Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 1, pp. 139–153
DOI : 10.47974/JDMSC-2313

On purity of ideals of n-ary semigroups

Patchara Pornsurat [§]
Pakorn Palakawong na Ayutthaya [†]
Bundit Pibaljommee *
Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002
Thailand

Abstract

We begin by defining interior ideals of n-ary semigroups, followed by the introduction of j-pure ideals and interior pure ideals within this algebraic framework. Subsequently, we provide characterizations of softly left regular, softly right regular, and softly intra regular n-ary semigroups ($n \geq 3$) through the analysis of various purity conditions on their associated ideals.

Subject Classification (2020): *20N15, 20M12.*

Keywords: *n-ary semigroup, j-pure ideal, Regular n-ary semigroup.*

1. Introduction

A semigroups, a nonempty set with an associative binary operation, fundamental algebraic structures, serving as building blocks for more complex algebraic systems. One method to generalize the algebraic structure of a semigroup involves replacing the binary operation with an n-ary operation. "The algebraic structure with an n-ary operation was first studied by Kasner [11] in 1904". Later, in 1928, Dörnte [6] investigated the notion of an "n-ary group". On n-ary semigroups, "Sioson [14] studied properties of i-ideals of n-ary semigroups, introduced the notion of regular n-ary semigroups, and characterized regular n-ary semigroups using of their ideals". In 1980, "Dudek and Groździńska [7] gave more results of regular n-ary semigroups". Moreover, a new kind of elements, namely, "potent elements of an n-ary semigroup where

[§] *E-mail:* **p.patchara@kkumail.com**
[†] *E-mail:* **pakopa@kku.ac.th**
* *E-mail:* **banpib@kku.ac.th (Corresponding Author)**

$n \geq 3$ was introduced and studied by Dudek [10]". In 2020, Daengsaen and Leeratanavalee [4] first introduced the concepts of "softly left regularity, softly right regularity, and softly intra-regularity of ordered n-ary semihypergroups and also investigated their related properties". Furthermore, in 2021, "Daengsaen, Leeratanavalee, and Davvaz [5] defined the notion of a j-hyperideal of an n-ary semihypergroup, investigated the maximality [minimality] of j-hyperideals, and gave a construction of the j-hyperideal generated by a nonempty subset of an n-ary semihypergroup". Moreover, research on N-ary relation fuzzy-soft sets provides another perspective within the fuzzy-soft set framework [2]. During that year, Pornsurat, Palakawong na Ayutthaya and Pibaljommee [13] also gave the construction of the canonical j-ideal derived from a nonempty subset of "ordered n-ary semigroups". Concepts of purities are fascinating properties of ideals of semigroups that can be used as a valuable tool for characterizing various types of regularities within semigroup structures. Several algebraic properties related to pure ideals and purely prime ideals in semigroups were studied by Ahsan and Takahashi [1] following their introduction of these notions in 1989. Subsequently, Changphas and Sanborisoot [3] extended the concept of pure ideals by defining left [right] pure and left [right] weakly pure ideals within ordered semigroups, providing characterizations for these new structures.

In this work, we first examine the concepts of j-pure and interior pure ideals, then explore the connections between j-pure ideals and i-ideals. Finally, we utilize these purity properties to provide complete characterizations of softly left regular, softly right regular, and softly intra-regular n-ary semigroups for $n \geq 3$.

2. Preliminaries

Let $n \in \mathbb{N}$ and $[n] = \{1, \dots, n\}$. The pair $(S,\circ)$ constitutes an "n-*ary groupoid*" provided S is a nonempty set and $\circ: S^n \to S$ is an n-ary operation, with $n \geq 2$. Given $i, j \in [n]$ with $i \leq j$. The block $t_i, \dots, t_j \in S$ is written as t_i^j. If $t_i = t_{i+1} = \cdots = t_j = t$, the block is written as $\overset{j-i+1}{t}$. Additionally, $\{\overset{i}{t}, \overset{k}{z}\}^j$ means $\underbrace{\overset{i}{t}, \overset{k}{z}, \dots, \overset{i}{t}, \overset{k}{z}}_{j \text{ terms}}$. We set the notations $\overset{0}{t}$ and t_i^j where $j < i$ as the empty symbols.

For example, $\circ\,(t_1, t_2, \dots, t_i, \underbrace{y, \dots, y}_{j \text{ terms}}, t_{i+j+1}, \dots, t_n)$ can be written by $\circ\,(t_1^i, \overset{j}{y}, t_{i+j+1}^n)$.

In the case of sequences of nonempty subsets of S, we also designate their notations as those of sequences of elements defined above. For example, if $Y_1, \dots, Y_n$ is the sequence of nonempty subsets of an n-ary groupoid $(S,\circ)$, then it is simply written as Y_1^n. So, we define

$$\circ\,(Y_1^n) = \{\circ\,(t_1^n) | t_i \in Y_i, i \in [n]\}.$$

For $i \in [n]$, if $Y_i = \{t_i\}$, we replace the singleton in the i-th argument by t_i, yielding $f(Y_1^{i-1}, t_i, Y_{i+1}^n)$.

An n-ary operation $\circ$ on $(S,\circ)$ is *associative* if, no matter which position the inner operation $\circ$ is applied, the result is the same. Formally, for any $i, j \in [n], i \leq j$ and $t_1, \dots, t_{2n-1} \in S$, it must fulfill the identity:

$$\circ\,(t_1^{i-1}, \circ\,(t_i^{n+i-1}), t_{n+i}^{2n-1}) = \circ\,(t_1^{j-1}, \circ\,(t_j^{n+j-1}), t_{n+j}^{2n-1}).$$

"An n-*ary semigroup* is an n-ary groupoid $(S,\circ)$ where $\circ$ is associative". Let $m \in \mathbb{N}$ with $m \geq 2$. We define the map $\circ_m$ by

$$\circ_m\,(t_1^{m(n-1)+1}) = \underbrace{\circ\,(\circ\,(\dots, \circ\,(\circ}_{m\text{ terms}}\,(t_1^n), t_{n+1}^{2n-1}), \dots), t_{(m-1)(n-1)+2}^{m(n-1)+1})$$

for any $t_1, \dots, t_{j(n-1)+1} \in S$.

Let $(S,\circ)$ be an n-ary semigroup. We set the notation $\circ_0\,(t) = t$ and $\circ_0\,(Z) = Z$ for any $t \in S$, $\emptyset \neq Z \subseteq S$, respectively.

