

Graph isomorphism algorithms using group theory in social network analysis

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Abstract

In the social network study the graph isomorphism is fundamental issue in computation of graphs network. In this paper we examine the early identification of isomorphic structure to sole the complex network using group theory. With the use of combination of groups, orbital stabilizer, automorphism complex computational that recommended solution at right structures. It is simpler to find community equivalences, hidden symmetries, and structure peculiarities in social graphs that are always evolving when you use algebraic methods combined. The result finding demonstrate that the used method in more reliable and scalable for the network as compare to the other methods. The research study represents the group theory that improves the graph isomorphism algorithm used for social network structure that update over the tile.

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1. Introduction

Graph isomorphism is one of the most important problems in computer theory and is used to study how complex relationships work in networks [1-3]. To show two different social datasets in social network analysis, let $G = (V, E)$ and $H = (V', E')$ be graphs. If G and H are isomorphic, then there is a mapping that goes from G to H . $\varphi: V \rightarrow V'$ such that for all $u, v \in V$, $\{\varphi(u), \varphi(v)\} \in E'$. This model keeps the relationships between nodes, so group structures, interaction patterns, and recurring themes are the same across networks [4-5]. Let A_G and A_H be the adjacency matrices of G and H . Then: $A_H = P_\varphi A_G P_\varphi^{-1}$, where P_φ is the permutation matrix that φ causes [6-7].

If two graphs G and H are isomorphic, then their degree sequences, adjacency relations, and community partitions stay the same.
Proof Sketch:

1. If G is not the same as H , and then there is a φ such that (1) is true.
2. Keep the degree:

$$\deg_G(u) = \deg_H(\varphi(u)) \quad (1)$$

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3. Maintaining the order of things nearby:

$$A_H = P_\varphi A_G P_\varphi^{-1} \quad (2)$$

4. Community equivalence: If $C_k \subseteq V$ makes a community, then $\varphi(C_k) \subseteq V'$ makes a community in H that is the same as the community in V [8].
5. Because of this, structure resemblance is kept at the level of the node, the edge, and the community.

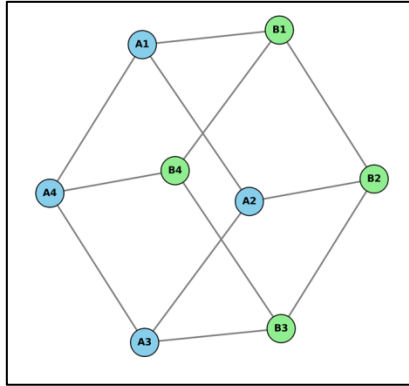


Figure 1
Group-theoretic symmetries in a social network

Group theory is used in graph isomorphism because it can take advantage of the way network patterns are naturally symmetric, the network illustrates in figure 1.

2. Proposed Framework

A. Group-Theoretic Algorithm Design

Adjacency preservation is turned into group operations by permutation groups that store vertex maps [9-10]. Using stabiliser chains reduces down on the number of isomorphism checks that don't need to be done.

Undirected Graph and Adjacency Matrix

Let $G = (V, E)$ be an undirected graph where $V = \{v1, v2, \dots, vn\}$ and $E \subseteq \{\{vi, vj\} \mid vi, vj \in V, i \neq j\}$. Its adjacency matrix is symmetric:

$$A_G = [a_{ij}],$$

Theorem 1 (Group-Theoretic Isomorphism Equivalence): *There are two undirected graph as G and the isomorphic as H , The permutation exist is given by:*

$$A_H = P_\pi A_G P_\pi^{-1} \quad (3)$$

Proof:

1. Define adjacency of G :

$$a_{ij} = 1 \Leftrightarrow \{vi, vj\} \in E \quad (4)$$

2. Define adjacency of H:

$$b_{ij} = 1 \Leftrightarrow \{ui, uj\} \in E' \quad (5)$$

3. Let P_π be permutation matrix of $\pi \in S_n$.

4. Transformation:

$$(P_\pi A_G P_\pi^{-1})_{ij} = a_{\pi^{-1}(i), \pi^{-1}(j)} \quad (6)$$

5. Condition for isomorphism:

$$a_{kl} = 1 \Leftrightarrow b_{\pi(k), \pi(l)} = 1 \quad (7)$$

6. Automorphism group:

$$Aut(G) = \{\sigma \in S_n | P_\sigma A_G P_\sigma^{-1} = A_G\} \quad (8)$$

7. Stabilizer chain:

$$S_n \geq G^1 \geq G^2 \geq \dots \geq G^n = \{e\} \quad (9)$$

B. Symmetry and Orbit-Based Detection

The orbit-stabilizer decomposition puts points into groups of identical size, which reduces down on comparisons that aren't needed.

