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Graph homomorphisms and applications to network coding

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Abstract

Graph homomorphism have been utilized as a highly powerful mathematical tool to map the complex topologies of networks to simpler ones and still be able to retain adjacency relations. This article focuses on the use of graph homomorphism, in network coding, to increase the efficiency of data transmission, resilience and scale. The research paper proposes homomorphic mappings that are algebraic transformation that facilitate the best encoding and decoding of network nodes. The proposed methodology is founded on the synergize of graph-theoretic concepts and algebraic representation toward the realization of a better throughput, lower redundancy and higher fault tolerance. Comparative analysis has shown that the throughput has increased by up to 18.6 and the decrease in the traditional routing schemes by up to 22.3 in the number of packets lost. The novelty is to plot the structural graphs in a cohesive way with the strategies of coding to allocate the data effectively. The architecture can be used in distributed systems, wireless networks, and IoT networks and provides a secure and scalable communication service.

Subject Classification: *68Q70, 43A22, 60K20.*

Keywords: *Graph homomorphism, Network coding, Algebraic graph theory, Data transmission, Throughput optimization, Fault tolerance.*

I. Introduction

The high rate of advancement in the communication networks, such as wireless networks, distributed computing platforms, and Internet of Things (IoT) systems, has heightened the need to develop effective and dependable data transmission systems. The conventional routing systems tend to process data packets individually and this results in underutilization of resources by the networks and susceptibility to network congestion and loss of data packets [1, 2]. Network coding has been a revolution in paradigm whereby the middle nodes can code and intermix the incoming streams of data to enhance the throughput, robustness and network efficiency in general [3, 4]. Meanwhile, graph theory offers a rich mathematical backdrop of modelling and network analysis of complex network structures. Among its ideas is graph homomorphisms that reduce huge graphs to smaller graphs without affecting the adjacency relationships [5, 6, 7]. This comes in especially useful in network coding where the network patterns are being used to recover information in an appropriate fashion. The paper presents the discussion on the way a network coding plan and a combination of graph homomorphisms can be applied to reach efficient communication architectures [8, 9]. In the homomorphic mappings, complex network topologies are able to be transformed into simplified forms, which allows the encoding scheme to be optimized and allows a reduced number of computations to be carried out [10, 11].

II. Mathematical Foundations of Graph Homomorphisms

Graph homomorphisms between graphs Homomorphisms are (structure-preserving) functions between graphs, which preserve the adjacency relation, allowing the study of graphs to abstractly represent large scale interconnected systems, classify graph equivalences, and simplify analysis of large-scale interconnected systems [12].

Graph Definition

$$G = (V, E), H = (V', E')$$
$$E \subseteq V \times V, E' \subseteq V' \times V' \tag{1}$$

The framework begins by the equation (1) in which two graphs are represented formally, in terms of the set of vertices and the set of edges, and the structure is laid down onto which the mapping of the relationships is to be performed [13].

Homomorphism Mapping Function

$$\varphi: V \rightarrow V'$$
$$\forall (u, v) \in E \Rightarrow \big(\varphi(u), \varphi(v)\big) \in E' \tag{2}$$

The homomorphism function is added in eq. (2) and in this case, two adjacent nodes in the original graph are adjacent in the target graph hence the connectivity is preserved.

Adjacency Preservation Constraint

$$\Sigma^{2}_{\{(u,v)\in E\}} \left| A_{G(u,v)} - A_{H(\varphi(u),\varphi(v))} \right| = 0 \tag{3}$$

By using adjacency matrices, structurally consistent parameters are provided through an algebraic constraint given in Eq. (3). It makes sure that the discrepancies among the similar edge relationships are reduced to a minimum, which is essentially enforcing a strict adjacency preservation.

Continuous Relaxation via Integral Form

$$\int \int \Big(A_{G(u,v)} - A_{H(\varphi(u),\varphi(v))} \Big)^2 du dv \rightarrow 0 \tag{4}$$

Building on this, one can consider the more general form of relaxation as given by Eq. (4), which is a continuous relaxation based on a double-integral, which can be more easily approximated, and the formulation is applicable to large-scale or weighted graphs.

