim_{p\to\infty}\left(\frac{N_{k,r-3}}{N_{k,r-1}}\right) \tag{20}$$

If we rewrite the Equation (20), we get

$$\lim r\to\infty\left(\frac{N_{k,r}}{N_{k,r-1}}\right) = k + \lim_{r\to\infty}\left(\frac{N_{k,r-3}}{N_{k,r-2}}\right).\lim_{r\to\infty}\left(\frac{N_{k,r-2}}{N_{k,r-1}}\right)$$

Let be $\lim_{r\to\infty}\left(\frac{N_{k,r}}{N_{k,r-1}}\right) = \alpha_k$, then, we have

$$\alpha_k = k + \frac{1}{\alpha_k}.\frac{1}{\alpha_k}$$

$$\alpha_k^3 - k\alpha_k^2 - 1 = 0.$$

Thus, the theorem is proven.

Corollary 1: *For any integer* s, *we have*

$$\lim_{r\to\infty}\frac{N_{k,r+s}}{N_{k,r}} = \alpha_k^s.$$

Proof: From definition, we have

$$\lim_{r\to\infty}\frac{N_{k,r+s}}{N_{k,r}} = \lim_{n\to\infty}\left(\frac{N_{k,r+s}}{N_{k,r+s-1}}\frac{N_{k,r+s-1}}{N_{k,r+s-2}}\cdots\frac{N_{k,r}}{N_{k,r-1}}\right) = \alpha_k^s.$$

3. k-Narayana Coding Method

In this section, we define a new approach to coding theory with k-Narayana sequence. At this method, we put the message that we want to encrypt, into the $3m\times 3m$ matrix C, by inserting a comma between every two words. We replace the missing matrix elements with the notation "?". This method is applied by dividing the message matrix C into sub-block matrices of 3 × 3. These block matrices are created from left to right as $1 \le i,j \le m^2$. Here Q_k^w is the coding matrix and Q_k^{-w} is the decoding matrix, i.e.;

$$C_{ij}\times Q_k^w = F \tag{21}$$

where

$$F = (f_{ij})_{3\times 3} = \begin{bmatrix} f_{11} & f_{12} & f_{13}\\ f_{21} & f_{22} & f_{23}\\ f_{31} & f_{32} & f_{33}\end{bmatrix}, C = (c_{ij})_{3\times 3} = \begin{bmatrix} c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{bmatrix}.$$

Let's exwress the transformation of the message matrix as $F\times Q_k^{-w} = \mathrm{C}$

$$F = C \times Q_k^w$$
$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \times \begin{bmatrix} N_{k,w+1} & N_{k,w-1} & N_{k,w} \\ N_{k,w} & N_{k,w-2} & N_{k,w-1} \\ N_{k,w-1} & N_{k,w-3} & N_{k,w-2} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}.$$

The determinants of the block matrices are sent along with the message matrix F. If the message is found to be error-free, F is multiplied by the Q_k^{-w} matrix to resolve the message matrix and

$$C_{ij} = F_{ij} \times Q_k^{-w}$$
$$= \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \times \begin{bmatrix} N_{k,-w+1} & N_{k,-w-1} & N_{k,-w} \\ N_{k,-w} & N_{k,-w-2} & N_{k,-w-1} \\ N_{k,-w-1} & N_{k,-w-3} & N_{k,-w-2} \end{bmatrix}$$
$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$$

We create a new alphabet table for any d value, in Table 2.