We adopt the convention that the n-ary semigroup $(S,\circ)$ is denoted by S. By definition, $\emptyset \neq Y \subseteq S$ is an n-ary subsemigroup of S if $\circ\,(a_1^n) \in Y$ whenever $a_1^n \in Y$. Many notations occurring above can be found in [7, 8, 9, 10].

Definition 2.1: "A nonempty subset I of S is designated an i-ideal if $\circ\,(\overset{i-1}{S}, I, \overset{n-i}{S}) \subseteq I$". The subset I qualifies as an ideal when it is simultaneously an i-ideal for all positions $i \in [n]$.

Now, we denote $ID(S)$ the set of all ideals of S.

Let $\emptyset \neq Y \subseteq S$. For any $i \in [n]$, the i-ideal of S generated by Y, denoted $J_i(Y)$, is *the smallest i-ideal of S which contains Y* (equivalently, $J_i(Y) = \cap\,\{K \in ID(S) | Y \subseteq K\}$). If $Y = \{t\}$, we write $J_i(t)$.

"An n-ary semigroup structurally embeds into an n-ary semihypergroup provided its induced hyperoperation yields only singleton sets". Moreover, every n-ary semigroup naturally satisfies the definition of an ordered n-ary semigroup when the partial order relation is taken to be equality. So, by constructions of "the i-hyperideal in an n-ary semihypergroup [4] and of the i-ideal in an ordered n-ary semigroup [13] which are generated by its nonempty subsets", the ensuing theorem is established.

Theorem 2.2: *Let $\emptyset \neq Y \subseteq S$. Then*

$$J_i(Y) = \bigcup_{\rho=1}^{n-1} \circ_\rho\,(\overset{\rho(i-1)}{S}, Y, \overset{\rho(n-i)}{S}) \cup Y.$$

The following remark can be obtained as a special case of its extensions in [4] and [13].

Remark 2.3: Let $\emptyset \neq Y \subseteq S$ and $i \in [n]$. Then

$$\bigcup_{\rho=1}^{n-1} \circ_\rho\,(\overset{\rho(i-1)}{S}, Y, \overset{\rho(n-i)}{S})$$

is an i-ideal of S.

Corollary 2.4: *Let $\emptyset \neq Y \subseteq S$. Then the results are established.*

(i) $J_1(Y) = \circ\,(Y, \overset{n-1}{S}) \cup Y$.

(ii) $J_n(Y) = \circ\,(\overset{n-1}{S}, Y) \cup Y$.

(iii) *If $i \in [n]$ and $i = \frac{n+1}{2}$ is an integer, then*

$$J_i(Y) = \circ\,(\overset{i-1}{S}, Y, \overset{n-i}{S}) \cup \circ_2 (\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup Y.$$

Given $\emptyset \neq Y \subseteq S$, *the ideal of S generated by Y*, denoted $J(Y)$, is the least ideal of S that contains Y. We abbreviate $J(\{t\})$ as $J(t)$. Its construction is investigated as follows.

Theorem 2.5: *Let $\emptyset \neq Y \subseteq S$. Then*

$$J(Y) = \bigcup_{\rho=1}^{n} \circ\,(\overset{\rho-1}{S}, Y, \overset{n-\rho}{S}) \cup \circ_2 (\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup Y.$$

Proof: Let $\emptyset \neq Y \subseteq S$ and $\mathcal{J} = \bigcup_{\rho=1}^{n} \circ\,(\overset{\rho-1}{S}, Y, \overset{n-\rho}{S}) \cup \circ_2 (\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup Y$. First, we show that $\mathcal{J} \in ID(S)$. We assert that $\circ\,(\overset{i-1}{S}, \mathcal{J}, \overset{n-i}{S}) = \bigcup_{\rho=1}^{n} \circ_2 (\overset{i+\rho-2}{S}, Y, \overset{2n-i-\rho}{S}) \cup \circ_3 (\overset{n+i-2}{S}, Y, \overset{2n-i-1}{S}) \cup \circ\,(\overset{i-1}{S}, Y, \overset{n-i}{S})$. Our argument now branches into the cases below.

Case: either $i = 1$ or $i = n$. Assume that $i = 1$. Then

$$\begin{aligned}
\circ\,(\mathcal{J}, \overset{n-i}{S}) &= \bigcup_{\rho=1}^{n} \circ_2 (\overset{\rho-1}{S}, Y, \overset{2n-1-\rho}{S}) \cup \circ_3 (\overset{n-1}{S}, Y, \overset{2n-2}{S}) \cup \circ\,(Y, \overset{n-1}{S}) \\
&= \bigcup_{\rho=1}^{n} \circ\,(\overset{\rho-1}{S}, Y, \circ(\overset{n}{S}), \overset{n-1-\rho}{S}) \cup \circ_2 (\overset{n-1}{S}, Y, \circ(\overset{n}{S}), \overset{n-2}{S}) \cup \circ\,(Y, \overset{n-1}{S}) \\
&\subseteq \bigcup_{\rho=1}^{n} \circ\,(\overset{\rho-1}{S}, Y, \overset{n-\rho}{S}) \cup \circ_2 (\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup \circ\,(Y, \overset{n-1}{S}) \\
&\subseteq \mathcal{J}.
\end{aligned}$$

If $i = n$, then the proof is analogous.

Case: $i \neq 1$ and $i \neq n$. If $i + \rho - 2 \geq n$ and $2n - i - \rho < n$, then

$$\begin{aligned}
&\bigcup_{\rho=1}^{n} \circ_2 (\overset{i+\rho-2}{S}, Y, \overset{2n-i-\rho}{S}) \cup \circ_3 (\overset{n+i-2}{S}, Y, \overset{2n-i-1}{S}) \cup \circ\,(\overset{i-1}{S}, Y, \overset{n-i}{S}) \\
&= \bigcup_{\rho=1}^{n} \circ\,(\circ(\overset{n}{S}), \overset{i+\rho-2-n}{S}, Y, \overset{2n-i-\rho}{S}) \cup \circ\,(\overset{i-2}{S}, \circ(\overset{n}{S}), Y, \circ(\overset{n}{S}), \overset{n-i-1}{S}) \cup \circ\,(\overset{i-1}{S}, Y, \overset{n-i}{S}) \\
&\subseteq \bigcup_{\rho=1}^{n} \circ\,(\overset{i+\rho-1-n}{S}, Y, \overset{2n-i-\rho}{S}) \cup \circ\,(\overset{i-1}{S}, Y, \overset{n-i}{S}) \\
&\subseteq \mathcal{J}.
\end{aligned}$$

If $i + \rho - 2 < n$ and $2n - i - \rho \geq n$, then the result follows analogously.

