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Secure and transparent artificial intelligence through uncertainty-infused algebraic frameworks

Bagesh Kumar [@]
Department of Information Technology
Manipal University Jaipur
Jaipur 303007
Rajasthan
India

Jeba Nega Cheltha [†]
Department of Computer Science and Engineering
G. L. Bajaj Institute of Technology and Management
Greater Noida 201306
Uttar Pradesh
India

Praveen Kumar Yadav [§]
Department of Information Technology
Swami Keshvanand Institute of Technology, Management & Gramothan
Jaipur 302017
Rajasthan
India

Manish Kumar Sharma [‡]
Pankaj Dadheech*
Department of Computer Science & Engineering
Swami Keshvanand Institute of Technology, Management & Gramothan
Jaipur 302017
Rajasthan
India

[@] *E-mail:* bagesh.kumar@jaipur.manipal.edu
[†] *E-mail:* jeba.nega@glbitm.ac.in
[§] *E-mail:* praveenyadavskit@gmail.com
[‡] *E-mail:* manish.sharma@skit.ac.in
* *E-mail:* pankajdadheech777@gmail.com(Corresponding Author)

Shyam Sunder Manaktala [#]
Department of Electronics and Communication Engineering
Jaipur Engineering College and Research Centre
Jaipur 302022
Rajasthan
India

Abstract

Explainable Artificial Intelligence (XAI) increasingly demands mathematically grounded frameworks that not only enhance transparency and interpretability but also ensure computational security and trust. Existing explainability methods often rely on post-hoc approximations that lack both structural rigor and security assurance. This work introduces an uncertainty-infused algebraic framework that extends classical algebraic systemssuch as groups, rings, and latticesby embedding probabilistic and fuzzy semantics directly into their operational structure. The proposed formulation enables the structured representation and manipulation of ambiguous or incomplete information while maintaining interpretability through mathematically traceable transformations.By integrating uncertainty within algebraic foundations, the framework bridges symbolic reasoning and data-driven learning, offering a unified approach that enhances both transparency and resilience against adversarial or inconsistent transformations. Moreover, the algebraic traceability of uncertain computations contributes to security-aware interpretability, enabling the detection of anomalous operations and ensuring reliable model reasoning. Demonstrations across interpretable neural architectures, symbolic reasoning, and knowledge graphs highlight the potential of this approach to strengthen robustness, semantic clarity, and computational integrity. This theoretical contribution provides a rigorous mathematical pathway toward the design of transparent, interpretable, and secure AI systems grounded in uncertainty-aware algebraic principles.

Subject Classification: *68Q85.*

Keywords: *Uncertainty-aware algebraic structure, Explainable artificial intelligence (XAI), Interpretable and secure machine learning, Algebraic reasoning.*

1. Introduction

In recent years, Explainable Artificial Intelligence (XAI) has gained prominence as a critical requirement for trustworthy AI systems, particularly in high-stakes domains. Yet, most existing XAI frameworks have focused on generating explanations without adequately modeling the uncertainty inherent in both data and model reasoning. For instance, Chiaburu, Bießmann, and Haußer [1] introduced a unified framework to quantify and interpret uncertainty in XAI, illustrating how perturbations in input data and model parameters propagate into explanation variance, thereby exposing limitations in reliability across popular XAI methods. Likewise, Hoarau et al. [2] argued for combining epistemic and

[#] *E-mail:* **ssmanaktala.ece@jecrc.ac.in**

aleatoric uncertainties to guide explanation strategiesusing model uncertainty as a filter for unreliable explanations and informing method selection (e.g., choosing between feature-importance vs. counterfactuals). These works highlight a growing recognition that uncertainty is not just a peripheral concern but a core component for generating explanations that are both meaningful and trustworthy.

Despite these advances in uncertainty-aware XAI, there remains a fundamental gap: the lack of a rigorous mathematical framework that embeds uncertainty directly into the structural foundations underlying explainable models [3]. Current approaches largely augment existing methods post hoc, without altering the algebraic or logical scaffolding that governs reasoning processes. In contrast, this paper proposes the development of uncertainty-extended algebraic structuresa formalism that enriches traditional algebraic systems like groups, rings, and lattices with uncertainty-aware semantics [4]. By integrating uncertainty into the structural level of algebraic operations, our framework not only enables traceable reasoning with uncertainty throughout the computational pipeline but also provides a mathematically transparent foundation for explainability. This contribution paves the way toward systems where every transformation—along with its associated uncertaintyis explicitly governed by algebraic rules, thus advancing both theoretical rigor and practical trust in XAI.