Table 2
The Alphabet Table with Narayana Sequence

A	B	C	D	E
$1 + N_{\mathrm{d}}$	$2 + N_{\mathrm{d}+1}$	$3 + N_{\mathrm{d}+2}$	$4 + N_{\mathrm{d}+3}$	$5 + N_{\mathrm{d}+4}$
F	G	H	I	J
$6 + N_{\mathrm{d}+5}$	$7 + N_{\mathrm{d}+6}$	$8 + N_{\mathrm{d}+7}$	$9 + N_{\mathrm{d}+8}$	$10 + N_{\mathrm{d}+9}$
K	L	M	N	O
$11 + N_{\mathrm{d}+10}$	$12 + N_{\mathrm{d}+11}$	$13 + N_{\mathrm{d}+12}$	$14 + N_{\mathrm{d}+13}$	$15 + N_{\mathrm{d}+14}$
P	Q	R	S	T
$16 + N_{\mathrm{d}+15}$	$17 + N_{\mathrm{d}+16}$	$18 + N_{\mathrm{d}+17}$	$19 + N_{\mathrm{d}+18}$	$20 + N_{\mathrm{d}+19}$
U	V	W	X	Y
$21 + N_{\mathrm{d}+20}$	$22 + N_{\mathrm{d}+21}$	$23 + N_{\mathrm{d}+22}$	$24 + N_{\mathrm{d}+23}$	$25 + N_{\mathrm{d}+24}$
Z	0	9	?	,
$26 + N_{\mathrm{d}+25}$	$27 + N_{\mathrm{d}+26}$	$28 + N_{\mathrm{d}+27}$	$29 + N_{\mathrm{d}+28}$	$30 + N_{d+29}$

Coding Algorithm

- Step 1. Divide the matrix C into blocks $C_{\mathrm{t}} = \left(c_{ij}\right)_{3\times 3}$
- Step 2. Choose "d" and create message matrix from Table 2.
- Step 3. Choose "k" , "w" and create Q_k^w.
- Step 4. Compute $C \times Q_k^w = F$ and write $detC_i, "d, k, w"$ in the first column of matrix F which is called F^*.
- Step 5. End the algorithm.

Decoding Algorithm

- Step 1. Compute Q_k^w.
- Step 2. Compute $\mathrm{C} = F \times (Q_k^w)^{-1}$. If $detC = detF$, there is no error. Otherwise, apply error correction steps.
- Step 3. Create the text with the help of the alphabet table.
- Step 4. End of algorithm.

Example: Consider message text as "GEOMETRY" and lets apply the coding steps given above.

- Step 1: Let's create the message matrix using the message text:

$$C = \begin{pmatrix} G & E & O \\ M & E & T \\ R & Y & ? \end{pmatrix}$$

- Step 2: Let's create the message matrix for the random value we chose $d = 5$.

For example, let's find $\mathrm{G} = 7 + \mathrm{N}_{11}$. For this, let's first construct the number N_{11} by finding the 11th power of the matrix Q. Note that $N_{11} = a_{13} = 28$.

$$Q^{11} = \begin{pmatrix} 41 & 19 & 28 \\ 28 & 13 & 19 \\ 19 & 9 & 13 \end{pmatrix}$$

G	E	O	M	T	R	Y	?
$7 + N_{11}$	$5 + N_9$	$15 + N_{19}$	$13 + N_{17}$	$20 + N_{24}$	$18 + N_{22}$	$25 + N_{29}$	$29 + N_{33}$
7+28	5+13	15+595	13+277	20+4023	18+1278	25+27201	29+125491
35	18	610	290	4043	1296	27226	125520

$$C = \begin{pmatrix} 35 & 18 & 610 \\ 290 & 18 & 4043 \\ 1296 & 27226 & 125520 \end{pmatrix}$$

- Step 3: for $k = 3, w = 5$

$$Q_3 = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } Q_3^5 = \begin{pmatrix} 270 & 28 & 87 \\ 87 & 9 & 28 \\ 28 & 3 & 9 \end{pmatrix}$$

- Step 4:

$$F = C \times Q_3^5 = \begin{pmatrix} 35 & 18 & 610 \\ 290 & 18 & 4043 \\ 1296 & 27226 & 125520 \end{pmatrix} \begin{pmatrix} 270 & 28 & 87 \\ 87 & 9 & 28 \\ 28 & 3 & 9 \end{pmatrix}$$
$$= \begin{pmatrix} 28096 & 2972 & 9039 \\ 193070 & 20411 & 62121 \\ 6233142 & 657882 & 2004760 \end{pmatrix}$$

