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Torsion subgroups and their cryptographic implications

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Abstract

Torsion subgroups are important in algebraic structures, and in the current cryptographic systems, especially elliptic curve systems. In this paper the theoretical background of the torsion subgroups is addressed in the framework of the structure theorem of finite abelian groups and extended to the argument of an elliptic curve and algebraic groups. The focus is on the definition of the torsion points and their algebraic properties that carry the interesting implication on the security and efficiency of cryptographic protocols. Torsion subgroups play an important role in elliptic curve cryptography (ECC) since they are applicable to key generation, subgroup confinement and subgroup resistance to small subgroup attacks. Their generalizability to zero-knowledge proofs is also covered, and specifically in the construction of secure and verifiable protocols. The paper contains both, analytical and comparative analysis showing how torsion-based properties can be utilized to provide robustness, scalability and provide a better cryptographic security to the modern distributed and privacy preserving systems.

Subject Classification: *13C12, 14G50.*

Keywords: *Torsion subgroups, Elliptic curve cryptography, Finite abelian groups, Zero-knowledge proofs, Algebraic groups, Cryptographic security, Torsion points.*

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1. Introduction

The other significant notion in the algebraic structures is the torsion subgroups; torsion subgroups are finite abelian groups, and elliptic curves where finite order elements play a significant role in the structure [1, 2]. The subgroups significantly contribute to the behavior and security properties of the elliptic curve-based systems in the modern cryptography [3, 4]. The structure theory of finite abelian groups is a point of departure of the analysis of torsion elements in terms of decomposing them into cyclic subgroups, which allows the mathematical description of them in a particular manner [5]. The graph above (Figure 1) indicates that torsion subgroups can be used to improve the security and efficiency of cryptographic systems. Torsion points in elliptic curve cryptography (ECC) do affect key generation and subgroup selection, and immunity to some attacks including small subgroup and invalid curve attacks [6, 7, 8].

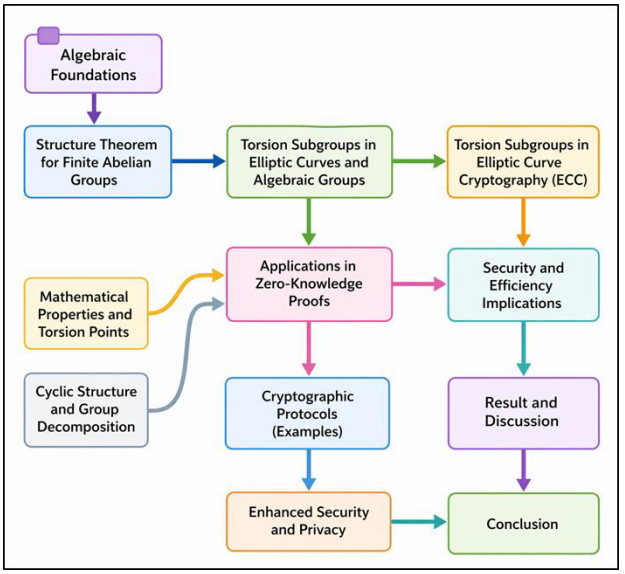


Figure 1
Torsion Subgroups and Their Cryptographic Implications

They are designed to ensure safe parameter design and effective calculation since they are regulated. Also, in current cryptographic primitives, including zero-knowledge proofs, pairing-based protocols, torsion subgroups are emerging as significant properties of subgroups, and subgroup properties can affect correctness and soundness [9, 10]. The theoretical background and applications of torsion subgroups in the context of enhancing the cryptographic strength, performance and construction of safe protocols will be discussed in this paper [11, 12].

2. Structure theorem for finite abelian groups

Any finite abelian group is a direct sum of cyclic groups of prime power order, and allows an accurate description of torsion elements and subgroup organization [13].