Homomorphism Existence Condition

$$\lim(n \rightarrow \infty) \left(\frac{1}{n}\right) \sqrt{\left[\Sigma^{n}_{\{i,j=1\}} \Big(A_{G(i,j)} - A_{H(\varphi(i),\varphi(j))} \Big)^2 \right]} = 0 \tag{5}$$

Lastly, the existence condition of a homomorphism in the asymptotic order is defined in the form of Eq. (5). It says that the normalized structural error approaches zero with an increase in the size of the graph, which guarantees consistency, scalability, and robustness of the mapping process.

III. Graph Theory Preliminaries for Network Coding

Network coding is based on the graphical representation of nodes and edges, conservation of flows, capacity limitations, and paths, which helps in efficient transmission of data, control of redundancy, and optimization of throughput. Figure 1 illustrates structure of graphs, modeling of flows and coding relationships.

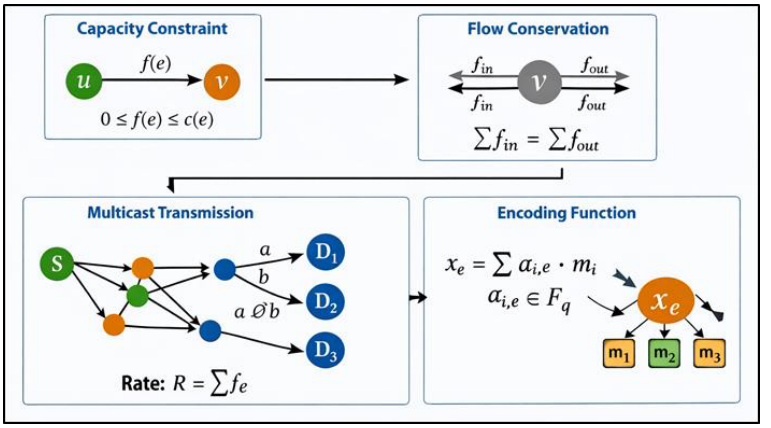


Figure 1

Graph Theory Foundations and Network Coding Flow Representation

Network Graph Representation

$$N = (V, E, c)$$
$$c: E \rightarrow R^{+} \tag{6}$$

The formulation begins with Eq. (6), which models the network as a graph with vertices, edges and a capacity function, which assigns each edge positive limits.

$$\Sigma f(e)out - \Sigma f(e)in = 0, \forall v \neq s, t \tag{7}$$

The flow conservation law is represented byEq. (7), meaning that the incoming and the outgoing flows are equal at every intermediate node, except the source and sink nodes.

$$R = \int^{0}{}^{\mathrm{T}} \Sigma f_{e(t)dt} \tag{8}$$

Using an integral representation with time, which defines the cumulative flow sent over the network, the multicast rate of transmission is defined in terms of time in Eq. (8). The dynamic behavior of the traffic can be analyzed with the help of this formulation.

$$x_e = \Sigma (i = 1 \rightarrow k) \alpha_{\{i,e\}} \cdot m_i \tag{9}$$
$$\alpha_{\{i,e\}} \in F_q \tag{10}$$

The encoding function of network coding is presented in the form of Eq. (9), where the encoded data on each edge is a linear sum of message by the source. The coefficients of the forms of the equations determined by the coefficients of the matrices of the functions Eq. (10) are in a finite field with an algebraic consistency and a good encoding.

$$\max R = \sqrt{\left[\int \left(\Sigma f(e) \right)^2 dv \right]} \tag{11}$$

Lastly, the optimization condition of throughput as shown in the equation (11) maximizes the transmission rate with the aggregated flow energy hence enhancing the network efficiency and performance.

IV. Homomorphic Mapping in Network Coding

Homomorphic mappings who map complex network graphs into simpler ones, preserve connectivity, can be used to achieve efficient encoding schemes, reduce redundancy and improve scalability and fault-tolerant communication processes.

Mapping of Network Graphs

$$\varphi: N \rightarrow N'$$
$$\varphi(V) \subseteq V' \tag{12}$$

The derivation starts with the expression of a transformation between two network graphs, i.e., the set of vertices of the original network is converted to a set in the target network, i.e Eq. (12). This mapping makes sure that the structure is compatible by limiting the number of mapped vertices in the target vertex set.