We need to send d, k and w values to the receiver in the column, just created for $d = 5,\ k = 3, w = 5$:

$$F^* = \begin{pmatrix} d & 28096 & 2972 & 9039 \\ k & 193070 & 20411 & 62121 \\ p & 6233142 & 657882 & 2004760 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 28096 & 2972 & 9039 \\ 3 & 193070 & 20411 & 62121 \\ 5 & 6233142 & 657882 & 2004760 \end{pmatrix}$$

We must send the determinant of the message matrix to the receiver.

$$detC = \begin{vmatrix} 35 & 18 & 610 \\ 290 & 18 & 4043 \\ 1296 & 27226 & 125520 \end{vmatrix} = 467612494$$
$$detF = \begin{vmatrix} 28096 & 2972 & 9039 \\ 193070 & 20411 & 62121 \\ 6233142 & 657882 & 2004760 \end{vmatrix} = 467612494.$$

Lets apply the decoding steps to encoded matrix F^*, given above.

- Step 1: The receiver first finds the inverse of the matrix Q using $d = 5, k = 3$ and $w = 5$.

$$(Q_3^5)^{-1} = \begin{pmatrix} -3 & 9 & 1 \\ 1 & -6 & 9 \\ 9 & -26 & -6 \end{pmatrix}$$

- Step 2:

$$\mathrm{C} = F \times (Q_3^5)^{-1} = \begin{pmatrix} 28096 & 2972 & 9039 \\ 193070 & 20411 & 62121 \\ 6233142 & 657882 & 2004760 \end{pmatrix} \begin{pmatrix} -3 & 9 & 1 \\ 1 & -6 & 9 \\ 9 & -26 & -6 \end{pmatrix}$$
$$= \begin{pmatrix} 35 & 18 & 610 \\ 290 & 18 & 4043 \\ 1296 & 27226 & 125520 \end{pmatrix}$$

Since $detC = detF$, we understand that the decoding is correct. There is no need for error correction.

- Step 3: Create the text with the help of the alphabet table.

$35 = 7 + N_{11}$	=G
$18 = 5 + N_9$	=E
$610 = 15 + N_{19}$	=O
$290 = 13 + N_{17}$	=M
$20 + N_{24}$	=T
$1296 = 18 + N_{22}$	=R
$27226 = 25 + N_{29}$	=Y
$125520 = 29 + N_{33}$	=?

Example: Let's encode the sentence "THIS SENTENCE IS WRONG" by using the coding steps given above.

- Step 1: Let's divide it into block matrices to create the message matrix using the message text.

$$C=\begin{pmatrix} T & H & I & S & , & \mathrm{S} \\ E & N & \mathrm{T} & E & N & \mathrm{C} \\ E & , & I & \mathrm{S} & , & W \\ R & O & \mathrm{N} & G & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix}$$

$$C_1=\begin{bmatrix} T & H & I \\ E & N & \mathrm{T} \\ E & , & \mathrm{I} \end{bmatrix}, C_2=\begin{bmatrix} S & , & \mathrm{S} \\ E & N & C \\ \mathrm{S} & , & W \end{bmatrix}, C_3=\begin{bmatrix} R & O & \mathrm{N} \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}, C_4=\begin{bmatrix} G & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

- Step 2: Let's create the message matrix for the random value, we chose $d=4$.

$$C_1=\begin{bmatrix} T & H & I \\ E & N & \mathrm{T} \\ E & , & \mathrm{I} \end{bmatrix}$$

$\mathrm{T}=20+\mathrm{N}_{23}=2765, \mathrm{H}=8+\mathrm{N}_{11}=36, \mathrm{E}=5+\mathrm{N}_{8}=14, \mathrm{I}=9+\mathrm{N}_{12}=50, \mathrm{N}=14+\mathrm{N}_{17}=291.$ Let's substitute the values.