If instead $2n - i - \rho < n$, then the condition $i + \rho - 2 < n$ implies $i + \rho = n + 1$. So,

$$
\begin{aligned}
&\bigcup_{\rho=1}^{n} \circ_2 (\overset{i+\rho-2}{S}, Y, \overset{2n-i-\rho}{S}) \cup \circ_3 (\overset{n+i-2}{S}, Y, \overset{2n-i-1}{S}) \cup \circ (\overset{i-1}{S}, Y, \overset{n-i}{S}) \\
&= \circ_2 (\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup \circ (\overset{i-2}{S}, \circ(\overset{n}{S}), Y, \circ(\overset{n}{S}), \overset{n-i-1}{S}) \cup \circ (\overset{i-1}{S}, Y, \overset{n-i}{S}) \\
&\subseteq \circ (\overset{i-1}{S}, Y, \overset{n-i}{S}) \\
&\subseteq \mathcal{J}.
\end{aligned}
$$

Now, we have established that $\mathcal{J} \in ID(S)$ and contains Y. To show it is the smallest such ideal, let K be any other ideal of S with $Y \subseteq J$. Since $K \in ID(S)$ and by the meaning of $\mathcal{J}$, we arrive $\mathcal{J} \subseteq K$. Therefore, $\mathcal{J} \in ID(S)$ which is generated by Y.

The following definition could be considered as a particular case of one defined on ordered n-ary semihypergroups [5].

Definition 2.6: An element $s \in S$ is called

(i) *softly left-regular* if $s = \circ_2 (t_1^{n-1}, s, t_n^{2n-3}, s)$ for some $t_1^{2n-3} \in S$,

(ii) *softly right-regular* if $s = \circ_2 (s, t_1^{n-2}, s, t_{n-1}^{2n-3})$ for some $t_1^{2n-3} \in S$,

(iii) *softly intra-regular* if $s = \circ_3 (t_1^{n-1}, s, t_n^{2n-3}, s, t_{2n-2}^{3n-4})$ for some $t_1^{3n-4} \in S$.

The semigroup S is *softly regular* (in all three senses: *left, right, and intra-regular*) if and only if every element in S is also softly regular in those respective manners.

We note that in a ternary semigroup [12], softly left (right) regularity and lightly left (right) regularity are coincidence.

The ensuing remark is a direct consequence of Definition 2.6.

Remark 2.7: For any given n-ary semigroup S, the assertions below are valid.

(i) S is softly left regular iff $s \in \circ_2 (\overset{n-1}{S}, s, \overset{n-2}{S}, s)$ for each $s \in S$.

(ii) S is softly right regular iff $s \in \circ_2 (s, \overset{n-2}{S}, s, \overset{n-1}{S})$ for each $s \in S$.

(iii) S is softly intra-regular iff $s \in \circ_3 (\overset{n-1}{S}, s, \overset{n-2}{S}, s, \overset{n-1}{S})$ for each $s \in S$.

Evidently, softly left-regular or softly right-regular n-ary semigroups are softly intra-regular.

3. Main results

Initializing our study, we define interior ideals within n-ary semigroups ($n \geq 3$). Also, we formalize the concepts of j-pure and interior pure ideals. Using these tools, we characterize the structure of n-ary semigroups that are softly left, right, and intra-regular in terms of their ideal purity.

Definition 3.1: Let $\emptyset \neq E \subseteq S$. Then E is called an interior ideal of S if $\circ_2(\overset{n-1}{S}, E, \overset{n-1}{S}) \subseteq E$.

Here, we set $Int(S)$ the set of all interior ideals of S. Let $\emptyset \neq Y \subseteq S$. Denote $I(Y)$ the interior ideals of S generated by Y, i.e., $I(Y) = \cap \{M \in Int(S) | Y \subseteq M\}$. We give the construction of $I(Y)$ as follows.

Theorem 3.2: *Let $\emptyset \neq Y \subseteq S$. Then $I(Y) = Y \cup \circ_2(\overset{n-1}{S}, Y, \overset{n-1}{S})$.*

Proof: Let $\emptyset \neq Y \subseteq S$ and $\mathcal{I} = Y \cup \circ_2(\overset{n-1}{S}, Y, \overset{n-1}{S})$. First, we demonstrate $\mathcal{I}$ an interior ideal of S. We consider

$$\begin{aligned}
\circ_2(\overset{n-1}{S}, \mathcal{I}, \overset{n-1}{S}) &= \circ_2(\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup \circ_4(\overset{2n-2}{S}, Y, \overset{2n-2}{S}) \\
&= \circ_2(\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup \circ_2(\circ(\overset{n}{S}), \overset{n-2}{S}, Y, \overset{n-2}{S}, \circ(\overset{n}{S})) \\
&\subseteq \circ_2(\overset{n-1}{S}, Y, \overset{n-1}{S}) \cup \circ_2(\overset{n-1}{S}, Y, \overset{n-1}{S}) \\
&\subseteq \mathcal{I}.
\end{aligned}$$

So, $\mathcal{I} \in Int(S)$. Next, let $K \in Int(S)$ with $Y \subseteq K$. Then $Y \cup \circ_2(\overset{n-1}{S}, Y, \overset{n-1}{S}) \subseteq K$. Therefore, $I(Y) = \mathcal{I}$.

Definition 3.3: Let $j \in [n]$ and $E \in ID(S)$. Then E is called j-pure ideal if $t \in \circ_{n-1}(\overset{(n-1)(j-1)}{E}, t, \overset{(n-1)(n-j)}{E})$ for any $t \in E$.

We say that $A \in ID(S)$ is defined as a *1-pure ideal* if $t \in \circ(t, \overset{n-1}{A})$ for any $t \in A$, and analogously, it is an *n-pure ideal* if $t \in \circ(\overset{n-1}{A}, t)$ for any $t \in A$.

Throughout the remainder, S is taken an n-ary semigroup with $n \geq 3$. Here, we present a characterization of an n-pure ideal of S in terms of its n-ideals, given below as Lemma 3.4.