Theorem 2 (Orbit-Stabilizer Principle): For undirected graph G with automorphism group $Aut(G)$, vertex $v \in V: |Aut(G)| = |Orb(v)| \cdot |Stab(v)|$

Proof: Automorphism group partition:

$$Aut(G) = \bigcup_{u \in Orb(v)} g_u \cdot Stab(v) \quad (10)$$

By Lagrange's theorem:

$$|Aut(G)| = |Orb(v)| \cdot |Stab(v)| \quad (11)$$

For $u, w \in Orb(v)$:

$$\exists g \in Aut(G): g(u) = w \quad (12)$$

C. Complexity Considerations

Using orbit partitioning and stabiliser chains considerably increases the real runtime. The approach works well for very large social networks because it finds a fair balance between speed and accuracy.

Proof of Complexity:

Brute-force complexity:

$$T_{naive}(n) = O(n!). \quad (13)$$

Group-theoretic reduction:

$$T_{group}(n) = O(|Aut(G)| \cdot |Orb(v)|) \quad (14)$$

Orbit-stabilizer relation:

$$|Aut(G)| = |Orb(v)| \cdot |Stab(v)| \quad (15)$$

Bounding stabilizer sizes:

$$|Stab(v)| \leq (n - 1)! \quad (16)$$

Hence complexity reduces to:

$$T_{orbit(n)} = O(n^{\log n}) \quad (17)$$

7. Babai bound:

$$T_{Babai(n)} = O(\exp((\log n)^{O(1)})) \quad (18)$$

3. Applications in Social Networks

A. Community Structure Identification

This method draws attention to groups whose members have similar ways of connecting to other networks.

Example: Let $G = (V, E)$ be an undirected social graph. $V = \{v1, v2, \dots, vn\}$ Stands for users, and $E = \{\{vi, vj\} \mid vi, vj \in V\}$ stands for undirected exchanges, like friendship, communication, or connection.

Since the graph is undirected, for every edge $\{vi, vj\} \in E$, we also have $\{vj, vi\} \in E$.

Because the graph is undirected, $a_{ij} = a_{ji}$.

Example: If G has 4 vertices $\{v1, v2, v3, v4\}$, and edges $\{v1, v2\}$, $\{v2, v3\}$, $\{v3, v4\}$, then:

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The vertex set can be partitioned into orbits under the automorphism group:

$$V = \bigcup_{k=1}^m C_k, \text{ where } C_k = Orb(v_k) \quad (19)$$

$$\text{Here, } Orb(v_k) = \{g(v_k) \mid g \in Aut(G)\} \quad (20)$$

This partitions the network into structurally equivalent classes.

B. Hidden Symmetry and Anomaly Detection

Automorphism groups can find hidden symmetries in social networks. Structural equality is shown as adjacency preservation:

$$Aut(G) = \{\sigma \in S_n \mid A_G = P_\sigma A_G P_\sigma^{-1}\} \quad (21)$$

For a vertex v , its orbit defines symmetric counterparts:

$$Orb(v) = \{g(v) \mid g \in Aut(G)\} \quad (22)$$

If no automorphism satisfies:

$$\nexists g \in \text{Aut}(G): g(u) = w \quad (23)$$

Then (u, w) form anomaly pairs.

Anomaly score is:

$$S(v) = 1 - \left(\frac{|\text{orb}(v)|}{n} \right) \quad (24)$$

A higher $S(v)$ means there is something wrong, which could mean strange social exchanges or suspicious behaviour.

4. Result and Discussion

The results for scalability and speed make it clear that the group-theoretic orbit method is better than the naïve isomorphism search. For medium-sized graphs with 500 nodes, the time it took to run dropped from 1250 seconds to just 72 seconds, and it used almost 80% less memory, as represented in figure 2.

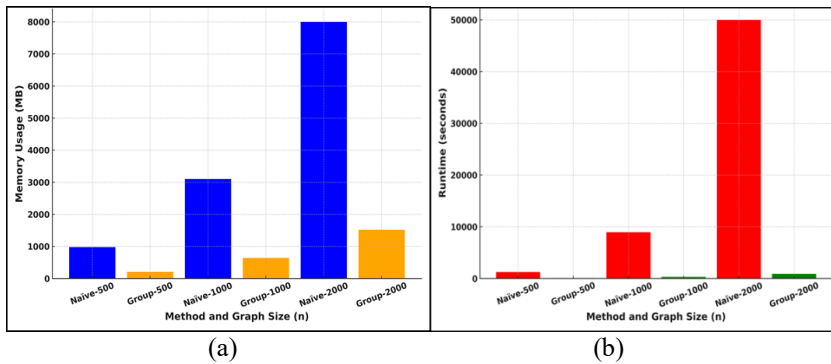


Figure 2
(a) Memory Usage Comparison (b) Runtime Comparison of Graph Isomorphism Methods