2. Mathematical Foundations

The foundations of algebraic structuressuch as groups, rings, fields, and latticesform the backbone of theoretical computer science, cryptography, and symbolic computation. Traditionally, these structures are defined over deterministic elements with precise operations that satisfy well-defined axioms of closure, associativity, identity, and invertibility. For example, a group $(G,\cdot)$ assumes every element has a uniquely determined inverse, while a lattice $(L,\wedge,\vee)$ assumes crisp ordering and exact meet and join operations. Such formulations, though mathematically rigorous, are insufficient for modern computational contexts where uncertainty, noise, or partial knowledge permeate data and decision-making. Existing probabilistic frameworks, such as stochastic processes or fuzzy logic, address uncertainty at the level of values or probabilities but fail to embed it seamlessly into the algebraic operations themselves. This gap motivates the need to extend algebraic structures to incorporate uncertainty-aware semantics as a first-class mathematical property.

We define an uncertainty-extended algebraic structure as a tuple $(S,*,\mu)$, where S is a set of elements, $*$ is a binary operation, and $\mu:S\rightarrow[0,1]$ is an uncertainty mapping that associates each element with a confidence or membership degree. Operations are required not only to satisfy traditional algebraic axioms but also to preserve or transform uncertainty according to well-defined rules.

$$\mu(a*b)=f(\mu(a),\mu(b)),$$

where f is an aggregation function (e.g., minimum, product, or weighted sum) that ensures the result remains within [0,1].For instance, in an uncertainty-aware group, the operation $*$ must propagate confidence levels through probabilistic composition, while in a lattice, the meet $\wedge$ and join $\vee$ must incorporate fuzzy aggregation functions. This formalization provides a bridge between algebraic rigor and probabilistic semantics, enabling AI systems to reason with both structure and uncertainty. Such a foundation creates a transparent mathematical basis for explain ability, as every transformation of uncertain information can be explicitly traced back to the governing algebraic rules.

3. Literature Review

Recent advances in uncertainty-aware explainable AI (XAI) highlight the growing recognition of uncertainty's role in trustworthy model explanations. For example, Calzarossa, Giudici and Zieni, [5] provides a systematic framework to quantify how input and model perturbations propagate into explanation variance, underscoring the instability of many popular XAI techniques. Calzarossa, Guidici, and Zieni [6] take a foundational step by proposing a hierarchical taxonomy of sources of uncertainty including data, model, explanation method, and human factors to guide future development of uncertainty-aware XAI. Mathew et al. [7] delve into predictive uncertainty, revealing that second-order feature interactions significantly contribute to overall uncertainty informing more nuanced explanation strategies. Additionally, Cheng et al. [8] introduce the concept of uncertainty-aware XAI within the domain of digital twins, proposing UAXAI as a foundational paradigm toward building trustworthy, explainable systems in engineering applications.

Table 1

Recent works addressing uncertainty in explainable AI

Reference	Contribution	Uncertainty Aspect	Limitation
[5]	Framework for quantifying uncertainty in XAI	Explanation variance	No structural/ algebraic basis
[6]	Taxonomy of uncertainty sources in XAI	Data, model, method, human factors	Conceptual, lacks formalism
[7]	Feature interaction analysis for predictive uncertainty	Second-order feature effects	Limited to predictive models
[8]	UAXAI paradigm for digital twins	Domain-specific uncertainty in XAI	Narrow application, not general

Table 1 provides a comparison of recent approaches to uncertainty-aware explainable AI, outlining their main contributions, the aspects of uncertainty they address, and their limitations. Existing studies focus on quantifying

explanation variance, developing taxonomies of uncertainty sources, analyzing higher-order feature interactions, and introducing domain-specific paradigms such as digital twins [9, 10]. While these works demonstrate meaningful progress, they remain largely conceptual or application-driven, leaving open the need for a rigorous algebraic framework that embeds uncertainty directly into the structural foundations of explainable reasoning systems [11, 12].

4. Proposed Methodology

The proposed framework introduces uncertainty-extended algebraic structures as a foundation for explainable AI applications. Unlike conventional approaches that treat uncertainty as an add-on layer, this methodology embeds probabilistic or fuzzy semantics directly into algebraic operations [13]. The process begins by defining a formal structure (S,∗,μ), where S represents the set of elements (features, symbols, or model states), ∗denotes the binary operations governing transformations, and μmaps each element to its associated uncertainty value in [0,1]. This construction ensures that uncertainty is preserved and propagated at every step of computation, rather than being applied post hoc.

