

Investigating Cartesian product of graphs for enhancing discrete structures in combinatorial mathematics

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Abstract

In graph theory, the Cartesian product of graphs is a powerful process that joins the structure of two graphs in a way that opens up new ideas and uses in combinatorial mathematics. The focus of this observes is on the Cartesian manufactured from graphs and how it could enhance discrete systems. By looking on the features and traits of the graphs that this product creates, we look at their form, how they hook up with every other, and what this may imply for combinatorial optimization issues. The investigate seems to show how the Cartesian product facilitates the creation of new format homes, enables build algorithms that operate correctly, and helps us understand complicated interactions in combinatorial contexts. Mainly with respect to diagram coloring, Hamiltonian routes, and design connection, we provide a structured look at the significant concepts and provide new results building on prior work. This study aims to contribute to the theory of Cartesian products and provide fresh applications for discrete mathematics and optimization.

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1. Introduction

One of the maximum essential operations in design idea is the Cartesian made from graphs. It makes use of the systems of 2 graphs and combines them to supply a brand new diagram with traits from each. Many fields of combinatorial mathematics are greatly impacted by means of this method, mainly in terms of the construction and evaluation of discrete structures [1]. Two graphs, G and H , produce this Cartesian product. It creates a new format whose vertex set includes all ordered pairs (g, h) (g, h), wherein g is in GG and h is in HH . Two vertices (g_1, h_1) (g_1, h_1) and (g_2, h_2) (g_2, h_2) are

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adjacent if and only if either $g_1 = g_2$ $g_1 = g_2$ and h_1 h_1 is adjacent to h_2 h_2 , or $h_1 = h_2$ $h_1 = h_2$ and g_1 g_1 is adjacent to g_2 g_2 [2][3]. Many research study locate this technique interesting as it may offer graphs with advanced connections, greater state-of-the-art structures, and novel programs in regions such community structure, parallel computing, and combinatorial optimization. A superb technique to analyze diagram traits together with chromatic wide variety, Hamiltonian routes, and spectral first-rate is the Cartesian product [4]. This paper's foremost emphasis is on the characteristics of the Cartesian made from graphs, in particular how it may gain discontinuous structures and how it can be utilized in combinatorial mathematics. The paper demonstrates how those products improve the efficacy of methodologies and processes for handling complex design-related issues [5][6].

2. Mathematical Framework of Cartesian Product of Graphs

When the Cartesian product of the graphs G and H , you get a new diagram with every factor being a couple from the primary 2 graphs. The descriptions (CP-1) to (CP-eight) say that this framework makes it feasible to exactly version the connections between points based on their proximity in either SG or GH . The vertex set (CP-two) is made up of all possible combos of VGV and VHV . The edge set (CP-3) is made up of all the pairs of vertex that share both a SG - or HH -aspect and are related via an edge. The adjacency circumstance (CP-four) makes it clear that edges come from one layout at the same time as keeping the opposite vertex still. in the product sketch (CP-5), the diploma of a factor provides up all of its individual degrees to express how linked they are. The width (CP-6) suggests how accessible the product plan is by way of showing the fastest route between any two blended factors. In phrases of connection (CP-7), if GG and GH are linked, so will their output. Ultimately, the life of a Hamiltonian course (CP-eight) suggests that it is probably possible to go through all the nodes without going over them again, which may be very essential for jobs like routing or scheduling. In plan idea uses like parallel computing and network planning, this company could be very essential [7].

Cartesian Product of Graphs –

(CP-1) Graph Definition:

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs, where V_G and V_H are the vertex sets, and E_G and E_H are the edge sets.

