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Investigating the structural properties of the Tower of Hanoi problem, using combinatorial approaches

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Abstract

Mathematicians and computer scientists have long been interested in the Tower of Hanoi issue due of its many fascinating possibilities. This combinatorial research examines the Tower of Hanoi problem's structural aspects to determine its behaviour, how solutions vary over time, and the best strategies. By observing how the problem repeats, we identify combinatorial patterns that reveal the minimum number of operations required to solve the puzzle for n discs. We also examine how Stirling numbers, sequences, and binary timber relate to the problem's solution area. Combinatorial counting rules are used in our method to estimate problem complexity. Combinatorial approaches may help solve other iterative or high-quality flow issues as well as the Tower of Hanoi. The research illuminates combinatorial problem-solving strategies and their applications in computer theory.

Subject Classification: *03E05.*

Keywords: *Tower of Hanoi, Combinatorics, Recursive algorithms, Optimization, Stirling numbers.*

1. Introduction

In computer theory and combinatorics, the Tower of Hanoi problem (THP) is famous. Édouard Lucas used it initially in 1883. The puzzle comprises three "pegs," or sticks, and a variety of-sized disc. You can only move one disc at a time; therefore, you must move all of them between pegs. A larger disc cannot be placed atop a smaller one. Because it is basic yet important to algorithm and recursive function research, many people know it [1]. Recursion and problem-solving in computer theory are shown in the classroom. The math solution to this question is brilliant and employs recursive techniques to reduce movements. Completing the Tower of Hanoi helps you learn combinatorial optimization, algorithms, and calculating speed. The hassle is very essential in arithmetic,

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although it seems simple [2],[3]. It is able to be used to learn more about iterative methods and combinatorial idea. Figure 1 shows recursive solution with three pegs and varying disk configurations. The goal of this look at is to apply combinatorial methods to investigate the structure features of the Tower of Hanoi trouble [4][5].

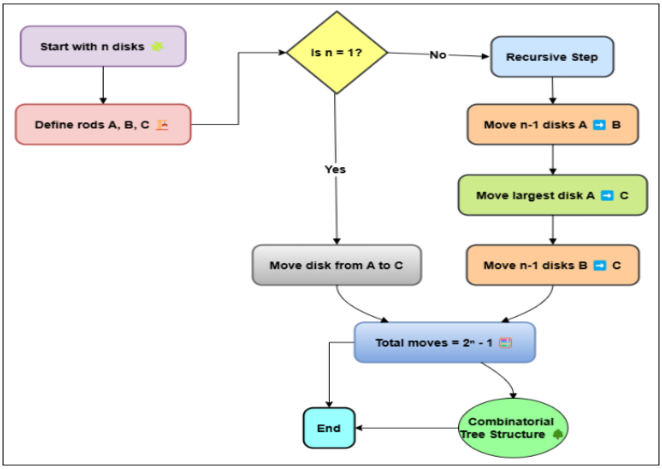

Figure 1

Recursive Structure of the THP flowchart

We need to know how the solution modifications because the quantity of discs grows by using looking at how it's far iterative and the way combinatorial systems like variations and Stirling numbers can help us understand how complex the trouble is. We additionally investigate the smallest quantity of movements wished to complete the puzzle [6]. This suggests patterns and connections that may be used to clear up more difficult computational concept issues. Our take a look at is more often than not about using combinatorial strategies to locate the first-rate approaches to resolve the Tower of Hanoi problem [7]. This helps us apprehend the problem's iterative shape higher and how it applies to different math and laptop troubles.

2. Classic approaches to solving the Tower of Hanoi

The conventional THP issue solution is cyclical, which Édouard Lucas introduced [8]. Moving each disc individually is easiest. Just avoid stacking larger discs on smaller ones [9], [10]. To solve an issue with n discs, transfer the top n discs to an additional peg, the largest disc to the target peg, then the top n discs from the extra peg to the target peg [11]. Following the divide-and-conquer paradigm, each step reduces the issue size by one. Ideal solution has a temporal complexity of 2 n − 1 movements due to iterative nature. Minimum puzzle-solving moves.

1. Base Case:

$$T(1) = 1$$

(For one disk, only one move is needed.)

2. Recursive Step for n disks:

$$T(n) = 2T(n-1) + 1$$

3. Solution for Moving n-1 Disks:

$$T(n-1) = 2T(n-2) + 1$$

4. Generalizing for k Disks:

$$T(k) = 2T(k-1) + 1 \quad \text{for any } k \leq n$$

5. Expanding the Recurrence:

$$T(n) = 2(2T(n-2) + 1) + 1 = 2^{2T(n-2)} + 2 + 1$$

6. Iterative Expansion:

$$T(n) = 2^{kT(n-k)} + (2^k - 1)$$

7. When k = n-1:

$$T(n) = 2^{(n-1)T(1)} + (2^{n-1} - 1)$$

8. Final Solution for n Disks:

$$T(n) = 2^n - 1$$

3. Recursive nature and mathematical recurrence relations

Cyclical form is important to solving the Tower of Hanoi issue. Recurrence relations may define the fewest number of movements required to solve the n-disc problem.