Lemma 3.4: *Let $Z \in ID(S)$. Then Z is n-pure iff $E \cap L \subseteq \circ(\overset{n-1}{Z}, L)$ for any n-ideal L of S.*

Proof: Let $Z \in ID(S)$ be n-pure and L an n-ideal of S and $t \in Z \cap L$. Then $t \in \circ(\overset{n-1}{Z}, L)$.

Conversely, assume that $Z \cap L \subseteq \circ(\overset{n-1}{Z}, L)$ for any n-ideal L of S. Let $c \in Z$. By Corollary 2.4, we arrive

$$\begin{aligned}
t &\in Z \cap J_n(c) \\
&= \circ(\overset{n-1}{Z}, J_n(c)) \\
&= \circ(\overset{n-1}{Z}, \circ(\overset{n-1}{S}, c) \cup t)
\end{aligned}$$

$$\begin{aligned} &= \circ(\overset{n-1}{Z}, \circ(\overset{n-1}{S}, c)) \cup \circ(\overset{n-1}{Z}, c) \\ &= \circ(\overset{n-2}{Z}, \circ(Z, \overset{n-1}{S}), c) \cup \circ(\overset{n-1}{Z}, c) \\ &\subseteq \circ(\overset{n-1}{Z}, c). \end{aligned}$$

Hence, Z is an n-pure.

Following the duality implied by the lemma, a characterization is presented: An ideal is 1-pure in an n-ary semigroup precisely via its 1-ideals.

Lemma 3.5: *Let $I \in ID(S)$. Then I is 1-pure iff $I \cap R \subseteq \circ(R, \overset{n-1}{I})$ for any 1-ideal R of S.*

Definition 3.6: Let $J \in ID(S)$. Then J is called interior pure if $t \in \circ_2(\overset{n-1}{J}, d, \overset{n-1}{J})$ for any $d \in J$.

Next lemma provides a characterization of interior pure ideals based on interior ideals in S.

Lemma 3.7: *Let $C \in ID(S)$. Then C is interior pure iff $C \cap Y \subseteq \circ_2(\overset{n-1}{C}, Y, \overset{n-1}{C})$ for any $Y \in Int(S)$ of S.*

Proof: Let C be interior pure. Let $Y \in Int(S)$ and $t \in C \cap Y$. By assumption, we get that $t \in \circ_2(\overset{n-1}{C}, Y, \overset{n-1}{C})$.

Conversely, assume that $C \cap Y \subseteq \circ_2(\overset{n-1}{C}, Y, \overset{n-1}{C})$ for any $Y \in Int(S)$ of S. Let $t \in C$. By Theorem 3.2, we arrive that

$$\begin{aligned} t &\in C \cap I(t) \\ &= \circ_2(\overset{n-1}{C}, t \cup \circ_2(\overset{n-1}{S}, t, \overset{n-1}{S}), \overset{n-1}{C}) \\ &= \circ_2(\overset{n-1}{C}, t, \overset{n-1}{C}) \cup \circ_4(\overset{2(n-1)}{C}, \overset{n-1}{S}, t, \overset{n-1}{S}, \overset{n-1}{C}) \\ &= \circ_2(\overset{n-1}{C}, t, \overset{n-1}{C}) \cup \circ_2(\overset{n-2}{C}, \circ(C, \overset{n-1}{S}), t, \circ(\overset{n-1}{S}, C), \overset{n-2}{C}) \\ &\subseteq \circ_2(\overset{n-1}{C}, t, \overset{n-1}{C}) \cup \circ_2(\overset{n-1}{C}, t, \overset{n-1}{C}) \\ &= \circ_2(\overset{n-1}{C}, t, \overset{n-1}{C}). \end{aligned}$$

Hence, C is interior pure.

We next present a theorem that characterizes a softly left-regular n-ary semigroups using its n-pure ideals.

Theorem 3.8: *S is softly left regular iff any ideal of S is n-pure.*

Proof: Let S be softly left regular. Let $Z \in ID(S)$ and L be an n-ideal of S. Let $t \in Z \cap L$. By assumption, we get that

$$\begin{aligned}
t &\in \circ_2(\overset{n-1}{S}, t, \overset{n-2}{S}, t) \\
&\subseteq \circ_2(\overset{n-1}{S}, \circ_2(\overset{n-1}{S}, t, \overset{n-2}{S}, t), \overset{n-2}{S}, t) \\
&= \circ_4(\overset{2n-2}{S}, \{t, \overset{n-2}{S}\}^2, t) \\
&\ \ \vdots \\
&\subseteq \circ_{2n-2}(\overset{n^2-2n+1}{S}, \{t, \overset{n-2}{S}\}^{n-1}, t) \\
&= \circ(\circ_{n-1}(\overset{n^2-2n+1}{S}, t), \circ(\overset{n-2}{S}, t, S), \circ(\overset{n-3}{S}, t, \overset{2}{S}), \ldots, \circ(\overset{2}{S}, t, \overset{n-3}{S}), \circ(S, t, \overset{n-2}{S}), t) \\
&\subseteq \circ(\overset{n-1}{Z}, L).
\end{aligned}$$

So, $Z \cap L \subseteq \circ(\overset{n-1}{Z}, L)$. By Lemma 3.4, Z is n-pure.

For the converse, we consider the case where every ideal of S is n-pure. Let $c \in S$. By assumption, Corollary 2.4, Theorem 2.5 and Lemma 3.4, it turns out that

$$\begin{aligned}
c &\in J(z) \cap J_n(c) \\
&\subseteq \circ(\overset{n-1}{J(c)}, J_n(z)) \\
&\subseteq \circ(J(c), \overset{n-2}{S}, J_n(c)) \\
&= \circ\left(\bigcup_{\rho=1}^{n} \circ(\overset{\rho-1}{S}, Y, \overset{n-\rho}{S}) \cup \circ_2(\overset{n-1}{S}, c, \overset{n-1}{S}) \cup c, \overset{n-2}{S}, \circ(\overset{n-1}{S}, c) \cup c\right)
\end{aligned}$$

These six cases are considered.

Case 1: If $z \in \circ(z, \overset{n-2}{S}, z)$, then $c \in \circ(c, \overset{n-2}{S}, z) \subseteq \circ(\circ(c, \overset{n-2}{S}, c), \overset{n-2}{S}, c) \subseteq \circ_2(\overset{n-1}{S}, c, \overset{n-2}{S}, c)$.

