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A bipartite graph associated to Hamiltonian degrees of finite groups

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Abstract

In this article, we employ the notion of Hamiltonian degree to connect group theory with graph theory. To do this, we present a new graph associated with a finite group G represented by the symbol $\Gamma_p(G)$. We explore $\Gamma_p(G)$'s structural characteristics, such as its connectedness, planarity, vertex degrees, regularity, and the characteristics of its generated subgraphs.

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1. Introduction

The exploration of relationships between algebraic structures and graph theory has garnered considerable attention in mathematics. Researchers have discovered novel methods to depict and evaluate group-theoretic characteristics by associating graphs with groups using graph-theoretic techniques. These connections improve our comprehension of group theory and offer novel approaches to apply graph theory in algebraic contexts. The first significant work linking these two theories is often attributed to Arthur Cayley. In 1878, he introduced what are now known as "Cayley graphs" [7]. These graphs visually represent the structure of a group by using the group elements as vertices and their relation with the generating set as edges. This pioneering work laid the foundation for the rich interplay between group theory and graph theory that continues to be explored in modern mathematics. In addition to classical developments, more recent studies have expanded the understanding of graph structures and their generalizations. For instance, Reddy and Hemavathi [16] proposed a generalization of bipartite graphs that broadened the traditional concept by considering extended structural properties. Similarly, Swadi and Najim [17] examined the generalized k-connectivity of equally complete bipartite graphs and their line graphs, establishing new results that strengthen the connection between connectivity measures and graph completeness. The exploration of Hamiltonian cycles within Cayley graphs has long been a fascinating topic in mathematical research, attracting the attention of numerous scholars. Among the notable contributions, Dobson et al. [10] delved into automorphism groups that possess cyclic commutator subgroups. They uncovered specific conditions under which the associated Cayley graphs exhibit Hamiltonian cycles, shedding light on the deep connections between group properties and the structural features of these graphs. Their findings have become a cornerstone for understanding how algebraic characteristics influence graph behavior.

Building on this foundation, Jones and Morris [13] expanded the scope of this research by focusing on normal Cayley graphs. Their work offered a detailed exploration of the structural factors necessary for Hamiltonicity, providing a clearer picture of the interplay between group symmetry and graph connectivity. Meanwhile, Morris and Witte [15] took a different approach by studying Cayley graphs of smaller order. They provided concrete examples and conducted an in-depth analysis of their properties, creating an important link between specific cases and broader theoretical conditions.

Adding to these advancements, Fitzgerald and Morris [12] turned their attention to Hamiltonian paths within Cayley graphs generated by alternating groups. Their research uncovered intriguing relationships between the structural aspects of alternating groups and the presence of Hamiltonian paths and cycles in their associated graphs. This work has further enriched our understanding of how group-theoretic intricacies can shape the behavior of Cayley graphs. Recently, Al-Kaseasbeh and Coykendall [1] explored adjacency-like conditions in induced ideal graphs, revealing how algebraic structures can influence graph properties. Anderson and Al-Kaseasbeh [4] discussed intersection subgroup graphs, highlighting the connections between subgroup interactions and graph theory. Das, Saha, and Al-Kaseasbeh, [8] examined co-maximal subgroup graphs, providing insights into how subgroup relationships can be represented graphically. Additionally, Al-Kaseasbeh and Erfanian [2] investigated bipartite graphs associated with elements and cosets of finite groups, demonstrating the utility of graph theory in understanding group coset structures. Furthermore, Al-Kaseasbeh and Erfanian [3] analyzed the structure of Cayley graphs of dihedral groups with the application of graph theory to specific group types.

In this paper, we investigate an original connection between group theory and graph theory by concentrating on the notion of Hamiltonian degree, which characterizes the probability that a finite group is Hamiltonian. We create a graph $\Gamma_P(G)$ associated with a finite group G. To simplify this investigation, we focus on the graph $\Gamma_P(G)$, where the vertices and edges encode information about the group's structure, particularly about Hamiltonian properties. Our goal is to explore the structural features of $\Gamma_P(G)$, such as vertex degrees, connectivity, planarity, and specific induced subgraphs. By analyzing these aspects, we aim to uncover how the algebraic characteristics of G influence the graph's geometric and topological properties. This investigation not only deepens the understanding of the interplay between groups and graphs but also lays the groundwork for future research into the graphical representation of algebraic probabilities.