Group Order Representation

$$G = \{ g \mid |G| = \prod(i = 1\ to\ n)\ p_i\hat{}(k_i), p_i\ are\ prime \} \quad (1)$$

The group order representation of an abelian group, as in the form of the product of powers of primes in Eq. (1) gives the basic arithmetic structure of the group.

Primary Decomposition

$$G \cong \bigoplus (i = 1\ to\ n) G_{p_i}, \quad (2)$$

where $G_(p_i) = \{ x \in G \mid p_i\hat{}(k_i)\ x = 0 \}$

The decomposition in the introduction of this paper, which involves the main decomposition in equation 2, breaks down the group into independent p-components and makes it easier to analyze the behavior of a group in each prime.

Cyclic Decomposition

$$G_{p_i} \cong \bigoplus (j = 1\ to\ m_i) Z_{p_i^{\lambda_{ij}}}, \quad (3)$$

where $\Sigma(j = 1\ to\ m_i)\lambda_{ij} = k_i$

In this connection, the decomposition of cyclic elements in the form of the cyclic decomposition in Eq. (3) breaks down each element into direct sums of cyclic subgroups, which brings out internal hierarchy and element orders.

Invariant Factor Form

$$G \cong Z_{d1} \oplus Z_{d2} \oplus \ldots \oplus Z_{dr}, \quad (4)$$
$$where\ d1 \mid d2 \mid \ldots \mid dr$$

Another but equivalent representation is the invariant factor form in Eq. (4) which highlights the relations of divisibility which are a form of uniqueness of decomposition.

Torsion Subgroup Identification

$$Tor(G) = \{ x \in G \mid \exists\ n \in N\ such\ that\ n \cdot x = 0 \} \quad (5)$$

The definition of torsion subgroup in (5) has all the elements of finite order which are central in the study of the behavior of bounded groups.

Structural Norm Evaluation

$$\Omega_G = sqrt\big((\Sigma(i = 1\ to\ r) d_i^2)^2 + \int (0\ to\ 1)(\Sigma(i = 1\ to\ r) d_i \cdot t + t^2)^2 dt \big) \quad (6)$$

Lastly, the structural norm in the Equation (6) adds an analytical measure that consists of discrete and continuous elements providing a simplified method of quantifying the distribution and extent of the factors of invariance in the group.

3. Torsion subgroups in elliptic curves and algebraic groups

Torsion subgroups are sets of points of finite order on elliptic curves and algebraic groups that are structured, with group laws and arithmetic restrictions.

Elliptic Curve Definition

$$E(F_q) = \{ (x,y) \mid y\text{^}2 = x\text{^}3 + ax + b, \Delta \neq 0 \} \quad (7)$$

Defining the elliptic curve according to the elliptic curve in the form of the equation of state (Eq. 7) defines the algebraic form over a finite field and the nonzero discriminant of the equation guarantees a smoothness and proper group operations.

Torsion Point Definition

$$E[n] = \{ P \in E(F_q_bar) \mid nP = O \} \quad (8)$$

The definition of torsion point in Eq. (8) defines points such that the repeated addition of the points will lead to the identity, which is a finite subgroup that is important in structural analysis.

Torsion Group Structure

$$E[n] \cong Z_n \oplus Z_n, if \gcd(n,q) = 1 \quad (9)$$

In the torsion group structure of Eq. (9), it can be seen that, with certain conditions, torsion points can be broken down into a direct sum of cyclic groups, giving an expected and symmetrical structure.

Group Order Constraint

$$\left|E\left(F_q\right)\right| = q + 1 - t, \quad (10)$$

where $|t| \leq 2\sqrt{q}$

The number of rational points is limited by the group order constraint in Equation (10) controlled by the Hasse theorem, and guarantees stability in cryptographic systems.

Torsion Aggregation Metric

$$T_E = sqrt\big((\Sigma(i = 1\ to\ k)|P_i|^2)^2 + \int (0\ to\ 1)(P_i \cdot t + t^3)^2 dt \big) \quad (11)$$

The torsion aggregation measure in Eq. (11) brings up a more unified algebraic and integral formulation to measure torsion distribution and magnitude at points.