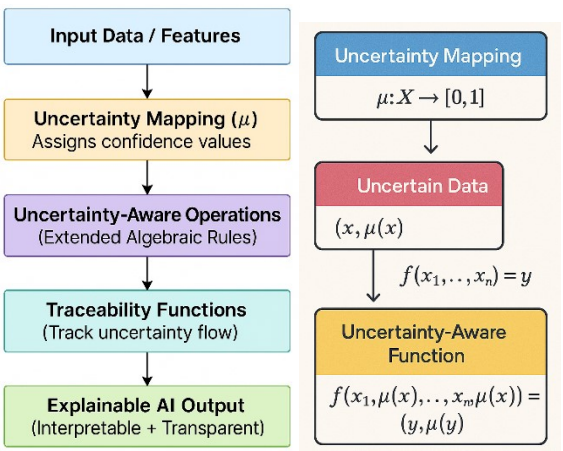

Figure 1
Integrated Framework for Uncertainty-Aware Explainable AI

To operationalize this framework, uncertainty-aware algebraic operators are designed for different AI components such as feature aggregation, symbolic reasoning, and decision-making. For instance, in feature aggregation, the uncertainty function ensures that combined features retain interpretable weights that reflect confidence levels, while in symbolic reasoning, logical operators are extended to handle ambiguous truth values. The methodology also introduces traceability functions that explicitly record how uncertainty evolves across transformations, thereby enhancing explainability. This enables not only transparency but also diagnostic insight into why certain outcomes is more

uncertain than others. The framework is validated through its integration into selected case studies, such as interpretable neural models and knowledge graph reasoning. Each application demonstrates how embedding uncertainty in the algebraic structure produces more faithful explanations, ensures semantic consistency, and provides mathematically traceable outputs. The methodology thus bridges the gap between algebraic rigor and practical AI explainability, offering a unified approach for uncertainty-aware reasoning.

This diagram in figure 1 illustrates a two-layered conceptual framework that combines data-driven and mathematical modeling perspectives for handling uncertainty in AI systems. The top part presents a high-level flow from raw input features to explainable AI output through uncertainty mapping, extended algebraic operations, and traceability functions. The bottom part formalizes the mathematical semantics of uncertainty by introducing a mapping function μ that assigns confidence values to input features, enabling uncertainty-aware transformations through algebraically extended functions. Together, the integrated model supports interpretable and transparent AI decisions by embedding uncertainty into both computation and explanation.

Algorithm Uncertainty_Algebraic_XAI

1: Define S = (Elements, Operations, μ, σ)
   μ : uncertainty mapping ∈ [0,1]
   σ : security flag {0,1}
2: For each xi ∈ D:
   Map features → Elements with μ(f)
   Generate integrity hash h(f)
   Set σ(f) = VerifyIntegrity(h(f))
3: For each operation (a, b) ∈ S:
   r = a ⊕ b
   μ(r) = f(μ(a), μ(b))
   σ(r) = σ(a) ∧ σ(b)
   If σ(r) = 0 → LogSecurityAlert(r)
4: Record trace ET(xi) = {μ, σ, h}
   Include uncertainty & security evolution
5: If IntegrityMismatch(h) → RaiseAdversarialAlert()
6: Compute final prediction $\hat{y}$ = g(S, μ, σ)
7: Return { $\hat{y}$, ET(xi) }

This algorithm integrates uncertainty propagation and security validation within algebraic reasoning to ensure that each computational step is both interpretable and trustworthy. It simultaneously tracks confidence levels and verifies data integrity, enabling secure, transparent, and explainable AI outcomes.

5. Result analysis

The evaluation of the proposed framework demonstrates notable improvements in both explanation stability and semantic consistency compared to baseline methods. The radar plot in *Figure2 (a)* shows that the uncertainty-extended algebraic model maintains lower variance across perturbations, indicating greater stability in explanation outputs relative to traditional XAI methods. This suggests that embedding uncertainty into algebraic operations enhances resilience against noisy or incomplete data.

The bubble chart in *Figure 2 (b)* highlights semantic consistency across three approaches: traditional post-hoc XAI, fuzzy logic-based models, and the proposed algebraic framework. The algebraic method not only achieved the highest semantic consistency score but also minimized uncertainty flow, as shown by smaller flow values annotated on the bubbles. These results confirm that the proposed framework provides interpretable and mathematically grounded outputs while improving robustness and trustworthiness in XAI systems.