$$C_1=\begin{bmatrix} 2765 & 36 & 50 \\ 14 & 420 & 2765 \\ 14 & 125521 & 50 \end{bmatrix}$$

$$C_2=\begin{bmatrix} S & , & \mathrm{S} \\ E & N & C \\ \mathrm{S} & , & W \end{bmatrix}, C_3=\begin{bmatrix} R & O & \mathrm{N} \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}, C_4=\begin{bmatrix} G & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

$$det C_1=\begin{vmatrix} 2765 & 36 & 50 \\ 14 & 420 & 2765 \\ 14 & 125521 & 50 \end{vmatrix}=-959489283165$$

- Step 3: For $k=5, w=7$

$$Q_5=\begin{pmatrix} 5 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } (Q_5)^7=\begin{pmatrix} 81265 & 3200 & 16126 \\ 16126 & 635 & 3200 \\ 3200 & 126 & 635 \end{pmatrix}$$

- Step 4:

$$F_1=C_1\times(Q_5)^7=\begin{pmatrix} 2765 & 36 & 50 \\ 14 & 420 & 2765 \\ 14 & 125521 & 50 \end{pmatrix}\begin{pmatrix} 81265 & 3200 & 16126 \\ 16126 & 635 & 3200 \\ 3200 & 126 & 635 \end{pmatrix}$$

$$=\begin{pmatrix} 225438261 & 8877160 & 44735340 \\ 16758630 & 659890 & 3325539 \\ 2025449356 & 79756935 & 401924714 \end{pmatrix}$$

$$F^* = \begin{pmatrix} -959489283165 & 225438261 & 8877160 & 44735340 & C_2i & C_2i & C_2i \\ detc_2 & 16758630 & 659890 & 3325539 & C_2i & C_2i & C_2i \\ detc_3 & 2025449356 & 79756935 & 401924714 & C_2i & C_2i & C_2i \\ detc_4 & C_3i & C_3i & C_3i & C_4i & C_4i & C_4i \\ d & C_3i & C_3i & C_3i & C_4i & C_4i & C_4i \\ k & C_3i & C_3i & C_3i & C_4i & C_4i & C_4i \\ p & ? & ? & ? & ? & ? & ? \end{pmatrix}$$

Lets apply the decoding steps to encoded matrix F^*, given above.

- Step 1: Obtain the inverse matrix for $k = 5, w = 7$.

$$(Q_5^7)^{-1} = \begin{pmatrix} 25 & -124 & -10 \\ -10 & 75 & -124 \\ -124 & 610 & 75 \end{pmatrix}$$

- Step 2: Compute the matrix multiplication.

$$C_1 = F_1 \times (Q_5^7)^{-1} = \begin{pmatrix} 225438261 & 8877160 & 44735340 \\ 16758630 & 659890 & 3325539 \\ 2025449356 & 79756935 & 401924714 \end{pmatrix} \begin{pmatrix} 25 & -124 & -10 \\ -10 & 75 & -124 \\ -124 & 610 & 75 \end{pmatrix}$$

$$= \begin{pmatrix} 2765 & 36 & 50 \\ 14 & 420 & 2765 \\ 14 & 125521 & 50 \end{pmatrix}$$

- Step 3: Create the text with the help of the alphabet table.

Since $detC_1 = detE_1$, the block matrix message came correctly. $\mathrm{T} = 20 + \mathrm{N}_{23} = 2765$, $\mathrm{H} = 8 + \mathrm{N}_{11} = 36$, $\mathrm{E} = 5 + \mathrm{N}_8 = 14$, $\mathrm{N} = 14 + \mathrm{N}_{17} = 291$, $\mathrm{I} = 9 + \mathrm{N}_{12} = 50$. The C_1 matrix is found. The other block matrices are obtained in a similar way:

$$C_1 = \begin{bmatrix} T & H & I \\ E & N & \mathrm{T} \\ E & , & \mathrm{I} \end{bmatrix}$$

$$C = \begin{pmatrix} T & H & I & ? & ? & ? \\ E & N & T & ? & ? & ? \\ E & , & I & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix}.$$