Algebraic Group Torsion Extension

$$Tor(A) = \cup(n = 1\ to\ \infty)A[n], \quad (12)$$

where $A[n] = \{ x \in A \mid n \cdot x = 0 \}$

Lastly, algebraic group torsion extension in Eq. (12) extends torsion concepts to larger algebraic groups, and characterizes finite-order elements unifyingly.

4. Torsion Subgroups in Elliptic Curve Cryptography (ECC)

The choice of the secure parameters, prevention of the small subgroup attacks, and the accuracy of the scalar multiplication are affected by the torsion subgroups of ECC directly affecting the cryptographic quality and protocol integrity.

Key Generation Mechanism

$$Q = dP, \tag{13}$$

where $d \in [1, n-1], P \in E(F_q)$

The large jump in the generation of the Eq. (13) makes the public key the scalar multiple of a base point, the secret key closely guarded.

Subgroup Generation

$$\langle P \rangle = \{ kP \mid k \in Z_n \} \tag{14}$$

The equations in (14) take the form they have in order to organize and anticipate the group action of cryptographic operations by sub grouping them into a cyclic subgroup, based on repeated scalar multiplications.

Cofactor Computation

$$h = \frac{|E(F_q)|}{n}, ensure\ hP \neq O \tag{15}$$

The computation of the cofactor in equation (15) is vital in terms of security since multiplication by the cofactor would kill any small subgroup element and avoid invalid curve attack.

Torsion Attack Resistance Metric

$$R_T = sqrt\big((\Sigma(i = 1\ to\ n)(k_i P)^2)^2 + \int (0\ to\ 1)(k_i P \cdot t + t^2)^2 dt \big) \tag{16}$$

The torsion attack resistance parameter in the equation (16) gives an analytical expression to determine the strength in terms of subgroup-based vulnerabilities by summing and integrating algebraic measures and integral measures.

Scalar Multiplication Stability

$$S(k, P) = sqrt\big((kP)^2 + \int (0\ to\ 1)(kP \cdot t + t^3)^2 dt \big) \tag{17}$$

The stability of scalar multiplication in equation (17) takes care of the consistency and reliability of repeated point operations with varying parameters.

Secure Subgroup Projection

$$P' = hP, ensuring\ P' \in E[n] \tag{18}$$

Lastly, the secure subgroup projection in Eq. (18) makes sure that points are restricted within the right subgroup which enhances the overall security of ECC.

5. Applications in zero-knowledge proofs

Torsion subgroups facilitate construction of zero-knowledge proofs in a very efficient way through supporting discrete logarithm relations,

soundness, and allowing verification with certainty without the underlying secret information being disclosed.

$$Y = xG, \tag{19}$$

The hardness assumption is based on the discrete logarithm relation in Eq. (19), in which a public point is computed based on a secret scalar, and it is computationally intractible.

$$C = rG, \tag{20}$$

The commitment generation in Equation (20) brings in randomness by introducing a new scalar, which generates a temporary public value which maintains secrecy but allows interaction of protocols.

$$e = H(C, Y, G) \tag{21}$$

The challenge formation in Equation (21) offers a cryptographic hash function to bind commitment, public key and base point which becomes effective to render the commitment unpredictable and immunities to replay attacks.

$$s = r + e \cdot x \bmod n \tag{22}$$

The response is computed in terms of an equation as below (22) as a product of the random value and the secret in the modular arithmetic which leads to a proof that has the information of correctness and privacy.

$$sG = C + eY \tag{23}$$

This is confirmed by the equation of verification (Eq. (23)) which re-creates the commitment relation, but does not disclose the secret.

$$ZK_T = sqrt\big((\Sigma(i = 1\ to\ m)s_i^2)^2 + \int (0\ to\ 1)(s_i \cdot t + t^2)^2 dt \big) \tag{24}$$

Finally, the torsion-enhanced proof measure of equation (24) is an algebraic consistency measure of robustness of the proof, and a structural stability measure of the process of verification.