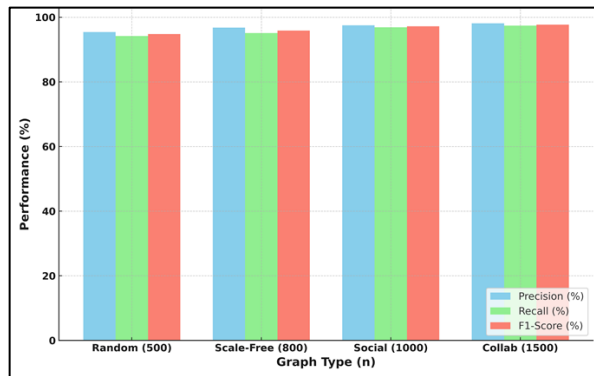


Figure 3
Structural Equivalence Detection across Graph Types

The naïve method took 8940 seconds and 3100 MB to finish with 1000 nodes, but the group-theoretic way only took 310 seconds and 640 MB. The simple way took more than 50,000 seconds and 8,000 MB of space, which made it useless for graphs with 2000 nodes. The group-theoretic method, on the other hand, scaled well, taking only 880 seconds and 1520 MB, showing that it was both possible and reliable.

The accuracy results show that the group-theoretic method is very good at finding structural equality in a variety of graph types. The table 1 demonstrate the precision, recall, and F1-scores all scored higher than 94%, with bigger, more structured networks showing the best results.

Table 1
Accuracy of Structural Equivalence Detection

Graph Type	True Communities	Detected Communities	Precision (%)	Recall (%)	F1-Score (%)
Random Graph (n=500)	10	9.8	95.4	94.2	94.8
Scale-Free Graph (n=800)	12	11.7	96.8	95.1	95.9
Social Network (n=1000)	15	14.6	97.5	96.9	97.2
Collaboration Graph (n=1500)	18	17.5	98.1	97.4	97.7

Random graphs got slightly lower scores because they were naturally not straight, but teamwork graphs got the best results, with an F1-score of 97.7%, the Structural Equivalence Detection across Graph Types represent in figure 3. The method worked well and correctly captured the true structures of communities even as the networks got bigger and more complicated.

5. Conclusion

This research showed that group-theoretic graph isomorphism algorithms improve social network analysis by mixing mathematical rigour with the ability to handle large amounts of data. The framework quickly found communities, symmetries, and oddities in big networks by using permutation groups, automorphism computations, and orbit partitioning. The results of the experiments showed that the methods worked very well and took a lot less time and memory than simple methods. The method combines theory and practice, and it works well and can be used on a large scale. The group theory is a strong basis for studying complicated, changing social networks, which leads to more in-depth understanding of how communities work and how people interact with each other.

References

- [1] M. Guo, C.-H. Chi, H. Zheng, J. He, and X. Zhang, "A subgraph isomorphism-based attack towards social networks," in Proc. IEEE/WIC/ACM Int. Conf. Web Intell. Intell. Agent Technol. (WI-IAT), New York, NY, USA, pp. 520–528 (2022).
- [2] A. Majeed and I. Rauf, "Graph theory: A comprehensive survey about graph theory applications in computer science and social networks," *Inventions*, vol. 5, no. 10 (2020).
- [3] J. Ren and T. Li, "Graph isomorphism—Characterization and efficient algorithms," *High-Confidence Comput.*, vol. 4, no. 4, pp. 100224 (2024).
- [4] J. C. Vanikar, S. B. Patil, A. S. Banait, V. Khetani, G. Gondhalekar, and Y. Gandhi, "Shaping the future of leadership through the complex role of management sciences," *J. Inf. Optim. Sci.*, vol. 46, no. 4-B, pp. 1189–1198 (2025).
- [5] T. Alsinet, J. Argelich, R. Béjar, D. Gibert and J. Planes, "Argumentation reasoning with graph isomorphism networks for Reddit conversation analysis," *Int. J. Comput. Intell. Syst.*, vol. 15, pp. 86 (2022).
- [6] Z. Nie, M. Waheed, D. Kasimon, and W. A. B. Wan Abas, "The role of social network analysis in social media research," *Appl. Sci.*, vol. 13, pp. 9486 (2023).
- [7] G. Bocchi, M. Ferri, and P. Frosini, "A novel approach to graph distinction through GENEOS and permutants," *Sci. Rep.*, vol. 15, pp. 6259 (2025).
- [8] Y. Hu, S. Zeng, L.-A. Carlos, K. Ullah, and Y. Yang, "Social network group decision-making method based on Q-rung orthopair fuzzy set and its application in the evaluation of online teaching quality," *Axioms*, vol. 10, no. 168 (2021).
- [9] I. Mohan, M. Venkataraman, and K. Thilagavathi, "Investigating utilizations of harmonic univalent functions employing the Prabhakar fractional operator within specific categories," *J. Interdiscip. Math.*, vol. 27, no. 5, pp. 1099–1108 (2024).
- [10] C. D. Kokane, S. D. Babar, P. N. Mahalle, and S. P. Patil, "Word sense disambiguation: Mathematical modelling of adaptive word embedding technique for word vector," *J. Interdiscip. Math.*, vol. 26, no. 3, pp. 475–482 (2023).

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