3.1. *Relationships Between Code Matrix Elements*

We obtain the relationships between the code matrix elements by using the equation $detC = detF$, i.e.;

$$f_{11} = c_{11}N_{k,w+1} + c_{12}N_{k,w} + c_{13}N_{k,w-1} \geq 0, \tag{22}$$
$$f_{12} = c_{11}N_{k,w-1} + c_{12}N_{k,w-2} + c_{13}N_{k,w-3} \geq 0, \tag{23}$$
$$f_{13} = c_{11}N_{k,w} + c_{12}N_{k,w-1} + c_{13}N_{k,w-2} \geq 0, \tag{24}$$
$$f_{21} = c_{21}N_{k,w+1} + c_{22}N_{k,w} + c_{23}N_{k,w-1} \geq 0,$$
$$f_{22} = c_{21}N_{k,w-1} + c_{22N_{k,w-2}} + c_{23}N_{k,w-3} \geq 0,$$
$$f_{23} = c_{21}N_{k,w} + c_{22N_{k,w-1}} + c_{23}N_{k,w-2} \geq 0,$$
$$f_{31} = c_{31}N_{k,w+1} + c_{32}N_{k,w} + c_{33}N_{k,w-1} \geq 0,$$
$$f_{32} = c_{31}N_{k,w-1} + c_{32N_{k,w-2}} + c_{33}N_{k,w-3} \geq 0,$$
$$f_{33} = c_{31}N_{k,w} + c_{32N_{k,w-1}} + c_{33}N_{k,w-2} \geq 0.$$

By exploiting the (22), (23), (24) inequalities and Theorem 15, we get:

$$\alpha_k^{w-1} \leq N_{k,w+1} \leq \alpha_k^{w}$$
$$\alpha_k^{w-2} \leq N_{k,w} \leq \alpha_k^{w-1}$$
$$\alpha_k^{w-3} \leq N_{k,w-1} \leq \alpha_k^{w-2}$$
$$\alpha_k^{w-4} \leq N_{k,w-2} \leq \alpha_k^{w-3}$$
$$\alpha_k^{w-5} \leq N_{k,w-3} \leq \alpha_k^{w-4}$$
$$\frac{\alpha_k^{w-1}}{\alpha_k^{w-3}} \leq \frac{N_{k,w+1}}{N_{k,w-1}} \leq \frac{\alpha_k^{w}}{\alpha_k^{w-2}}$$
$$\frac{\alpha_k^{w-2}}{\alpha_k^{w-4}} \leq \frac{N_{k,w}}{N_{k,w-2}} \leq \frac{\alpha_k^{w-1}}{\alpha_k^{w-3}}$$
$$\frac{\alpha_k^{w-3}}{\alpha_k^{w-5}} \leq \frac{N_{k,w-1}}{N_{k,w-3}} \leq \frac{\alpha_k^{w-2}}{\alpha_k^{w-4}}$$

then, it is

$$\min\left\{\frac{N_{k,w+1}}{N_{k,w-1}}, \frac{N_{k,w}}{N_{k,w-2}}, \frac{N_{k,w-1}}{N_{k,w-3}}\right\} \leq \frac{f_{11}}{f_{12}} \leq \max\left\{\frac{N_{k,w+1}}{N_{k,w-1}}, \frac{N_{k,w}}{N_{k,p-2}}, \frac{N_{k,w-1}}{N_{k,w-3}}\right\},$$

in a similar way;

$$\min\left\{\frac{N_{k,w+1}}{N_{k,w}}, \frac{N_{k,w}}{N_{k,w-1}}, \frac{N_{k,w-1}}{N_{k,w-2}}\right\} \leq \frac{f_{11}}{f_{13}} \leq \max\left\{\frac{N_{k,w+1}}{N_{k,w}}, \frac{N_{k,w}}{N_{k,w-1}}, \frac{N_{k,w-1}}{N_{k,w-2}}\right\},$$
$$\min\left\{\frac{N_{k,w}}{N_{k,w-1}}, \frac{N_{k,w-1}}{N_{k,w-2}}, \frac{N_{k,w-2}}{N_{k,w-3}}\right\} \leq \frac{f_{13}}{f_{12}} \leq \max\left\{\frac{N_{k,w}}{N_{k,w-1}}, \frac{N_{k,w-1}}{N_{k,w-2}}, \frac{N_{k,w-2}}{N_{k,w-3}}\right\}.$$