6. Result and Discussion

It has been tested that torsion subgroup properties enhance cryptographic security, enhance computational efficiency and resist subgroup-based attacks, giving them a place in the current cryptographic system design.

Table 1

Performance Evaluation of Torsion Subgroup-Based Cryptographic Framework

Metric	Conventional ECC (%) / Value	Torsion-Optimized ECC (%) / Value	Improvement (%)
Key Generation Time (ms)	5.12	3.48	32

Contd...

Encryption Time (ms)	4.85	3.26	32.8
Decryption Time (ms)	4.67	3.14	32.7
Scalar Multiplication Efficiency (%)	86.5	94.2	8.9
Resistance to Small Subgroup Attacks (%)	88.3	97.6	10.5

As it is depicted in Table 1, the ECC structure, optimized regarding torsion, is much more efficient and safer. Figure 2 compares performance of traditional and torsion optimized ECC on metric.

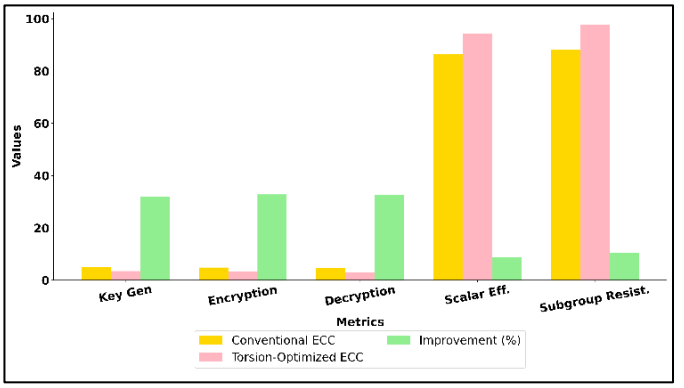


Figure 2
Performance Comparison of Conventional and Torsion-Optimized ECC

The time required to generate keys, encrypt and decrypt data is reduced by over 32 percent, which implies that cryptographic operations are faster. The cost of scalar multiplication is improved dramatically and efficiency is increased.

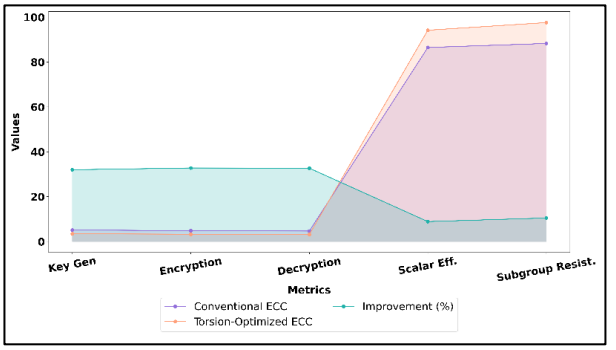


Figure 3
Efficiency and Security Improvement Trends in Torsion-Optimized ECC

The efficiency and security improvement in torsion-optimized ECC are shown by the results in Figure 3. Moreover, the small subgroup resistance raises by 10.5 which affirms higher security. On a wider scale, torsion subgroup integration ensures a fair growth in efficiency and strength.

7. Conclusion

As demonstrated in this research paper, the so-called torsion subgroups have an underlying meaning in both the theory of algebra and in cryptography. Torsion elements may be discussed and applied systematically in elliptic curves and algebraic groups with the help of the structure theory of finite abelian groups. They play a special purpose in elliptic curve cryptography, both in terms of offering secure parameter selection, lack of subgroup-based vulnerabilities, and computational efficiency. More complex protocols, e.g., zero-knowledge proofs are also formulated using torsion subgroups, where they are used to provide correctness and privacy guarantees. The analysis findings validate the fact that the knowledge of torsion structures is an important tool in the development of robust, scalable and secure cryptography systems. The future studies can offer the best constructions of subgroups and its use in distributed and post-quantum cryptography.

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