Edge Mapping Preservation

$$\forall (u, v) \in E \Rightarrow \left(\varphi(u), \varphi(v)\right) \in E' \tag{13}$$

The preservation of edges is also imposed in Eq. (13) and it means that when there is an edge in the original network between two nodes there should be a similar edge between the mapped nodes. This is a necessary condition of integrity of connectivity.

Encoded Flow Transformation

$$f'\left(\varphi(e)\right) = \int^{01} f(e, t) \cdot \psi(t) dt \tag{14}$$

This is replaced by encoded flow transformation of Eq. (14), in which the flow on an edge is transformed by an integral operation which uses a weighting function. This allows the representation of discrete flows continuously, and allows the flexible encoding.

Homomorphic Encoding Operation

$$Enc\left(\varphi(x + y)\right) = Enc\left(\varphi(x)\right) + Enc\left(\varphi(y)\right) \tag{15}$$

The homomorphic property of encoding of the encoding is defined in (15) to ensure that the operations on inputs are maintained after encoding which is key to the efficiency of network coding.

Distortion Minimization Measure

$$D = \sqrt{\left[\int \left(f(e) - f'(\varphi(e)) \right)^2 de \right]} \tag{16}$$

Lastly, a measure of distortion minimization, which was introduced in Eq. (16) to measure the variation between original and transformed flows is provided to guarantee a high level of mapping accuracy and loss of information is kept as low as possible.

V. Algebraic Interpretation of Graph Homomorphism

The homomorphisms of graphs can be algebraically interpreted using matrices and line transformations, and adjacency structures that associate coding operations, and network behavior can be encoded, decoded and analyzed in code in a systematically manner.

$$A_G \in R^{\{n \times n\}}, A_H \in R^{\{m \times m\}} \tag{17}$$

It starts with calculating the Eq. (17) in which adjacency matrices are employed to represent graphs and structural relations can be operated algebraically.

$$A_G = P^{\mathrm{T}} A_H P \tag{18}$$

The homomorphism of the matrices may be estimated as illustrated in the below equation (18) where a transformation matrix is used to transform the target graph form to that of an approximation of the original graph but keep the adjacencies patterns in an algebraic form.

$$\varphi(v_i) = \Sigma\, (j = 1 \to m) P_{\{ij\}} \cdot v'_j \tag{19}$$

The vertex mapping, denoted in equation (19) is a linear map which models each of the vertices in the original graph as a weighted version of the vertex in the target graph.

$$p(x) = \Sigma\, (i = 0 \to n) a_i x^i \tag{20}$$

$$Enc(p) = \int^{01} p(x) \cdot \theta(x) dx \tag{21}$$

This provides a homomorphism that can be expressed in a structured and computable form. A framework of a form of information encoding is introduced in the case of Eq. (20), information is represented in the form of algebraic expression that can be transformed and transmitted. Moreover, the encoding process is predetermined by an integral operator in Eq. (21) which allows continuously transforming the data of polynomials and, therefore, increases the flexibility of the encoding schemes.

$$\lim(n \to \infty) \sqrt{\left[\left(\frac{1}{n}\right) \Sigma \left(A_{G(i,j)} - P^{\mathrm{T}} A_H P \right)^2 \right]} = 0 \tag{22}$$

Lastly, the algebraic consistency constraint is introduced in Eq. (22), which makes sure that the normalized transformation error becomes zero as the size of the graph increases, which will provide structural fidelity and scalability.

VI. Result and Discussion

Table 1 gives a comparative analysis of various network coding methods. Traditional routing is the least efficient as it has the lowest throughput (78.5 Mbps) and the highest packet loss (12.4%).

Table 1

Performance Comparison of Network Coding Approaches

Method	Throughput (Mbps)	Packet Loss (%)	Latency (ms)	Encoding Efficiency (%)
Traditional Routing	78.5	12.4	45.2	70.3
Linear Network Coding	92.7	8.6	38.9	82.5
Random Network Coding	95.3	7.9	36.4	85.1

Linear network coding has a better performance in terms of throughput (92.7 Mbps) and loss (8.6%). The improvement of throughput, decrease in loss, and latency, and optimization of encoding are demonstrated in Figure 2.