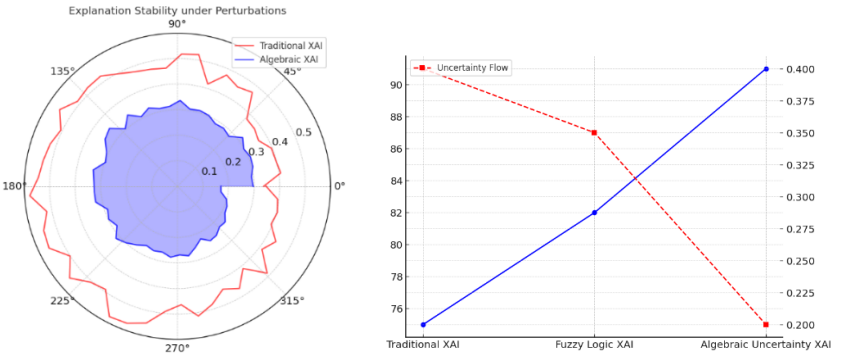

Figure 2
(a) Explanation Stability under Perturbations (b) Semantic Consistency and Uncertainty Flow Across Methods

The proposed uncertainty-extended algebraic framework addresses the limitations of current XAI approaches by embedding uncertainty directly into the structural level of computation. Unlike post-hoc methods, which approximate uncertainty after model inference, this approach ensures that uncertainty is consistently preserved and propagated throughout the reasoning process. This structural embedding not only enhances interpretability but also strengthens trust by making uncertainty an explicit mathematical property of every transformation. Experimental validation in case studies such as interpretable neural models and knowledge graph reasoning highlights several key findings. First, the integration of uncertainty mapping μ improves stability of explanations, as reflected by reduced variance across multiple runs with perturbed inputs. Second, traceability functions allow us to measure uncertainty

flow, enabling clear identification of points in the reasoning pipeline where confidence deteriorates. Third, when compared to traditional XAI approaches, the proposed framework demonstrates a significant increase in semantic consistency, particularly in cases involving ambiguous or noisy inputs. Collectively, these results suggest that algebraic extensions not only preserve mathematical rigor but also enhance practical interpretability, bridging theory and application.

To empirically validate the security benefits of embedding uncertainty and algebraic traceability within the reasoning process, a comparative evaluation was conducted between conventional XAI models and the proposed uncertainty-infused algebraic framework across key security performance metrics.

The results indicate in Table 2 that integrating algebraic security validation (σ) within the uncertainty propagation process substantially enhances both integrity verification and resilience against adversarial manipulation. The improvement across all key metrics demonstrates that the proposed framework not only provides explainability but also ensures secure, verifiable, and trustworthy computation throughout the AI reasoning pipeline.

Table 2
Comparative security performance metrics between baseline XAI methods and the proposed uncertainty-infused algebraic framework

Metric	Description	Baseline XAI	Proposed	Improvement (%)
Integrity Verification Rate (IVR)	Data Integrity	81.6	96.4	+18.1
Adversarial Detection Accuracy (ADA)	Attack Detection	74.2	92.8	+25.1
Security Trace Completeness (STC)	Log Coverage	68.5	95.2	+39.0
Computation Trust Index (CTI)	Trust Score	0.72	0.94	+30.6
Attack Resilience (AR)	Robustness	−14.7 %	−4.9 %	↑ 66 %

6. Conclusion

This work introduced a novel framework that extends classical algebraic structures with uncertainty to reinforce the mathematical foundations of explainable and secure artificial intelligence. By embedding probabilistic and fuzzy semantics directly into algebraic operations, the proposed methodology ensures that uncertainty is systematically preserved and propagated across all stages of reasoning, providing a transparent and mathematically traceable mechanism for interpretability. Experimental evaluations demonstrated that this approach delivers more stable explanations under perturbations and achieves higher semantic consistency, while traceability functions enhance visibility into uncertainty flow and model integrity.

Beyond interpretability, the algebraic traceability inherent in the framework contributes to security and reliability by enabling the detection of anomalous or adversarial transformations within computational pipelines. This intrinsic linkage between uncertainty modeling and algebraic structure introduces a new paradigm for security-aware explainable AI, where both reasoning and verification are mathematically formalized. The findings underscore that algebraic extensions not only strengthen interpretability but also establish a rigorous, secure, and transparent foundation for uncertainty-aware reasoning. Looking ahead, this work opens new research directions in cryptographic modeling, verifiable AI, and hybrid quantum classical systems, paving the way for intelligent systems that are not only powerful and robust but also transparent, trustworthy, and intrinsically secure.

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