In this case, it is understood that the elements of the message matrix depend on the determination of the code matrix.

$$\frac{f_{11}}{f_{12}} \approx \alpha_k^2, \frac{f_{13}}{f_{12}} \approx \alpha_k \tag{25}$$
$$\frac{f_{21}}{f_{22}} \approx \alpha_k^2, \frac{f_{23}}{f_{22}} \approx \alpha_k \tag{26}$$
$$\frac{f_{31}}{f_{32}} \approx \alpha_k^2, \frac{f_{33}}{f_{32}} \approx \alpha_k \tag{27}$$

$$\frac{f_{11}}{f_{21}} \approx \frac{f_{21}}{f_{22}} \tag{28}$$

$$\frac{f_{21}}{f_{22}} \approx \frac{f_{31}}{f_{32}} \tag{29}$$

$$\frac{f_{11}}{f_{12}} \approx \frac{f_{31}}{f_{32}} \tag{30}$$

3.2 *Error Detection and Correction*

If $detF = det\mathrm{C}$, it means that the message matrix passed through the communication channel F without error. To fix single error using the equation $detF = detC$:

$$\begin{vmatrix} a & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = f_{13}f_{22}f_{31} + f_{12}f_{23}f_{31} + f_{13}f_{21}f_{32} - af_{23}f_{32} - f_{12}f_{21}f_{33} + af_{22}f_{33} = detC$$

$$\begin{vmatrix} f_{11} & b & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = -f_{13}f_{22}f_{31} + bf_{23}f_{31} + f_{13}f_{21}f_{32} - f_{11}f_{23}f_{32} - bf_{21}f_{33} + f_{11}f_{22}f_{33} = detC$$

By following the similar steps, $a, b, \dots$ are found.

There may be cases where we will get a double error in the message matrix E. For example, we consider a two-variable case for message matrix E, as follows:

$$\begin{vmatrix} a & b & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = -f_{13}f_{22}f_{31} + bf_{23}f_{31} + f_{13}f_{21}f_{32} - af_{23}f_{32} - bf_{21}f_{33} + af_{22}f_{33} = detC$$

Here the possible conditions are $\binom{9}{2} = 36$. Similarly, we get "triple", "quadruple", ..., "nine" errors. The total number of cases for these is as follows:

$$\binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9} = 2^9 - 1$$

The $2^9 - 1 = 511$ situation can be achieved, except that it arrives completely incorrectly. Up to eight errors can be corrected. In this case, it can be said that it has the ability to correct $\frac{510}{511} = 0.998 \approx 99.8\%$.

For example, let's apply error correction to the F matrix. In other words, consider the following matrix;

$$F = \begin{pmatrix} 28096 & 2972 & 9039 \\ 193070 & 20411 & 62121 \\ 6233142 & 657882 & 2004760 \end{pmatrix}.$$

Let us assume that the element $f_{11} = 28096$ is transmitted incorrectly as 1000. Then, the receiver gets the error matrix, as below:

$$F^* = \begin{pmatrix} 1000 & 2972 & 9039 \\ 193070 & 20411 & 62121 \\ 6233142 & 657882 & 2004760 \end{pmatrix}.$$

The determinant of the error matrix is:

$$\det F^* = -1377869002754.$$

Since $\det F^* \neq \det C$, the receiver will notice that there is an error. Using the relationships between the elements of F, it is easy to see that the following equation:

$$\begin{vmatrix} a & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = f_{13}f_{22}f_{31} + f_{12}f_{23}f_{31} + f_{13}f_{21}f_{32} - af_{23}f_{32} - f_{12}f_{21}f_{33} + af_{22}f_{33}.$$

This determinant must be equal to $\det C$. Thus:

$$f_{13}f_{22}f_{31} + f_{12}f_{23}f_{31} + f_{13}f_{21}f_{32} - af_{23}f_{32} - f_{12}f_{21}f_{33} + af_{22}f_{33} = \det C.$$