The paper is structured as follows. In Section 2, we introduce the key concepts and notations required for our study, covering essential definitions related to graphs, Hamiltonian cycles, and the algebraic structures underlying our work. This section establishes a clear foundation for the discussions in the subsequent sections. Section 3 presents the main contribution of our research: a new graph structure associated with the Hamiltonian degree of finite groups. We describe the construction of the graph, its motivation, and its relevance. Additionally, we explore the interplay between the algebraic properties of finite groups and graph-theoretic concepts, illustrating the meaningful connections that emerge between these two areas.

2. Preliminaries

This section introduces the fundamental concepts from graph theory and group theory that form the basis of our work. Readers unfamiliar with some of these topics may consult standard references such as [9, 6, 5] for further background.

Definition 2.1: ([9]). A graph $\Gamma = (V, E)$ consists of a non-empty set V of vertices and a set E of edges, where each edge is an unordered pair of distinct vertices. A graph is said to be simple if it contains no loops or multiple edges. In this paper, we consider only simple undirected graphs.

Definition 2.2: ([9]). A graph Γ is called connected if there exists a path between every pair of vertices. Otherwise, the graph is disconnected.

Definition 2.3: ([9]). The girth of a graph is defined as the length of its shortest cycle. If the graph has no cycles, its girth is said to be infinite.

Definition 2.4: ([9]). For a vertex v in a graph Γ, the degree of v, denoted $\deg(v)$, is the number of edges incident to v. In the context of directed graphs, one can also define the in-degree and out-degree, depending on the direction of edges.

Definition 2.5: ([9]). A graph Γ is said to be planar if it can be drawn on a plane such that its edges intersect only at their endpoints.

Theorem 2.6: (Kuratowskis Theorem, [14]). *A graph is planar if and only if it does not contain a subgraph that is a subdivision of either K_5 (the complete graph on five vertices) or $K_{3,3}$ (the complete bipartite graph with three vertices in each part).*

Definition 2.7: ([9]). The diameter of a connected graph is the greatest distance between any two vertices, where distance is defined as the length of the shortest path connecting them.

It follows that a graph with diameter one must be a complete graph. The complete graph on n vertices is denoted K_n.

Definition 2.8: A graph $\Gamma = (V, E)$ is bipartite if the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to a vertex in V_2. No edges exist between vertices within the same part.

Each part in a bipartite graph is an *independent set*, meaning that no two vertices within it are adjacent. The size of the largest independent set in a graph is called its *independence number*. A *complete bipartite graph*, denoted $K_{n,m}$, is one where every vertex in V_1 is connected to every vertex in V_2. If either $n = 1$ or $m = 1$, the graph is referred to as a *star graph*. It is well known that a graph is bipartite if and only if it contains no cycle of odd length.

We now recall a group-theoretic concept that will play a key role in this paper.

Definition 2.9: ([11]). Let G be a finite group, and let $L(G)$ denote the set of all subgroups of G. The Hamiltonian degree of G, also referred to as the probability that an element commutes with a subgroup, is defined by:

$$P(G) = \frac{|\{(x,H) \in G \times L(G) \mid xH = Hx\}|}{|G||L(G)|}.$$

It is easy to see that $P(G) = 1$ if and only if every subgroup of G is normal. Groups with this property are called *Dedekind groups*, and the non-abelian ones are often referred to as *Hamiltonian groups*. Hence, the function $P(G)$ provides a numerical measure of how close G is to being Hamiltonian.

Example 1 ([11]).

- The quaternion group Q_8 is Hamiltonian, so $P(Q_8) = 1$.
- In contrast, S_4 and A_4 contain non-normal subgroups, so both satisfy $P(G) < 1$.
- or the symmetric group S_3, which has 3 normal subgroups out of 6, we compute:
$$P(S_3) = \frac{2}{3}.$$
- For the alternating group A_4, the Hamiltonian degree is:
$$P(A_4) = \frac{1}{2}.$$
- The dihedral group D_8, with several normal subgroups, satisfies:
$$P(D_8) = \frac{4}{5}.$$

For dihedral groups of order $2n$, the Hamiltonian degree satisfies $P(D_{2n}) < 1$, unless the group is Hamiltonian. For a particular class of dihedral groups where n is a power of an odd prime, an explicit formula for $P(G)$ has been given.

Theorem 2.10: ([11]). *Let p be an odd prime and $n \geq 1$. Then the dihedral group D_{2p^n} satisfies:*
$$P(D_{2p^n}) = \frac{2n+2}{n+2+p+p^2+\cdots+p^n}.$$

3. A graph associated to Hamiltonian degree

In this section, we employ the notion of Hamiltonian degree to connect group theory with graph theory. To do this, we present a new graph associated to a finite group G and represented by the symbol $\Gamma_P(G)$. We explore $\Gamma_P(G)$'s structural characteristics, such as its connectedness, planarity, vertex degrees, and the characteristics of its generated subgraphs. By studying these characteristics, we hope to shed light on how group-theoretic and graph-theoretic structures interact and propose an original perspective on the study of finite groups using graphical representations.