Case 2: If $c \in \circ(c, \overset{n-2}{S}, \circ(\overset{n-1}{S}, c))$, then $c \in \circ(c, \circ(\overset{n}{S}), \overset{n-3}{S}, c)) \subseteq \circ(c, \overset{n-2}{S}, c) \subseteq \circ_2(\overset{n-1}{S}, c, \overset{n-2}{S}, c)$.

Case 3: If $c \in \circ(\circ_2(\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{n-2}{S}, c)$, then $c \in \circ_2(\overset{n-1}{S}, c, \circ(\overset{n}{S}), \overset{n-3}{S}, c) \subseteq \circ_2(\overset{n-1}{S}, c, \overset{n-2}{S}, c)$.

Case 4: If $c \in \circ(\circ_2(\overset{n-1}{S}, z, \overset{n-1}{S}), \overset{n-2}{S}, \circ(\overset{n-1}{S}, c))$, then $c \in \circ_2(\overset{n-1}{S}, c, \circ_2(\overset{2n-1}{S}), \overset{n-3}{S}, c) \subseteq \circ_2(\overset{n-1}{S}, c, \overset{n-2}{S}, c)$.

Case 5: If $c \in \circ(\bigcup_{\rho=1}^{n} \circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{n-2}{S}, c)$, then $c \in \circ(\circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{n-2}{S}, c)$ for some $j \in [n]$.

If $\rho \neq 1$, then

$$\begin{aligned}
z &\in \circ_2 (\overset{\rho-1}{S}, c, \overset{2n-\rho-2}{S}, c) \\
&\subseteq \circ_2 (\overset{\rho-1}{S}, \circ_2 (\overset{\rho-1}{S}, c, \overset{2n-\rho-2}{S}, c), \overset{2n-\rho-2}{S}, c) \\
&= \circ_4 (\overset{2(\rho-1)}{S}, \{c, \overset{2n-\rho-2}{S}\}^2, c) \\
&\vdots \\
&\subseteq \circ_{2n-2} (\overset{(n-1)(\rho-1)}{S}, \{c, \overset{2n-\rho-2}{S}\}^{n-1}, c) \\
&= \circ_{2n-2} (\overset{(n-1)(\rho-1)}{S}, c, \overset{2n-\rho-2}{S}, \{c, \overset{2n-\rho-2}{S}\}^{n-2}, c) \\
&= \circ_2 (\circ_{\rho-2} (\overset{\rho n-\rho-2n+3}{S}), \overset{n-2}{S}, c, \circ_{2n-2-\rho} (\overset{2n^2-4n-\rho n+\rho+3}{S}), \overset{n-3}{S}, c) \\
&\subseteq \circ_2 (\overset{n-1}{S}, c, \overset{n-2}{S}, c).
\end{aligned}$$

If $\rho = 1$, then we also obtain that

$$\begin{aligned}
c &\in \circ_2 (c, \overset{2n-3}{S}, z)) \\
&\subseteq \circ_{2n-2} (\{c, \overset{2n-3}{S}\}^{n-1}, c) \\
&= \circ_{2n-2} (\{c, \overset{2n-3}{S}\}^{n-2}, c, \overset{2n-3}{S}, c) \\
&= \circ_2 (\circ_{2n-5} (\overset{2n^2-7n+6}{S}), \overset{n-2}{S}, c, \circ (\overset{n}{(S)}), \overset{n-3}{S}, c) \\
&\subseteq \circ_2 (\overset{n-1}{S}, z, \overset{n-2}{S}, c).
\end{aligned}$$

Case 6: If $z \in \circ (\circ (\overset{\rho-1}{S}, Y, \overset{n-\rho}{S}), \overset{n-2}{S}, \circ (\overset{n-1}{S}, c))$, then the proof proceeds analogously to the previous cases proofs.

The conditions established in Cases 1 to 6, supported by Remark 2.7, are sufficient to prove that S is softly left regular.

From the dual perspective of Theorem 3.8, one adopts the concept of a 1-pure ideal to characterize a softly right regular n-ary semigroup as follows.

Theorem 3.9: *S is softly right regular iff every ideal of S is 1-pure.*

The following result is given as a characterization of a softly intra-regular n-ary semigroup using its interior pure ideals.

Theorem 3.10: *S is softly intra-regular iff any ideal of S is interior pure.*

Proof: Let S be softly intra-regular, $I \in ID(S)$ and $Z \in Int(S)$. Let $c \in I \cap Z$. By assumption, we arrive that

$$c \in \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S})$$

$$
\begin{aligned}
&\subseteq \circ_3 (\overset{n-1}{S}, \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}), \overset{n-2}{S}, c, \overset{n-1}{S})\\
&= \circ_6 (\overset{2(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^2)\\
&\ \vdots\\
&\subseteq \circ_{3n} (\overset{n(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^n)\\
&= \circ_{3n} (\overset{n(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^{n-1}, \overset{n-2}{S}, c, \overset{n-1}{S})\\
&\subseteq \circ_{3n} (\overset{n(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^{n-1}, \overset{n-2}{S}, \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}), \overset{n-1}{S})\\
&= \circ_{3n+3} (\overset{n(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^{n-1}, \overset{n-2}{S}, \overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{2(n-1)}{S})\\
&\subseteq \circ_{3n+3} (\overset{n(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^{n-1}, \overset{n-2}{S}, \overset{n-1}{S}, c, \overset{n-2}{S}, \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}), \overset{2(n-1)}{S})\\
&= \circ_{3n+6} (\overset{n(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^{n-1}, \overset{n-2}{S}, \{\overset{n-1}{S}, c, \overset{n-2}{S}\}^2, c, \overset{3(n-1)}{S})\\
&\ \vdots\\
&\subseteq \circ_{6n-3} (\overset{n(n-1)}{S}, c, \{\overset{n-2}{S}, c, \overset{n-1}{S}\}^{n-1}, \overset{n-2}{S}, \{\overset{n-1}{S}, c, \overset{n-2}{S}\}^{n-1}, c, \overset{n(n-1)}{S})\\
&= \circ_2 (\circ_n (\overset{n^2-n}{S}, c), \circ (\overset{n-2}{S}, c, S), \circ_2 (\overset{2n-4}{S}, c, \overset{2}{S}), \circ_2 (\overset{2n-5}{S}, c, \overset{3}{S}), \dots, \circ_2 (\overset{n}{S}, c, \overset{n-2}{S}),\\
&\circ_5 (\overset{n-1}{S}, c, \overset{n-1}{S}, \overset{n-2}{S}, \overset{n-1}{S}, c, \overset{n-1}{S}), \circ_2 (\overset{n-2}{S}, c, \overset{n}{S}), \dots, \circ_2 (\overset{3}{S}, c, \overset{2n-5}{S}), \circ_2 (\overset{2}{S}, c, \overset{2n-4}{S}),\\
&\circ (S, c, \overset{n-2}{S}), \circ_n (c, \overset{n^2-n}{S}))\\
&\subseteq \circ_2 (\overset{n-1}{I}, Z, \overset{n-1}{I}).
\end{aligned}
$$

So, $I \cap Z \subseteq \circ_2 (\overset{n-1}{I}, Z, \overset{n-1}{I})$. By Lemma 3.7, I is interior pure.