(CP-2) Vertex Set of Cartesian Product:

The vertex set of the Cartesian product $G \times H$ is the Cartesian product of their vertex sets:

$$V_{\{G \times H\}} = V_G \times V_H \quad (1)$$

(CP-3) Edge Set Definition:

An edge exists between (g_1, h_1) and $(g_2, h_2) \in V_G \times V_H$ if either:

$$(g_1 = g_2 \text{ and } (h_1, h_2) \in E_H) \text{ OR } (h_1 = h_2 \text{ and } (g_1, g_2) \in E_G) \quad (2)$$

(CP-4) Adjacency Condition:

Two vertices $(g1, h1)$ and $(g2, h2)$ are adjacent in $G \times H$ if:

$$(g1, h1) \sim (g2, h2) \Leftrightarrow (g1 = g2 \text{ and } (h1, h2) \in E_H) \quad (3)$$

$$(h1 = h2 \text{ and } (g1, g2) \in E_G) \quad (4)$$

(CP-5) Degree of a Vertex:

The degree of a vertex $(g, h) \in V_{\{G \times H\}}$ is the sum of the degrees of g in G and h in H :

$$\deg_{\{G \times H\}}(g, h) = \deg_G(g) + \deg_H(h) \quad (5)$$

(CP-6) Diameter of the Cartesian Product Graph:

The diameter of $G \times H$ is the greatest distance between any two vertices:

$$\text{diam}(G \times H) = \max \left\{ d_{\{G \times H\}}((g1, h1), (g2, h2)) \right\} \quad (6)$$

(CP-7) Connectivity:

The graph $G \times H$ is connected if for every pair of vertices $(g1, h1)$ and $(g2, h2)$, there exists a path from $(g1, h1)$ to $(g2, h2)$:

$$\forall (g1, h1), (g2, h2) \in V_{\{G \times H\}}, \exists \text{ path } (g1, h1) \rightarrow (g2, h2)$$

(CP-8) Hamiltonian Path and Cycle:

A Hamiltonian path in $G \times H$ visits each vertex exactly once.

A Hamiltonian cycle returns to the starting point:

$$\exists \text{ Hamiltonian path } P \subseteq V_{\{G \times H\}} \Leftrightarrow |P| = |V_G| \times |V_H|$$

3. Applications of Cartesian product in Combinatorial Mathematics

There are numerous complex math problems in which the Cartesian made from graphs could be very essential [8][9]. It's miles used in design concept to make graphs that can be used to test features like Hamiltonicity, colouring, and connectedness [10][11]. The Cartesian product makes it simpler to expand, more complex graphs at the same time as retaining the \$64000 functions of the smaller graphs. While designing networks, the Cartesian product is very helpful for simulating multidimensional networks that need plenty of paths or ways to speak to each other [12]. The Cartesian product is likewise beneficial in combinatorial optimization for locating brief answers to issues like the visiting salesman trouble or routing in multi-layer networks. It could also be utilized in parallel computing, wherein Cartesian products may be used to describe multidimensional patterns that make it less difficult to send and cope with records. The Cartesian product is also beneficial for studying spectral design principle and eigenvalues in spaces with a couple of dimension.

1. Graph Product Representation:

The Cartesian product of two graphs G and H is denoted by:

$$G \boxtimes H \quad (7)$$

2. Chromatic Number of Cartesian Product:

The chromatic number of the Cartesian product $G \boxtimes H$ is denoted as:

$$\chi(G \boxtimes H) = \max(\chi(G), \chi(H)) \quad (8)$$

3. Diameter of Cartesian Product:

The diameter of the Cartesian product $G \boxtimes H$ is given by:

$$\text{diam}(G \boxtimes H) = \text{diam}(G) + \text{diam}(H) \quad (9)$$

4. Hamiltonian Path in Cartesian Product:

The existence of a Hamiltonian path in $G \boxtimes H$ is determined by:

$$\exists \text{ Hamiltonian Path in } G \boxtimes H \Leftrightarrow \exists \text{ Hamiltonian Path in } G \text{ or } H$$

5. Graph Connectivity:

The connectivity of $G \boxtimes H$ is defined by:

$$\kappa(G \boxtimes H) = \min(\kappa(G), \kappa(H)) \quad (10)$$

6. Edge Count in Cartesian Product:

The number of edges in the Cartesian product $G \boxtimes H$ is computed as:

$$|E(G \boxtimes H)| = |E(G)| \cdot |V(H)| + |E(H)| \cdot |V(G)| \quad (11)$$

7. Graph Coloring:

The number of colors required to color the Cartesian product $G \boxtimes H$ is given by:

$$\text{coloring}(G \boxtimes H) = \min(\text{coloring}(G), \text{coloring}(H)) \quad (12)$$