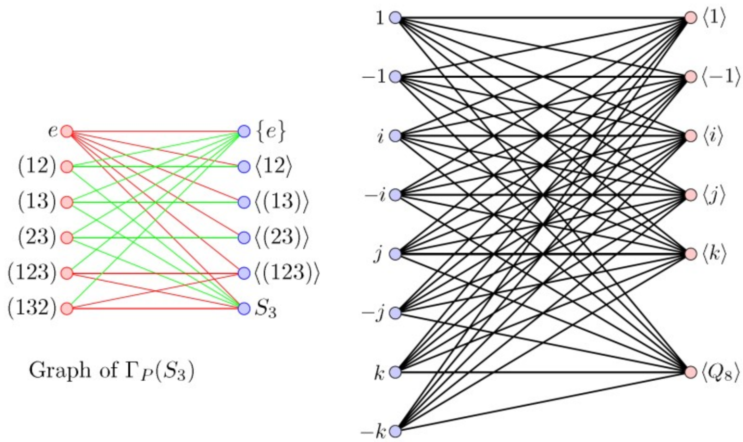
Definition 3.1: Let G be a finite group with $L(G)$ its set of subgroups. We define the bipartite graph $\Gamma_P(G)$ of G with $V(\Gamma_P(G)) = G \cup L(G)$ and $E(\Gamma_P(G))$ the set of edges defined as follows: A vertex x from G is connected by an edge to a vertex H from $L(G)$ if and only if $xH = Hx$, i.e., $x \in N_G(H)$.

It is clear that $\Gamma_P(G)$ is a simple and undirected graph. Moreover, for any group G, $|V(\Gamma_P(G))| \geq 2$ and $|E(\Gamma_P(G))| \geq 1$. Note that if $G = \{e\}$ is the trivial group, then $V(\Gamma_P(G)) = \{e\} \cup \{\{e\}\}$ and so $|V(\Gamma_P(G))| = 2$ and $E(\Gamma_P(G))| = 1$. If G is not trivial, then obviously $|V(\Gamma_P(G))| \geq 4$ and $E(\Gamma_P(G)| \geq 4$.

Note that the identity element is adjacent to all subgroups in $L(G)$ and the two subgroups $\{e\}$ and G are adjacent to all elements in G. The following lemma gives a relation between the $P(G)$ and the number of edges $|E(\Gamma_P(G)|$. The proof follows from definitions 2.9 and 3.1 directly.

Lemma 3.2: *Let G be a finite group with graph $\Gamma_P(G)$. Then the total number of edges in $\Gamma_P(G)$ is equal to $|G||L(G)|P(G)$.*

Example 2: The total number of edges in $\Gamma_P(S_3)$, $\Gamma_P(D_8)$, $\Gamma_P(Q_8)$, and $\Gamma_P(A_4)$ are equal to 24, 64, 48, and 60 respectively. (See Figure 1 and Figure 2.)



Graph of $\Gamma_P(S_3)$

Graph of $\Gamma_P(Q_8) = K_{8,6}$

Figure 1

Graph of $\Gamma_P(Q_8) = K_{8,6}$ Graph representations of $\Gamma_P(S_3)$ and $\Gamma_P(Q_8) = K_{8,6}$.

Corollary 3.3: *Let p be an odd prime and D_{2p^n} the Dihedral group with $2p^n$ elements. Then the total number of edges in $\Gamma_P(D_{2p^n})$ is equal to $(2n+2)2p^n$.*

Proof: The proof follows from Theorem 2.10 and Lemma 3.2

Corollary 3.4: *Let G be a Dedekind or Hamiltonian finite group. Then the total number of edges in $\Gamma_P(G)$ is equal to: $|G||L(G)|$, and so $\Gamma_P(G)$ is the complete bipartite graph $K_{|G|,|L(G)|}$.*