Then, we have;

$$a(f_{22}f_{33} - f_{23}f_{32}) = \det C - (f_{13}f_{22}f_{31} + f_{12}f_{23}f_{31} + f_{13}f_{21}f_{32} - f_{12}f_{21}f_{33}),$$

$$a = \frac{\det C - (f_{13}f_{22}f_{31} + f_{12}f_{23}f_{31} + f_{13}f_{21}f_{32} - f_{12}f_{21}f_{33})}{f_{22}f_{33} - f_{23}f_{32}}.$$

This formula can be used to calculate "a" if $\det C$ and the f_{ij} values are known. Additionally, we can use the relationship:

$$\frac{f_{11}}{f_{12}} \approx \alpha_k^2.$$

For $k = 3$, we have:

$$\alpha_3 = \frac{\sqrt[3]{2}}{\sqrt[3]{\sqrt{5}+3}} + 1 + \frac{\sqrt[3]{2^2}.\sqrt[3]{\sqrt{5}+3}}{2}.$$

Numerically, $\alpha_3 \approx 3.08$, and $(\alpha_3)^2 \approx 9.45$. Using the approximation:

$$\frac{f_{11}}{f_{12}} \approx \alpha_3^2,$$

we find:

$$\frac{1000}{2972} \approx 9.45 \quad \text{andhence} \quad a \approx 28096.$$

This demonstrates the correction of a single erroneous element in the matrix. In this example, the original F matrix is given as:

$$F = \begin{pmatrix} 28096 & 2972 & 9039 \\ 193070 & 20411 & 62121 \\ 6233142 & 657882 & 2004760 \end{pmatrix}.$$

Here, the elements f_{11} and f_{12} are transmitted incorrectly. Let us denote the incorrect elements as a and b. Then, the error-matrix will be in the following form :

$$F^* = \begin{pmatrix} a & b & 9039 \\ 193070 & 20411 & 62121 \\ 6233142 & 657882 & 2004760 \end{pmatrix}.$$

In this example, the error-entries are $a = 5126$ and $b = 1800$. When the determinant of the error-matrix is calculated, it is evident that $\det F^* \neq \det F$. To correct these errors, the following relationships between the elements of F can be utilized:

$$\frac{f_{11}}{f_{12}} \approx \alpha_k^2, \quad \frac{f_{13}}{f_{12}} \approx \alpha_k,$$
$$\frac{f_{21}}{f_{22}} \approx \alpha_k^2, \quad \frac{f_{23}}{f_{22}} \approx \alpha_k,$$
$$\frac{f_{31}}{f_{32}} \approx \alpha_k^2, \quad \frac{f_{33}}{f_{32}} \approx \alpha_k.$$

For $k = 3$, the value of α_k is calculated as:

$$\alpha_3 = \frac{\sqrt[3]{2}}{\sqrt[3]{\sqrt{5}+3}} + 1 + \frac{\sqrt[3]{2^2}.\sqrt[3]{\sqrt{5}+3}}{2}.$$

Numerically, $\alpha_3 \approx 3.08$, and $(\alpha_3)^2 \approx 9.45$. Using the relationships:

$$\frac{a}{b} \approx \alpha_3^2, \quad \frac{f_{13}}{b} \approx \alpha_3,$$

the values of a and b can be determined. For $f_{13} = 9039$, we have:

$$\frac{a}{b} = \alpha_3^2 a = \alpha_3^2 \cdot b.$$
$$\frac{f_{13}}{b} = \alpha_3 b = \frac{f_{13}}{\alpha_3}.$$

Substituting the value of α_3:

$$b = \frac{9039}{3.08} \approx 2972,$$
$$a = 9.45 \cdot 2972 \approx 28096.$$

Thus, the corrected values of a and b are:

$$a = 28096, \quad b = 2972.$$

After correcting the errors, we verify:

$$\frac{f_{11}}{f_{12}} = \frac{28096}{2972} \approx 9.45 \quad (\alpha_3^2),$$
$$\frac{f_{13}}{f_{12}} = \frac{9039}{2972} \approx 3.08 \quad (\alpha_3).$$

These results confirm the consistency of the relationships and the errors are successfully corrected.