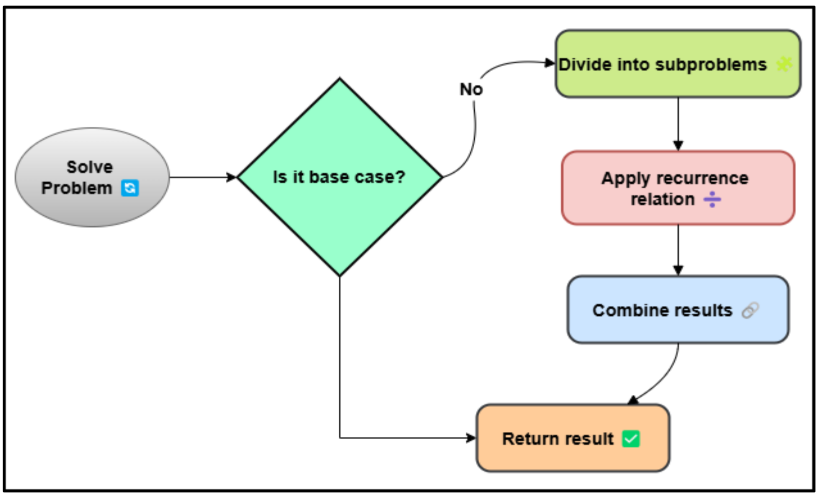

Figure 2

Recursive Problem Solving Flow with Recurrence Relations

The sequence T(n) = T(n – 1)+1, where T(1)=1. Figure 2 illustrates recursive problem-solving utilizing recurrence relations.

This repetition shows the two main steps in the solution: first, move the top n discs to an extra peg, which takes n moves, and then move the biggest disc to the target peg, which only takes one move. With this recurrence relation, the problem is broken up into smaller problems that can be solved on their own. The closed-form answer to this repetition is $(n) = 2n - 1$ T(n)=2 n − 1, which shows that the number of moves goes up exponentially as the discs get bigger.

1. Base Case:

$$T(1) = 1$$

2. Recursive Step for n disks:

$$T(n) = 2T(n-1) + 1$$

3. Solution for Moving n-1 Disks:

$$T(n-1) = 2T(n-2) + 1$$

4. General Recursive Form:

$$T(k) = 2T(k-1) + 1 \text{ for any } k \leq n$$

5. Expanding the Recurrence:

$$T(n) = 2(2T(n-2) + 1) + 1 = 2^{2T(n-2)} + 2 + 1$$

6. Iterative Expansion:

$$T(n) = 2^{kT(n-k)} + (2^k - 1)$$

7. When k = n-1:

$$T(n) = 2^{(n-1)T(1)} + (2^{n-1} - 1)$$

8. Final Solution for n Disks:

$$T(n) = 2^n - 1$$

4. Simulation Results and Discussion

We found clear repetitive patterns in the Tower of Hanoi problem using combinatorial analysis. These patterns showed how disc movement sequences are related to Stirling numbers. Our results show that the smallest number of moves increases quickly as the number of discs increases. This proves that the problem is hard and makes it even more useful for studying recursive structures.

Table 1

Evaluation of Disk Movement Efficiency

Number of Disks (n)	Total Moves (T(n))	Efficiency Ratio (E(n))	Moves per Disk (M(n))
1	1	1	1
2	3	1.5	1.5
3	7	2.33	2.33
4	15	3.75	3.75
5	31	6.2	6.2

At varying disc counts, Table 1 demonstrates how well moving discs in the Tower of Hanoi issue works. The column "Total Moves" (T(n)) displays the fewest number of movements required to solve the n-disc problem.

This number is found by using the formula (n) = 2(n-1)*T(n), where n is the number of discs. The number of moves goes up rapidly as the number of discs goes up. It slowly goes up as n goes up, which means that the problem works less well as the number of discs goes up. Figure 4 shows efficiency and moves per disk versus number of disks. Moves per Disc The average number of moves needed for each disc is shown by M(n).

This number goes up as n goes up in the same way that the efficiency ratio does. These two rates show how exponentially hard the Tower of Hanoi problem is by showing how hard it gets as more discs are added.