Lemma 3.5: *Let G be a finite group. Then the following statements hold.*

(i) *For $x \in G$, the degree of x is given by $deg(x) = P(x,G)|L(G)|$, where*

$$P(x,G) = \frac{|\{H \in L(G) | xH = Hx\}|}{|L(G)|}.$$

(ii) *For $H \in L(G)$, the degree of H is given by $deg(H) = |N_G(H)|$.*

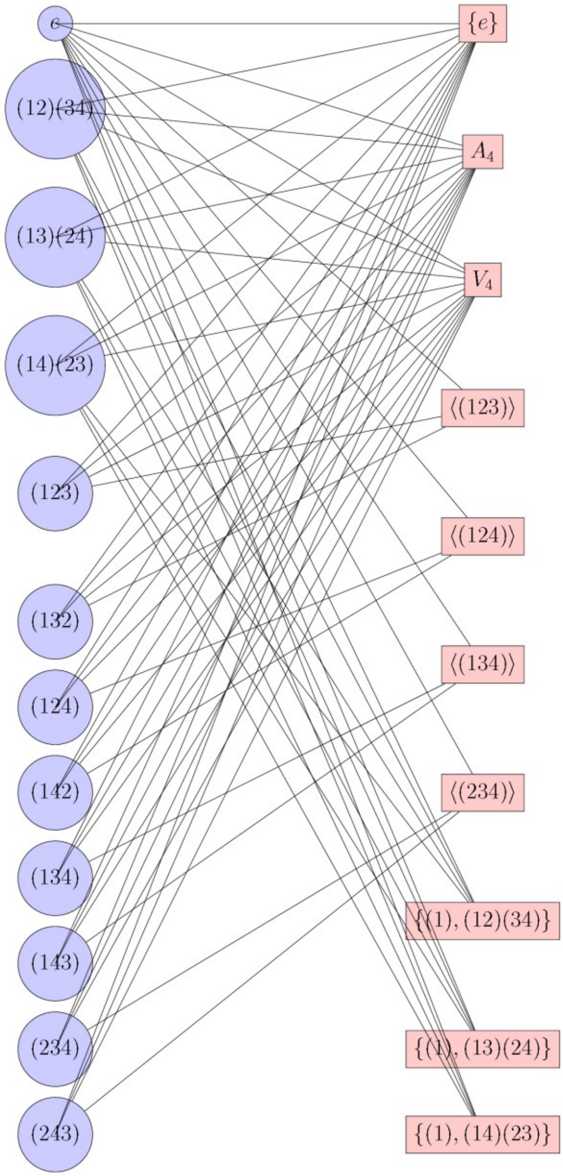


Figure 2
Bipartite graph for A_4

Proof:

(i) For $x \in G$, we have $\deg(x) = |\{H \in L(G) | xH = Hx\}|$. Since the number of such H depends on the function $P(x, G)$, which counts the proportion of elements in G that commute with x, and the size of $L(G)$, it follow that $\deg(x) = P(x,G)|L(G)|$.

(ii) For $H \in L(G)$, the degree $\deg(H)$ corresponds to the number of elements $x \in G$ such that $xH = Hx$. Equivalently, this is the number of elements $x \in G$ satisfying $xH = Hx$. This condition defines the normalizer of H in G, denoted as $N_G(H)$. Hence $\deg(H) = |N_G(H)|$.

Lemma 3.6: *Let G be a finite group. Then $\Gamma_P(G)$ has no isolated vertices. Moreover, if G is not the trivial group, then $\Gamma_P(G)$ has no pendant vertices.*

Proof: Let $x \in G$. Then x is adjacent to the trivial subgroup, so $deg(x) \geq 1$. Similarly, if $H \in L(G)$, then H is adjacent to the identity element e. Hence, $deg(x) \neq 0$ and $deg(H) \neq 0$ for every $x \in G$ and $H \in L(G)$ which implies that $\Gamma_P(G)$ has no isolated vertices. If $G \neq \{e\}$, then there exist $e \neq x \in G$. Therefore, $\{e, x\} \subseteq G$ and $\{\{e\}, G\} \subseteq L(G)$. Now if $x \in G$ and $H \in L(G)$ are arbitrary vertices, then x is adjacent to at least $\{e\}$ and G. Hence, $deg(x) \geq 2$. If $H = \{e\}$, then H is adjacent to at least e and x. If $H \neq \{e\}$, then $|H| \geq 2$ and so we have $deg(H) = |N_G(H)| \geq |H| \geq 2$. Therefore, there is no pendant vertex in $\Gamma_P(G)$.

As a consequence of lemma 3.6, we can see that if G is a non-trivial group then $\Gamma(G)$ cannot be a tree, because every tree has at least two pendant vertices. Moreover, $\Gamma_P(G)$ is a star graph if and only if G is the trivial group.