Conversely, assume that all ideals of S are interior pure. Let $c \in S$. By assumption, Lemma 3.7, Theorem 2.5 and 3.2, it provides that

$$
\begin{aligned}
c &\in J(c) \cap I(c)\\
&\subseteq \circ_2 (\overset{n-1}{J(c)}, I(c), \overset{n-1}{J(c)})\\
&\subseteq \circ_2 (J(c), \overset{2n-3}{S}, J(c))\\
&= \circ_2 (\textstyle\bigcup_{\rho=1}^{n} \circ (\overset{\rho-1}{S}, c, \overset{n-\rho}{S}) \cup \circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}) \cup c, \overset{2n-3}{S}, \bigcup_{\beta=1}^{n} \circ\\
&\quad (\overset{\beta-1}{S}, c, \overset{n-\beta}{S}) \cup \circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}) \cup c)
\end{aligned}
$$

These nine cases are considered.

Case 1: If $c \in \circ_2 (c, \overset{2n-3}{S}, c)$, then

$$
\begin{aligned}
c \in \circ_2 (c, \overset{2n-3}{S}, c) &\subseteq \circ_2 (\circ_2 (c, \overset{2n-3}{S}, c), \overset{2n-3}{S}, \circ_2 (c, \overset{2n-3}{S}, c))\\
&= \circ_3 (\circ (c, \overset{n-1}{S}), \overset{n-2}{S}, c, \circ (\overset{n}{S}), \overset{n-3}{S}, c, \overset{n-2}{S}, \circ (\overset{n-1}{S}, c))\\
&\subseteq \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}).
\end{aligned}
$$

Case 2: If $c \in \circ_2(c, \overset{2n-3}{S}, \circ_2(\overset{n-1}{S}, c, \overset{n-1}{S}))$, then

$$\begin{aligned}
c &\in \circ_2(c, \overset{2n-3}{S}, \circ_2(\overset{n-1}{S}, c, \overset{n-1}{S}))\\
&\subseteq \circ_2(c, \overset{2n-3}{S}, \circ_2(\overset{n-1}{S}, \circ_2(c, \overset{2n-3}{S}, \circ_2(\overset{n-1}{S}, c, \overset{n-1}{S})), \overset{n-1}{S}))\\
&= \circ_3(\circ_2(c, \overset{2n-2}{S}), \overset{n-2}{S}, c, \circ_2(\overset{2n-1}{S}), \overset{n-3}{S}, c, \circ(\overset{n}{S}), \overset{n-2}{S})\\
&\subseteq \circ_3(\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}).
\end{aligned}$$

Case 3: If $c \in \circ_2(c, \overset{2n-3}{S}, \bigcup_{\beta=1}^{n} \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))$, then $c \in \circ_2(c, \overset{2n-3}{S}, \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))$ for some $\beta \in [n], \beta \neq n$. Then it proceeds

$$\begin{aligned}
c &\in \circ_2(c, \overset{2n-3}{S}, \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))\\
&= \circ_3(c, \overset{2n+\beta-4}{S}, c, \overset{n-\beta}{S})\\
&\subseteq \circ_3(c, \overset{2n+\beta-4}{S}, \circ_3(c, \overset{2n+\beta-4}{S}, c, \overset{n-\beta}{S}), \overset{n-\beta}{S})\\
&= \circ_6(\{c, \overset{2n+\beta-4}{S}\}^2, c, \overset{2(n-\beta)}{S})\\
&\;\vdots\\
&\subseteq \circ_{3n-3}(\{c, \overset{2n+\beta-4}{S}\}^{n-1}, c, \overset{(n-1)(n-\beta)}{S})\\
&= \circ_{3n-3}(\{c, \overset{2n+\beta-4}{S}\}^{n-2}, c, \overset{2n+\beta-4}{S}, c, \overset{(n-1)(n-\beta)}{S})\\
&\subseteq \circ_{3n-3}(\{c, \overset{2n+\beta-4}{S}\}^{n-2}, \circ_3(c, \overset{2n+\beta-4}{S}, c, \overset{n-\beta}{S}), \overset{2n+\beta-4}{S}, c, \overset{(n-1)(n-\beta)}{S})\\
&\subseteq \circ_{3n}(\overset{2n^2-5n+\beta n-\beta+3}{S}, c, \overset{3n-4}{S}, c, \overset{n^2-n-\beta n+\beta}{S})\\
&= \circ_3(\circ_{2n+\beta-4}(\overset{2n^2-6n+\beta n-\beta+5}{S}), \overset{n-2}{S}, c, \circ_2(\overset{2(n-1)+1}{S}), \overset{n-3}{S}, c, \circ_{n-\beta-1}\\
&\qquad (\overset{n^2-2n-\beta n+\beta+2}{S}), \overset{n-2}{S})\\
&\subseteq \circ_3(\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}).
\end{aligned}$$

If $\beta = n$, then

$$\begin{aligned}
c &\in \circ_3(c, \overset{3n-4}{S}, c))\\
&\subseteq \circ_6(\{c, \overset{3n-4}{S}\}^2, c)\\
&\;\vdots\\
&\subseteq \circ_{3n}(\{c, \overset{3n-4}{S}\}^n, c)\\
&= \circ_{3n}(\{c, \overset{3n-4}{S}\}^{n-2}, c, \overset{3n-4}{S}, c, \overset{3n-4}{S}, c)\\
&\subseteq \circ_3(\circ_{3n-7}(\overset{3n^2-10n+8}{S}), \overset{n-2}{S}, c, \circ_2(\overset{2n-1}{S}), \overset{n-3}{S}, c, \circ_2(\overset{2n-1}{S}), \overset{n-2}{S})\\
&\subseteq \circ_3(\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}).
\end{aligned}$$

Case 4: If $c \in \circ_2 (\circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{2n-3}{S}, c)$, then the proof proceeds analogously to the proof of Case 2.