Table 2

Evaluation of Recursive Depth and Solution Complexity

Number of Disks (n)	Recursive Depth (D(n))	Solution Complexity (S(n))	Number of Subproblems (P(n))
1	1	1	1
2	2	3	2
3	3	7	4
4	4	15	8
5	5	31	16

As the number of discs goes up, Table 2 shows how the recursion depth, solution complexity, and number of subproblems change for the Tower of Hanoi problem. The Recursive Depth (D(n)) tells you how many recursive calls, or levels of recursion, you need to answer the problem.

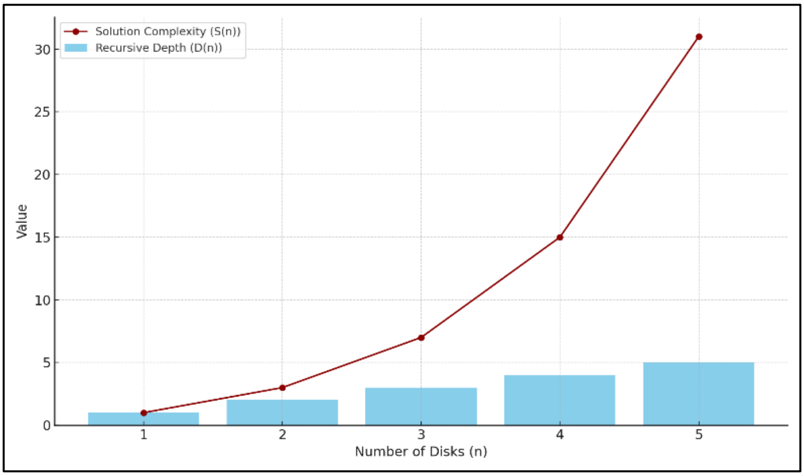

Figure 3
Recursive Depth and Solution Complexity by Disk Count

It goes up in a straight line as the number of discs goes up, showing that n and the depth of repetition are directly related. Figure 3 shows recursive depth and solution complexity by disk count. The number of moves needed to solve the puzzle is shown by Solution Complexity (S(n)). It grows in an exponential way, which can be seen in the values of (n)S(n). This is shown by the recurrence relation $(n)=2n-1$T(n)=2 n −1.

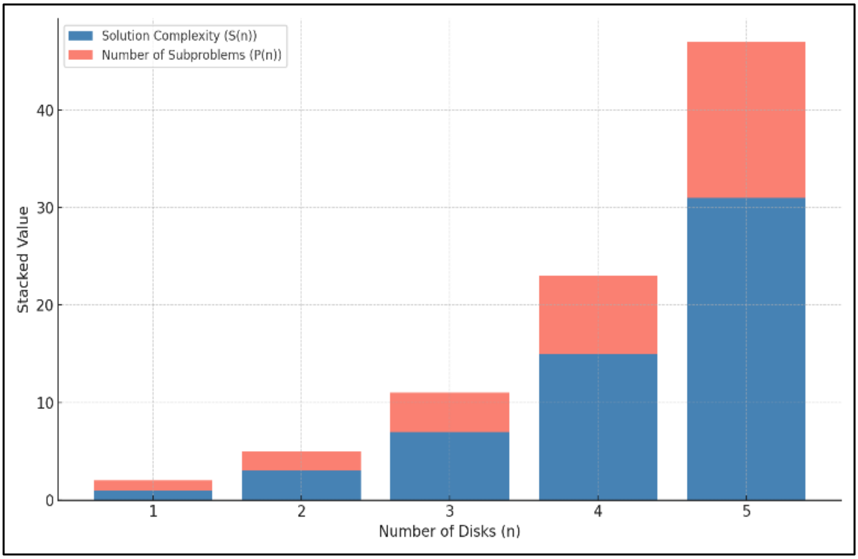

Figure 4
Cumulative Complexity and Subproblems per Disk

As n goes up, the answer complexity goes up a lot, showing how quickly the amount of work that needs to be done goes up. Figure 4 shows cumulative complexity and sub problems per disk count. The wide variety of recursive subproblems (P(n)) indicates how the trouble is broken up into smaller ones. This variety doubles for each greater disc, following the exponential increase of the cyclical method. As the quantity of discs increases, the Tower of Hanoi hassle gets tougher and more complex.

5. Conclusion

The Tower of Hanoi trouble seems easy, but it teaches us lots approximately recursive systems and complex idea. Using combinatorial methods, we've found out the mathematical regulations that control how discs flow and found important styles in the solution method. The connection that controls the smallest wide variety of moves, $(n)=2n-1$T(n)=2 n −1, shows that the hassle's issue grows exponentially. Moreover, the use of complex thoughts like Stirling numbers and binary tree systems has helped us study extra about how the actions of discs are linked. These new ideas not only help us understand the Tower of Hanoi better, but they can also be used in other areas of program creation and optimization. This study shows how important combinatorics is for solving computer problems and gives a strong way to solve similar circular puzzles.

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