In the following lemma, we determine the girth of the graph $\Gamma_P(G)$ and an upper bound for the diameter of G. Recall that the girth of $\Gamma_P(G)$ is the length of the shortest cycle in the graph, and the diameter is the greatest distance between any two vertices in the graph. The girth and the diameter of $\Gamma_P(G)$ are denoted by $girth(\Gamma_P(G))$ and $diam(\Gamma_P(G))$, respectively.

Lemma 3.7: *If G is a non-trivial group, then $girth(\Gamma_P(G)) = 4$ and $diam(\Gamma_P(G)) \leq 3$.*

Proof: Assume that G is a non-trivial group with identity element e. So, we have $|G| \geq 2$ and $|L(G)| \geq 2$. Let $x \in G$ such that $x \neq e$, and $\{e\}, G \in |L(G)|$, then we have the following cycle of length 4:

$$\mathrm{e} \_\_\_\_ \mathrm{G} \_\_\_\_ x \_\_\_\_ \{e\} \_\_\_\_ e$$

Since $\Gamma_P(G)$ has no cycle of length 3, then the above cycle is the smallest. Therefore, $girth(\Gamma_P(G)) = 4$. To prove $diam(\Gamma_P(G)) \leq 3$, we consider the following cases:

Case 1: Let $x, y \in G$. In this case, we have a path

$$x \_\_\_\_ G \_\_\_\_ y$$

. Hence, $d(x, y) = 2 < 3$.

Case 2: Let $H, K \in L(G)$. In this case, we have a path

$$H \_\_\_\_\_ e \_\_\_\_\_ K$$

. Hence, $d(H,K) = 2 < 3$.

Case 3: For any $x \in G$ and $H \in L(G)$ we may consider the following path

$$x \_\_\_\_\_ \{e\} \_\_\_\_\_ e \_\_\_\_\_ H$$

Thus, $d(x,H) = 3$. It follows that the greatest distance between any two vertices in the graph $\Gamma_P(G)$ is at most 3.

Corollary 3.8: *For trivial group $\{e\}$, $diam(\Gamma_P(\{e\})) = 1$. If G is non-trivial simple and abelian, then $diam(\Gamma_P(G)) = 2$.*

Example 3: Consider the group $G = \mathbb{Z}_3$, the cyclic group of order 3. The set of subgroups of G is $L(G) = \{\{0\}, \mathbb{Z}_3\}$. The graph $\Gamma_P(\mathbb{Z}_3)$ has vertices $V(\Gamma_P(\mathbb{Z}_3)) = \mathbb{Z}_3 \cup L(G) = \{0,1,2\} \cup \{\{0\}, \mathbb{Z}_3\}$. (See Figure 3)

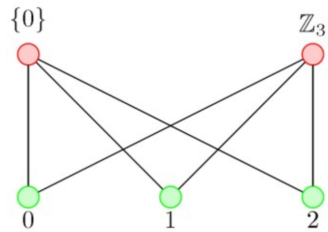


Figure 3
The diameter of $\Gamma_P(\mathbb{Z}_3)$ is 2.

In the following lemma, we show that $\Gamma_P(G)$ contains a complete bipartite subgroup $K_{|G|,|L_n(G)|}$, where $L_n(G)$ is the set of all normal subgroups of G.

Lemma 3.9: *Let G be a finite group. Then $\Gamma_P(G)$ has an induced bipartite subgraph $K_{|G|,|L_n(G)|}$.*

Proof: Let $V(\Gamma_P(G))$ denote the vertex set of $\Gamma_P(G)$, which consists of the elements of G and the subgroups of G. Put $V_1 = G$ and $V_2 = L_n(G)$, then for every $x \in G$ and $H \in L_n(G)$, we have $xH = Hx$. Since H is a normal subgroup, it follows that every element of G is adjacent to every normal subgroup in $L_n(G)$. So, the subgraph induced by $V_1 \cup V_2$ is a complete bipartite graph $K_{|V_1|,|V_2|}$, where $|V_1| = |G|$ and $|V_2| = |L_n(G)|$. Thus, $\Gamma_P(G)$ contains an induced subgraph that is isomorphic to $K_{|G|,|L_n(G)|}$.

Lemma 3.10: *Let G be a finite group*

- *If G is a Dedekind group, then $\Gamma_P(G)$ has a cycle of length $2t$, where $t = min\{|G|, |L(G)|\}$.*
- *If G is not a Dedekind group, then $\Gamma_P(G)$ has a cycle of length $2t + 2$, where $t = min\{|G|, |L_n(G)|\}$.*

Proof: Let $\{e\} \neq H$ be a subgroup of G. Lemma 3.9 asserts that the graph $\Gamma_P(G)$ has an induced bipartite subgraph $K_{|G|,|L_n(G)|}$. This bipartite structure guarantees the existence of a cycle of length $2|L_n(G)|$ or $2|G|$, as each vertex in one part can connect to all vertices in the other part.