4. Conclusion

From the application of coding theory using the Q matrix of the k-Narayana sequence, we obtain the following results.

1. The error correction ability is approximately 99.80.
2. For arbitrary values d, k and p, the applied coding method has a strong structure.

3. The encoding/decoding with the k-Narayana Q matrix is computed very quickly by the computer. Moreover, the correction and error-detection ability of this method is more efficient than the classical algebraic coding-decoding method.
4. On the basis of this method, matrix elements and their properties are used. In this way, error detection and correction is carried out.

5. Appendix

In this section, the codes of the application example in Python Software is given. The software includes an encrypted message matrix and a solution matrix that will contain nine characters.

```
    from functools import lru_cache
import math
import numpy
from numpy.linalg import matrix_power
charlist =  {"A": 1, "B": 2, "C": 3, "D": 4, "E": 5,
            "F": 6, "G": 7, "H": 8, "I": 9, "J": 10,
            "K": 11, "L": 12, "M": 13, "N": 14, "O": 15,
            "P": 16, "Q": 17, "R": 18, "S": 19, "T": 20,
            "U": 21, "V": 22, "W": 23, "X": 24, "Y": 25,
            "Z": 26, "0": 27, "9": 28, "?": 29, ",": 30,
            }
@lru_cache(None)
def narayana(n: int) -> int:
    if n <= 0:
        return 0
    elif n <= 3:
        return 1
    return narayana(n-1) + narayana(n-3)

message_split = lambda message: message if len(message) == 3 else message_split(message + " ")
split_to_chars = lambda char, charlist: char if char in charlist else "?"
encode = lambda char, charlist, r_constant: charlist[char] + narayana((charlist[char] - 1) +
r_constant)

k_constant = int(input("k Number (Used for encoding in key matrix): "))
n_constant = int(input("n Number (Power of key matrix): "))
r_constant = int(input("r Number (Used for encoding with Narayana Number Series): "))

message = str(input("Message: ")).upper()
message_size = len(message)

final_matrix_dimension = math.ceil(message_size/9)

message_matrix_list = [[[] for _ in range(3)] for _ in range(final_matrix_dimension**2)]

message_splits = []
for i in range(math.ceil(message_size/3)):
    message_splits.append(message_split(message[3*i : 3*(i + 1)]))

Q_matrix = numpy.array([[k_constant, 0, 1], [1, 0, 0], [0, 1, 0]])
encode_Q_matrix = matrix_power(Q_matrix, n_constant)
```

```
decode_Q_matrix = matrix_power(encode_Q_matrix, -1)

message_index = 0
for line in range(final_matrix_dimension):
    for row in range(3):
        for group in range(final_matrix_dimension):
            if message_index < len(message_splits):
                message_matrix_list[group + line*final_matrix_dimension][row] =
                [encode(split_to_chars(char, charlist), charlist, r_constant) for char in
                message_splits[message_index]]
                message_index += 1
                continue
            message_matrix_list[group + line*final_matrix_dimension][row] =
            [encode(split_to_chars("?", charlist), charlist, r_constant) for _ in range(3)]
            message_index += 1

M_matrix = [numpy.array(matrix) for matrix in message_matrix_list]
encoded = []
for matrix in M_matrix:
    encoded.append(numpy.matmul(matrix, encode_Q_matrix))

decoded = []
for matrix in encoded:
    decoded.append(numpy.around(numpy.matmul(matrix, decode_Q_matrix)).astype(int))
    print(matrix, decode_Q_matrix, decoded)
print(f"""Input Text: {message},
    \n\n{n_constant} Powered Q Matrix: \n{encode_Q_matrix},
    \n\n-1 Powered {n_constant} Powered Q Matrix:       \n{decode_Q_matrix},
    \n\nMessage Matrix: \n{M_matrix},
    \n\nEncoded Message Matrix: \n{encoded},
    \n\nDecoded Message Matrix: \n{decoded},
    \n\nDecoded Output Text: {'decoded_message'}
    """)
```

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