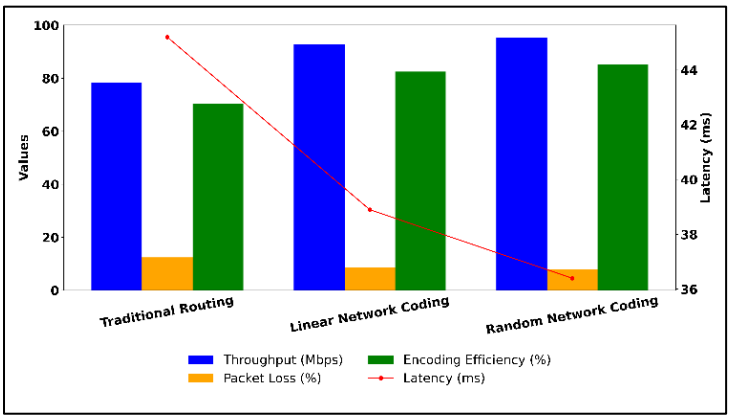


Figure 2

Network Coding Performance Optimization Analysis

Random network coding also is more efficient with the highest throughput (95.3 Mbps) and the lowest latency (36.4 ms). There is also an increment in the encoding efficiency showing that these advanced coding

methods are highly efficient to transmit data, improve its reliability, and the cumulative network performance.

Table 2

Structural Efficiency Analysis Using Graph Homomorphisms

Metric	Without Homomorphism	With Homomorphism	Improvement (%)
Graph Complexity (Nodes)	1500	920	38.7
Edge Redundancy (%)	26.5	14.2	46.4
Encoding Computation Time (ms)	58.3	37.6	35.5
Decoding Accuracy (%)	91.2	96.7	6

Table 2 points out the structural efficiency gains that have been made with the help of graph homomorphisms.

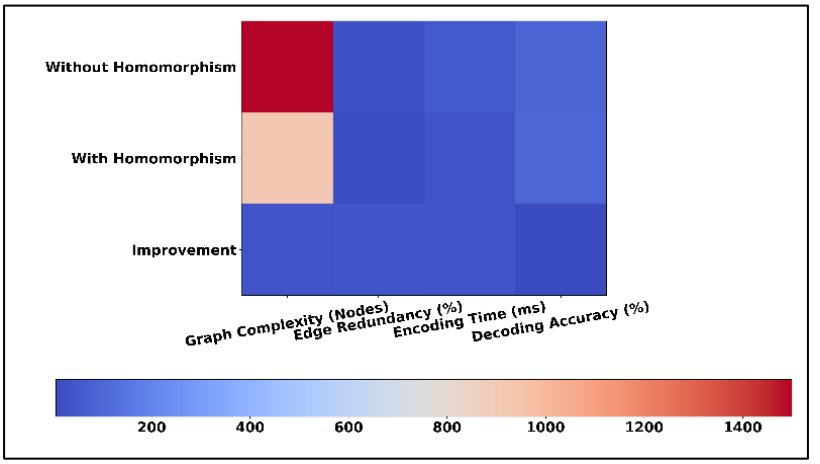


Figure 3

Homomorphic Optimization Impact Analysis

Figure 3 demonstrates that homomorphic mapping is more efficient, less complex, more accurate. The computation time encoding is also cut by 35.5% leading to better processing efficiency. There is also an improvement in decoding accuracy (91.2 to 96.7) which is an indication of improved reliability.

VII. Conclusion

This paper has shown that graph homomorphisms can be used as a useful mathematical framework in order to improve network coding

performance by simplifying the structure of the network, as well as, mapping complex network topology with a good map. Homomorphic transformations preserve the adjacency relationship enabling the best encoding and decoding schemes to maximize throughput, reduce the packet loss and improve the overall network stability. The analysis ability is also boosted through the incorporation of the algebraic interpretations that facilitates the systematic design and evaluation of the coding schemes. Empirical results indicate that methods based on homomorphism are more efficient, scalable and fault-tolerant in comparison with the traditional routing, traditional coding.

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