(i) If G is a Dedekind group (i.e., every subgroup of G is normal), then $|L_n(G)| = |L(G)|$. This means the induced bipartite subgraph is $K_{|G|,|L(G)|}$, and the cycle length becomes $2t$ where $t = \min\{|G|, |L(G)|\}$.

(ii) If G is not a Dedekind group (i.e., not all subgroups are normal), then we still have an induced subgraph $K_{|G|,|L_n(G)|}$. However, since not all subgroups are normal, we have a non-normal subgroup H of G. Now, we can extend the cycle of length $2t$, where $t = min\{|G|, L_n(G)\}$ to the cycle of length $2t + 2$ as the following:

Take elements e and h in H, then we have a path

$$e\_\_\_H\_\_\_h\_\_\_\{e\}.$$

If we replace an edge between e and $\{e\}$ in the cycle of the length $2t$ by the above path, then we will get a cycle of length $2t + 2$ as required.

Lemma 3.11: *Let G be a finite group. Then the independence number of the graph $\Gamma_P(G)$ satisfies:*

$$\alpha(\Gamma_P(G)) \geq \max\{|G|, |L(G)|\}.$$

Proof: Recall that the independence number $\alpha(\Gamma_P(G))$ is the size of the largest subset of vertices in which no two vertices are connected by an edge.

First, consider the subset of vertices corresponding to the group elements, i.e., $G \subset V(\Gamma_P(G))$. In the construction of $\Gamma_P(G)$, edges are only formed between elements of G and subgroups in $L(G)$. Thus, there are no edges connecting any two distinct elements of G, making the entire set G an independent set of size $|G|$.

Similarly, take the set of all subgroups $L(G)$. Since adjacency is also defined only between subgroups and elements not between subgroups themselves this set forms another independent set of size $|L(G)|$.

Hence, $\Gamma_P(G)$ contains at least two independent sets of sizes $|G|$ and $|L(G)|$. The largest of these determines a lower bound for the independence number, so we conclude:

$$\alpha(\Gamma_P(G)) \geq \max\{|G|, |L(G)|\}.$$

We conjecture that this inequality is, in fact, an equality. While several examples support this claim and no counterexample has been found, a general proof remains elusive.

Next, we investigate the planarity of the graph $\Gamma_P(G)$.

Lemma 3.12: *If G is a finite group that is not simple, then the graph $\Gamma_P(G)$ is non-planar.*

Proof: Suppose G is not simple, so it contains a nontrivial proper normal subgroup H with $H \neq \{e\}$. Let us choose elements $x \in H\backslash\{e\}$ and $y \in G\backslash H$. Then consider the subset of vertices $\{e, x, y\} \cup \{\{e\}, H, G\}$. Within $\Gamma_P(G)$, each element in $\{e, x, y\}$ is adjacent to each subgroup in the second set, due to the normality conditions or subgroup relationships.

This configuration yields a subgraph isomorphic to the complete bipartite graph $K_{3,3}$, which is known to be non-planar by Kuratowskiâ€™s theorem. Consequently, $\Gamma_P(G)$ cannot be drawn in the plane without edge crossings, and hence it is not planar.

Lemma 3.13: *Let G be a simple group.*

(i) *If G is abelian, then $\Gamma_P(G)$ is planar.*

(ii) *If G is not abelian and H is a non-normal subgroup with $|H| \geq 3$, then $\Gamma_P(G)$ is not planar.*

Proof:

(i) If G is abelian and simple, then G must be a group of order prime number p. In this case, we have $V(\Gamma_P(G) = \{e, x_2, x_3, \cdots, x_p\} \cup \{\{e\}, G\}$, and so $\Gamma_P(G)$ is $K_{2,p}$ which is planar.

(ii) If $|H| \geq 3$, then there are at least three element e, x, y in H. Hence, the subgraph induced to set $\{e, x, y\} \cup \{\{e\}, H, G\}$ is $K_{3,3}$. Therefore, $\Gamma_P(G)$ is not planar.

It is interesting to see whenever $\Gamma_P(G)$ is outerplanar, remember that any outerplanar graph is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing. Outerplanar graphs are characterized by Wagner's Theorem, which states that a graph is outerplanar if it has not K_4 and $K_{2,3}$ as a subgraph or subdivision of the graph. The following theorem determines $\Gamma_P(G)$ when it is an outerplanar.

Theorem 3.14: $\Gamma_P(G)$ is outerplanar if and only if $|G| = 1$ or 2.