Case 5: If $c \in \circ_2 (\circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{2n-3}{S}, \circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}))$, then $c \in \circ_6 (\overset{n-1}{S}, c, \overset{4n-5}{S}, c, \overset{n-1}{S}) = \circ_3 (\overset{n-1}{S}, c, \circ_3 (\overset{3n-2}{S}), \overset{n-3}{S}, c, \overset{n-1}{S}) \subseteq \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S})$.

Case 6: If $c \in \circ_2 (\circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{2n-3}{S}, \bigcup_{\beta=1}^{n} \circ (\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))$, then $c \in \circ_2 (\circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{2n-3}{S}, \circ (\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))$ for some $\beta \in [n]$.

If $\beta \neq n$, then it turns out that

$$
\begin{aligned}
c &\in \circ_2 (\circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{2n-3}{S}, \circ (\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))\\
&\subseteq \circ_2 (\circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{2n-3}{S}, \circ (\overset{\beta-1}{S}, \circ_2 (\circ_2 (\overset{n-1}{S}, c, \overset{n-1}{S}), \overset{2n-3}{S}, \circ (\overset{\beta-1}{S}, c, \overset{n-\beta}{S})), \overset{n-\beta}{S}))\\
&= \circ_{10} (\{\overset{n-1}{S}, c, \overset{n-1}{S}, \overset{2n-3}{S}, \overset{\beta-1}{S}\}^2, c, \overset{2(n-\beta)}{S})\\
&= \circ_{10} (\{\overset{n-1}{S}, c, \overset{3n+\beta-5}{S}\}^2, c, \overset{2(n-\beta)}{S})\\
&\ \ \vdots\\
&\subseteq \circ_{5n-5} (\{\overset{n-1}{S}, c, \overset{3n+\beta-5}{S}\}^{n-1}, c, \overset{(n-1)(n-\beta)}{S})\\
&= \circ_{5n-5} (\overset{n-1}{S}, c, \overset{3n+\beta-5}{S}, \{\overset{n-1}{S}, c, \overset{3n+\beta-5}{S}\}^{n-2}, c, \overset{(n-1)(n-\beta)}{S})\\
&\subseteq \circ_{5n-5} (\overset{n-1}{S}, c, \overset{4n^2-10n+\beta n-\beta+5}{S}, c, \overset{n^2-n-\beta n+\beta}{S})\\
&= \circ_3 (\overset{n-1}{S}, c, \circ_{4n+\beta-7} (\overset{4n^2-11n+\beta n-\beta+8}{S}), \overset{n-3}{S}, c, \circ_{n-\beta-1} (\overset{n^2-2n-\beta n+\beta+2}{S}), \overset{n-2}{S})\\
&\subseteq \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}).
\end{aligned}
$$

If $\beta = n$, then it turns out that

$$
\begin{aligned}
c &\in \circ_5 (\overset{n-1}{S}, c, \overset{4n-5}{S}, c)\\
&\subseteq \circ_{10} (\{\overset{n-1}{S}, c, \overset{4n-5}{S}\}^2, c)\\
&\ \ \vdots\\
&\subseteq \circ_{5n} (\{\overset{n-1}{S}, c, \overset{4n-5}{S}\}^n, c)\\
&= \circ_{5n} (\{\overset{n-1}{S}, c, \overset{4n-5}{S}\}^{n-2}, \{\overset{n-1}{S}, c, \overset{4n-5}{S}\}^2, c)\\
&\subseteq \circ_3 (\overset{n-2}{S}, \circ_{5n-10} (\overset{5n^2-15n+11}{S}), c, \overset{n-3}{S}, \circ_4 (\overset{4n-3}{S}), c, \overset{n-2}{S}, \circ_3 (\overset{3n-2}{S}))\\
&\subseteq \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}).
\end{aligned}
$$

Case 7: If $c \in \circ_2 (\bigcup_{\rho=1}^{n} \circ (\overset{\rho-1}{S}, c, \overset{n-\rho}{S})) \overset{2n-3}{S}, c)$, then the proof proceeds analogously to the proof of Case 3.

Case 8: If $c \in \circ_2 (\bigcup_{\rho=1}^{n} \circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{2n-3}{S}, \circ_2(\overset{n-1}{S}, c, \overset{n-1}{S}))$, then the proof proceeds analogously to the proof of Case 6.

Case 9: If $c \in \circ_2 (\bigcup_{\rho=1}^{n} \circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{2n-3}{S}, \bigcup_{\beta=1}^{n} \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))$, then $c \in \circ_2 (\circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{2n-3}{S}, \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S}))$ for some $\beta, \rho \in [n]$. We proceed by considering the subsequent four subcases.