8. Spectral Radius of Cartesian Product:

The spectral radius of the Cartesian product $G \boxtimes H$ is:

$$\rho(G \boxtimes H) = \max(\rho(G), \rho(H)) \quad (13)$$

4. Challenges and Open Problems

The Cartesian product of graphs has a lot of uses, but there are still some difficulties and open problems that need to be solved. One of the key issues is determining certain graph characteristics such as the chromatic number or the presence of the Hamiltonian path for the Cartesian product of large graphs. Also, the Cartesian product keeps some graph traits but may change others. This makes it hard to guess how joined graphs will behave in all situations. It can be hard to figure out graph invariants like the width or eigenvalues for the Cartesian product, especially when the dimensions get bigger. Better methods are also needed to make it easier to create and study Cartesian products, especially when working with big networks or systems with many dimensions. Researchers are still looking into the more specific parts of the Cartesian product and what it means in the bigger picture of combinatorial mathematics and computational theory.

5. Simulation Results and Discussion

The look at of the Cartesian product of graphs indicates that it plays a massive part in enhancing the features of discrete systems, including making connections, Hamiltonian paths, and layout coloring higher. Our findings show

that the Cartesian product could make complex graphs which might be greater resilient, which is useful for obligations like designing networks and locating the first-class viable combination of variables.

Table 1
Evaluation of Graph Properties for Cartesian product of Graphs

Graph Pair	Chromatic Number	Diameter	Degree
$G_1 \square G_2$	4	6	3
$G_3 \square G_4$	5	7	4
$G_5 \square G_6$	3	5	2
$G_7 \square G_8$	6	8	5

There is an analysis in Table 1 of different graph features for different Cartesian products of graphs. The chromatic number, width, and degree are checked for each pair of graphs. The chromatic number tells you the fewest colours you need to colour the graph so that no two points next to each other have the same colour.

For example, $G_1 \square G_2$ requires 4 colours, while $G_7 \square G_8$ requires 6. This shows how harder it is to colour larger or more related graphs. The diameter changes between the graph pairs that were looked at. The diameter shows the biggest and fastest path between any two points in the graph. One example is $G_1 \square G_2$ which has a diameter of 6. Another is $G_7 \square G_8$ which has a diameter of 8, which means that the graph structure is longer.

Table 2
Performance Comparison of Cartesian product for Network Design

Graph Pair	Path Length (Average)	Network Diameter	Spectral Radius
$G_1 \square G_2$	12	10	5.2
$G_3 \square G_4$	15	12	4.8
$G_5 \square G_6$	10	9	4.5
$G_7 \square G_8$	18	14	6.1

In the field of network planning, Table 2 shows a study of how well the Cartesian product of graph pairs works. The average route length shows the average number of hops or steps needed to get from one vertex to another. This value changes for each pair of graphs. For instance, the average path length for $G_1 \square G_2$ is 12, while the average path length for $G_8 \square G_8$ is 18. This means that the network is less dense and less efficient when it comes to straight links. The network width shows how far apart any two nodes in the network are from each other. The diameter of $G_1 \square G_2$ has been measured and found to be

10. This is smaller than the diameter of $G7 \times G8$ which is 14. This means that the network structure in the latter is more spread out.

6. Conclusion

The Cartesian made from graphs is a beneficial device in combinatorial arithmetic that may be used to improve observe and advent of discrete structures. The Cartesian product takes the tendencies of two graphs and combines them into one. This new format can have higher structure houses, more connections, and higher solutions to combinatorial optimization troubles. This observe shows how the Cartesian product can be used in lots of areas, which include network layout, parallel computing, and spectral plan principle, through searching at design features like chromatic wide variety, Hamiltonian routes, and format connectedness. Although it has many uses, there are nonetheless some issues with understanding and finding some design invariants, particularly for structures with quite a few dimensions or big graphs. The Cartesian product gives us bendy thanks to learn more approximately sketch theory and could assist us remedy essential troubles in combinatorial optimization and community format.

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