Proof: If $|G| = 1$ or 2, then $\Gamma_P(G) = K_{1,1}$ or $K_{2,2}$ and so it is outerplanar. Now, assume that $\Gamma_P(G)$ is outerplanar. It is clear that $|L(G)| \geq 2$, if $|G| \neq 1$. Thus, $\{e\}$ and G are adjacent to any element in G. If $|G| \geq 3$, then we have induced subgraph $K_{2,3}$ and so $\Gamma_P(G)$ is not outerplanar . Hence, $|G| \leq 2$ and the proof is completed.

Finally, let us deal with the regularity of $\Gamma_P(G)$. It seems that we do not have many groups G such that $\Gamma_P(G)$ is regular. Remember that a graph is $k-regular$ if the degree of every vertex is k.

Lemma 3.15: *If $\Gamma_P(G)$ is a regular graph then $|G| = |L(G)|$.*

Proof: It is clear that $deg(e) = |L(G)|$ and $deg(G) = |G|$. Since $\Gamma_P(G)$ is regular, we should have $deg(e) = deg(G)$, which implies that $|G| = |L(G)|$.

Proposition 3.16:

(i) Let G be a group of prime order p. Then $\Gamma_P(G)$ is regular if and only if $p = 2$.

(ii) $\Gamma_P(D_{2p})$ is not regular.

(iii) $\Gamma_P(D_{2p^2})$ is not regular.

Proof:

(i) It is clear that $|G| = p \neq |\{\{e\}, G\}| = |L(G)|$, whenever, $p \neq 2$ and hence, $\Gamma_P(G)$ is not regular. If $p = 2$, then we have $G = \{e, x\}$ and $L(G) = \{\{e\}, G\}$ and $\Gamma_P(G)$ is $K_{2,2}$ which is a $2-regular$ graph.

(ii) We know that $|D_{2p}| = 2p$ and $|L(D_{2p})| = p + 3$. So, $|D_{2p}| \neq |L(D_{2p})|$ when $p \neq 3$. Thus $\Gamma_P(D_{2p})$ is not regular whenever $p \neq 3$. Now, if $p = 3$, then $D_{2p} = D_6 \cong S_3$, and it is obvious that $\Gamma_P(S_3)$ is not regular. Note that $deg(e) = 6$, but $deg((12)) = 3$.

(iii) Similar to the second part, we can see that $|D_{2p^2}| = 2p^2$ and $|L(D_{2p^2})| = p^2 + p + 4$ are not equal, because the equation $2p^2 = p^2 + p + 4$ has no prime number solution. Consequently, $\Gamma_P(D_{2p^2})$ is not regular.

Corollary 3.17: *For $n \geq 2$, the graph $\Gamma_P(D_{2n})$ is not regular.*

Proof: Assume that $D_{2n} = \langle a, b | a^n = b^2 = e, bab = a^{-1} \rangle$. So, we have $D_{2n} = \{e, a, a^2, a^3, \cdots, a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b\}$. Consider two subgroups $\{e\}$ and $H = < b >$ in $L(D_{2n})$. It is clear that $deg(\{e\}) = |D_{2n}| = 2n$. We know that H is not a normal subgroup of D_{2n} and so $N(H) < D_{2n}$. Therefore, $deg(H) = N(H) \neq 2n$. Hence, $\Gamma_P(D_{2n})$ can not be regular, for $n \geq 3$. If $n = 2$, then D_4 is an abelian non-cyclic group of order 4. We have $|D_4| = 4$ and $|L(D_4)| = 5$ which implies that $\Gamma_P(D_4)$ is not regular, by Lemma 3.13.

Finally, we state the following conjecture about the regularity of $\Gamma_P(G)$ as:

Conjecture: $\Gamma_P(G)$ is regular if and only if $|G| = 1$ or 2.

It is obvious that:

- If $|G| = 1$, then $\Gamma_P(G) \cong K_{1,1}$ is 1-regular.
- If $|G| = 2$, then $\Gamma_P(G) \cong K_{2,2}$ is 2-regular.

Conversely, assume that $\Gamma_P(G)$ is regular, then we have the following cases:

Case 1: *G* is simple and abelian

In this case, the group G is a cyclic group of order a prime number p, and $\Gamma_P(G) \cong K_{2,|G|}$. So, $\Gamma_P(G)$ is regular if $|G| = 2$.

Case 2: G is simple and non-abelian

Since G is non-abelian and simple, so there exists a non-normal subgroup H of G.