If $\rho \neq 1$ and $\beta \neq n$, then

$$
\begin{aligned}
c &\in \circ_2 (\circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{2n-3}{S}, \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S})) \\
&\subseteq \circ_2 (\circ(\overset{\rho-1}{S}, \circ_2(\circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{2n-3}{S}, \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S})), \overset{n-\rho}{S}), \overset{2n-3}{S}, \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S})) \\
&= \circ_8 (\overset{2(\rho-1)}{S}, c, \{\overset{n-\rho}{S}, \overset{2n-3}{S}, \overset{\beta-1}{S}, c, \overset{n-\beta}{S}\}^2) \\
&= \circ_8 (\overset{2(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^2) \\
&\;\vdots \\
&\subseteq \circ_{4n-4} (\overset{(n-1)(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^{n-1}) \\
&= \circ_{4n-4} (\overset{(n-1)(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^{n-2}, \overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}) \\
&\subseteq \circ_{4n-4} (\overset{(n-1)(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^{n-2}, \overset{3n+\beta-\rho-4}{S}, \\
&\qquad \circ_2(\circ(\overset{\rho-1}{S}, c, \overset{n-\rho}{S}), \overset{2n-3}{S}, \circ(\overset{\beta-1}{S}, c, \overset{n-\beta}{S})), \overset{n-\beta}{S}) \\
&= \circ_{4n} (\overset{(n-1)(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^{n-2}, \overset{3n+\beta-\rho-4}{S}, c, \overset{3n+\beta-\rho-4}{S}, c, \overset{2(n-\beta)}{S}) \\
&\subseteq \circ_{4n} (\overset{(n-1)(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^{n-2}, \overset{3n+\beta-\rho-4}{S}, \overset{\rho-1}{S}, c, \overset{3n+\beta-\rho-4}{S}, \\
&\qquad \circ_4(\overset{\rho-1}{S}, c, \overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}), \overset{2(n-\beta)}{S}) \\
&= \circ_{4n+4} (\overset{(n-1)(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^{n-2}, \overset{3n+\beta-\rho-4}{S}, \overset{2n-3}{S}, \overset{\beta-1}{S}, \\
&\qquad \{\overset{\rho-1}{S}, c, \overset{3n+\beta-\rho-4}{S}\}^2, c, \overset{3(n-\beta)}{S}) \\
&\;\vdots \\
&\subseteq \circ_{8n-12} (\overset{(n-1)(\rho-1)}{S}, c, \{\overset{3n+\beta-\rho-4}{S}, c, \overset{n-\beta}{S}\}^{n-2}, \overset{3n+\beta-\rho-4}{S}, \\
&\qquad \{\overset{\rho-1}{S}, c, \overset{3n+\beta-\rho-4}{S}\}^{n-2}, c, \overset{(n-1)(n-\beta)}{S}) \\
&= \circ_3 (\overset{n-2}{S}, \circ_{\rho-2}(\overset{kn-2n-\rho+3}{S}), \overset{n-3}{S}, \circ_{n+\beta-\rho-12}(\overset{7n^2+(\beta-\rho-19)n+(\rho-\beta+12)}{S}), c, \\
&\qquad \circ_{n-\beta-1}(\overset{n^2-2n-\beta n+\beta+2}{S}), \overset{n-2}{S}) \\
&\subseteq \circ_3 (\overset{n-1}{S}, c, \overset{n-2}{S}, c, \overset{n-1}{S}).
\end{aligned}
$$

If $\rho = 1$ and $\beta \neq n$, then $c \in \circ_4(c, \overset{3n-4}{S}, \overset{\beta-1}{S}, c, \overset{n-\beta}{S}) = \circ_3(c, \circ(\overset{n}{S}), \overset{2n-4}{S}, \overset{\beta-1}{S}, c, \overset{n-\beta}{S}) \subseteq \circ_3(c, \overset{2n+\beta-4}{S}, c, \overset{n-\beta}{S})$. The proof runs identically to the proof of Case 3.

If $\rho \neq 1$ and $\beta = n$, then the argument mirrors that of the preceding case.

If $\rho = 1$ and $\beta = n$, then

$$\begin{aligned} c &\in \circ_2(\circ(c, \overset{n-1}{S}), \overset{2n-3}{S}, \circ(\overset{n-1}{S}, c)) \\ &= \circ_4(c, \overset{4n-5}{S}, c) \\ &= \circ_2(c, \circ(\overset{n}{S}), \circ(\overset{n}{S}), \overset{2n-5}{S}, c) \\ &\subseteq \circ_2(c, \overset{2n-3}{S}, c). \end{aligned}$$

The proof runs identically to the proof of Case 1. Combining Cases 1 through 9 and Remark 2.7, it turns out that S is softly intra-regular.

4. Conclusions

This study introduces j-pure ideals and interior pure ideals in the context of n-ary semigroups. The main result shows that an n-ary semigroup is softly left [right] regular and softly intra-regular if and only if all of its ideals are n-pure, 1-pure, and interior pure, respectively. Furthermore, we propose as open problems the generalization of these results to j-pure ideals where $1 < j < n$ for $n \geq 3$.

Acknowledgements

The research on purity of ideals of n-ary semigroups by "Khon Kaen University has received funding support from the National Science, Research and Innovation Fund (NSRF)".

References

[1] J. Ahsan and M. Takahashi, "Pure spectrum of a monoid with zero," *Kobe Journal of Mathematics*, vol. 6, pp. 163-182 (1989).

[2] Z. F. Abd Alhussain and A. F. Hassan, "N-ary relation fuzzy-soft set," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 7, pp. 1449-1454 (2023).

[3] T. Changphas, and J. Sanborisoot, "Pure ideals in ordered semigroups," *Kyungpook Mathematical Journal,* vol. 54, no. 1, pp. 123-129 (2014).

[4] J. Daengsaen and S. Leeratanavalee, "On minimal and maximal hyperideals in n-ary semihypergroups," *Mathematics*, vol. 8, article no. 1656 (2020).

[5] J. Daengsaen, S. Leeratanavalee, and B. Davvaz, “Regularities in ordered n-ary semihypergroups,” *Mathematics,* vol. 9, article no. 1857 (2021).

[6] W. Dörnte, “Untersuchungen über einen verallgemeinerten gruppenbegriff,” *Mathematische Zeitschrift,* vol. 29, pp. 1-19 (1928).

[7] W. A. Dudek and I. Groz’dzin’ska, “On ideals in regular n-ary semigroups,” *Matematicki Bilten,* vol. 4, pp. 35-44 (1980).

[8] W. A. Dudek, “Remarks on n-group,” *Demonstratio Mathematica,* vol. 13, pp. 165-181 (1980).

[9] W. A. Dudek, “On (i,j)-associative n-groupoids with the non-empty center,” *Ricerche di Matematica*, vol. 35, pp. 105-111 (1986).

[10] W. A. Dudek, “Idempotents in n-ary semigroups,” *Southeast Asian Bulletin of Mathematics,* vol. 25, no. 1, pp. 97-104 (2001).

[11] E. Kasner, “An extension of the group concepts,” *Bulletin of the American Mathematical Society,* vol. 10, pp. 290-291 (1904).

[12] P. Pornsurat and B. Pibaljommee, “Regularities in ordered ternary semigroups,” *Quasigroups and Related Systems*, vol. 27, pp. 107-118 (2019).

[13] P. Pornsurat, P. Palakawong na Ayutthaya, and B. Pibaljommee, “Prime i-ideals in ordered n-ary semigroups,” *Mathematics,* vol. 9, article no. 491 (2021).

[14] F. M. Sioson, “On regular algebraic systems,” *Proceeding of the Japan Academy*, Series B, vol. 39, pp. 283-286 (1963).

Received October, 2024