Thus, $\deg(\{e\}) = |G| \neq |N_G(H)| = \deg(H)$, hence $\Gamma_P(G)$ is not regular.

Case 3: G is not simple

- (a) If there exists a non-normal subgroup H of G, then similar to Case 2, we have $\Gamma_P(G)$ is not regular.
- (b) If all subgroups of G are normal, then G is Dedekind or Hamiltonian groups. In this case, $P(G) = 1$ and it implies that $\Gamma_P(G) \cong K_{|G|,|L(G)|}$. So, the graph is regular if and only if $|G| = |L(G)|$. Thus, the proof of the conjecture turns out to be the following problem:

Problem: Let G be a Dedekind or Hamiltonian group. Then $|G| \neq |L(G)|$.

In spite of the fact that we believe the problem is true, but we have not been able to prove or disprove it as yet.

4. Conclusion

The paper connected group theory and graph theory by defining the graph $\Gamma_P(G)$, where the vertices represent elements and subgroups of G, and the number of edges is determined by the $P(G)$ relation. Several properties of this graph were explored, and illustrative examples were provided. These findings offer theoretical insights and practical tools for analyzing complex group behaviors.

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References

[1] S. Al-Kaseasbeh and J. Coykendall, "Adjacency-like conditions and induced ideal graphs," *Communications in Algebra*, vol. 51, pp. 3708-3728 (2023).

[2] S. Al-Kaseasbeh and A. Erfanian, "A bipartite graph associated to elements and cosets of subgroups of a finite group," *AIMS Mathematics*, vol. 6, no. 10, pp. 10395-10404 (2021).

[3] S. Al-Kaseasbeh and A. Erfanian, "The structure of Cayley graphs of dihedral groups of valencies 1, 2 and 3," *Proyecciones* (Antofagasta), vol. 40, pp. 1683-1691 (2021).

[4] D. Anderson and S. Al-Kaseasbeh, "The intersection subgroup graph of a group," *Communications in Algebra,* vol. 51, pp. 3556-3573 (2023).

[5] C. Berge, Graphs and Hypergraphs. Amsterdam, The Netherlands: North-Holland (1973).

[6] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications. New York, NY, USA: Elsevier (1976).

[7] A. Cayley, "The theory of groups: Graphical representation," *American Journal of Mathematics*, vol. 1, no. 2, pp. 174-176 (1878), doi: 10.2307/2369306.

[8] A. Das, M. Saha, and S. Al-Kaseasbeh, "On co-maximal subgroup graph of a group," *Ricerche di Matematica,* vol. 73, pp. 2075-2089 (2024), doi: 10.1007/s11587-022-00718-0.

[9] R. Diestel, Graph Theory, 5th ed. Berlin, Germany: Springer (2017).

[10] E. Dobson, H. Gavlas, J. Morris, and D. Witte, "Automorphism groups with cyclic commutator subgroup and Hamilton cycles," *Discrete Mathematics*, vol. 189, no. 1-3, pp. 69-78 (Jul. 1998), doi: 10.1016/S0012-365X(98)00024-4.

[11] A. Erfanian, M. Al Tahan, and S. Al-Kaseasbeh, "Hamiltonian degree of finite groups," *International Journal of Group Theory,* vol. 15, no. 4, pp. 193-206 (2026).

[12] R. W. Fitzgerald and J. Morris, "Hamiltonian paths in Cayley graphs on alternating groups," *Journal of Combinatorial Theory,* Series B, vol. 97, no. 5, pp. 717-727 (2007), doi: 10.1016/j.jctb.2006.11.004.

[13] G. A. Jones and J. Morris, "Hamiltonian cycles in normal Cayley graphs," *Graphs and Combinatorics*, vol. 25, pp. 389-404 (2009), doi: 10.1007/s00373-009-0835-4.

[14] K. Kuratowski, "Sur le problme des courbes gauches en topologie," *Fundamenta Mathematicae*, vol. 15, pp. 271-283 (1930).

[15] J. Morris and D. Witte, "Hamiltonian cycles in Cayley graphs of small order," *Ars Mathematica Contemporanea*, vol. 1, pp. 42-56 (2008), doi: 10.26493/1855-3974.41.17e.

[16] P. Siva Kota Reddy and P. S. Hemavathi, "Generalization of bipartite graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 23, no. 3, pp. 787-793 (2020), doi: 10.1080/09720529.2019.1701269.

[17] S. A. Swadi and A. A. Najim, "The generalized k-connectivity of equally complete bipartite graphs and their line graphs," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 27, no. 5, pp. 1567-1573 (2024), doi: 10.47974/JDMSC-1939.

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