

On e^* SS-small submodules and e^* SS-hollow modules

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Abstract

The goal for this work is to submit a conception for e^* S-essential small submodules along with e^* SS-hollow modules as the popularization for singular small submodules and hollow modules, a few characteristics and features of these conceptions are proved.

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1. Introduction

Through this paper X will be a unitary left module, and R is the associative ring with identity. the submodules K of X is named a small submodule of X denote by $(K \ll X)$ if each submodule L for X like as, $X = L + K$, implies $X = L$, [1,2,3]. A submodule A of X is called essential in X denoted by $(A \leq_e X)$ if every non-zero submodules of X has a non-zero intersect with A , [1,4,5]. Cosingular submodule insert by özcan in [6], as follows $Z^*(X) = \{m \in X : mR \ll E(X)\}$ when $E(X)$ be injective hull for X . An R -module X is named cosingular if $Z^*(X) = X$. Baanoon and Khalid in [7], generalized a new

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submodule called e^* -essential as follows, for any nonzero co-singular submodule B of X if $A \cap B \neq 0$, A is named e^* -essential submodule in X , denoted by $A \leq_{e^*} X$. The singular submodules is define by, $Z(X) = \{m \in X : \text{ann}(m) \leq_e R\}$. If $Z(X) = X$, X is named a singular module and a nonsingular module if $Z(X) = 0$, [5,8]. The concept of e^* -essential submodule used by Kabban and Khalid [9], to introduce a generalization of singular module as follows: $Z_{e^*}(X) = \{m \in X : \text{ann}(m) \leq_{e^*} R\}$ if $Z_{e^*}(X) = X, (Z_{e^*}(X) = 0)$, then X is called e^* -singular (e^* -nonsingular), the concept of e^* -singular used by Kabban [10], to introduce e^*S -essential submodule, when submodule A for X is named e^*S -essential if for any non-zero e^* -singular submodule B , $A \cap B \neq 0$. We introduce a new generalization of a small submodule depends on the concept of e^*S -essential submodule called e^*SS -small and show properties of this type of submodules. It is known that a module $X (X \neq 0)$ named hollow, if all proper submodules for X is small in X , see [11,12].

We define e^*SS -hollow as follows, a module $X (X \neq 0)$, is e^*SS -hollow, if every $K \subseteq X$, $K \neq X$ is e^*SS -small in X . We prove among results that an epimorphic image for e^*SS -hollow be e^*SS -hollow.

2. e^*SS -Small submodule

We present the conception of e^* -singular essential small sub module as generalization of small sub module.

Definition 2.1: A sub-module T for the modules X called e^* -singular essential small sub module of X (signified by $T \ll_{e^*SS} X$), if whenever $X = T + H$, with H is e^* -singular essential sub module ($H \subseteq_{e^*S} M$), implies that, $X = H$.

Examples and Remarks 2.2:

1. Each small is e^* -singular essential small, yet opposite in general is not valid, to explain that in Z_6 as Z -module, $\langle \bar{2} \rangle$ and $\langle \bar{3} \rangle$ are e^* -singular essential small but not small.
2. If X be e^* -singular module and $K \subseteq X$. Then $K \ll_{e^*SS} X$, if and only if $K \ll X$.
3. Let X a module then, $X \ll_{e^*SS} X$ if and only if $X = \{0\}$.
4. $\{0\} \ll_{e^*SS} X$.
5. Z_4 as Z -module, all proper submodules of Z_4 is e^*SS -small submodules.
6. In Z as Z -module, the only e^*SS -small submodule is $\{0\}$.

The following are some main properties of e^*SS -small submodules.

Proposition 2.3: Suppose N a submodule of module X . The statements that follow are equivalent,

- 1) $N \ll_{e^*SS} X$

2) If $K + N = X$, in that case $K \subseteq_{\oplus} X$ with $\frac{X}{K}$ a semi simple module.

Proof: (1) $\Rightarrow$ (2). Let Y be a complement of K in X , so $K \oplus Y \subseteq_{e^*S} X$, since, $K + Y + N = X$, and $N \ll_{e^*SS} X$, then $K \oplus Y = X$. so $\frac{X}{K} \cong Y$, to show that Y is semi simple, let $A \leq Y$. Then $K + A + N = X$, by replacing $K + A$ by K , we have, $K + A = K \oplus A$, be direct summand for X , implying $A \subseteq_{\oplus} Y$, so $\frac{X}{K}$ is a semi simple.

(2) $\Rightarrow$ (1). Let $K \subseteq_{e^*S} X$ & $K + N = X$, then $K \subseteq_{\oplus} X$, so we have $K = X$. Thus we get $N \ll_{e^*SS} X$.

Proposition 2.4: Assume $C \subseteq W$, a sub-module of module X , if $C \ll_{e^*SS} W$, then $C \ll_{e^*SS} X$.

Proof: Suppose $C + N = X$, where $N \subseteq_{e^*S} X$, then $(C + N) \cap W = X \cap W$, by Modular Law, we obtain, $C + (W \cap N) = W$, $W \cap N \subseteq_{e^*S} N$, [10]. Then $(W \cap N) = W$, therefore $W \subseteq N$, since $C \subseteq W$, thus $C \subseteq N$, hence $C + N = N$, but $C + N = X$, thus $N = X$, therefore $C \ll_{e^*SS} X$.

Proposition 2.5: Let X be module,

- 1) Suppose T, K, J are submodules of X , with $K \subseteq T$,
 - a) If $T \ll_{e^*SS} X$, then $K \ll_{e^*SS} X$ and $\frac{T}{K} \ll_{e^*SS} \frac{X}{K}$.
 - b) $T + J \ll_{e^*SS} X$ if and only if $T \ll_{e^*SS} X$ and $J \ll_{e^*SS} X$.
- 2) If $K \ll_{e^*SS} X$ and $f: X \rightarrow N$, is a homomorphism, then: $f(K) \ll_{e^*SS} N$. Special case, if $K \ll_{e^*SS} X \subseteq N$, then $K \ll_{e^*SS} N$.
- 3) Assume that $C_1 \subseteq_{e^*S} X_1 \subseteq X$, $C_2 \subseteq_{e^*S} X_2 \subseteq X$ and $X = X_1 \oplus X_2$, then $C_1 \oplus C_2 \ll_{e^*SS} X_1 \oplus X_2$ iff $C_1 \ll_{e^*SS} X_1$ & $C_2 \ll_{e^*SS} X_2$.

Proof:

- 1) a) If $L \subseteq_{e^*S} X$ and $L + K = X$, then $L + T = X$.
 Thus $L = X$ for $T \ll_{e^*SS} X$, so $K \ll_{e^*SS} X$. If $J \leq X$, with $\frac{J}{K} \subseteq_{e^*S} \frac{X}{K}$ and $\frac{J}{K} + \frac{T}{K} = \frac{X}{K}$, then $J + T = X$ and $J \subseteq_{e^*S} X$ by [9], hence $J = X$ and $\frac{J}{K} = \frac{X}{K}$.
 b) $\Rightarrow$) From (a).
 $\Leftarrow$) Suppose $K \subseteq_{e^*S} X$, with $T + J + K = X$, then $J + K = X$, since $J + K \subseteq_{e^*S} X$ and $T \ll_{e^*SS} X$, thus $K = X$, since $J \ll_{e^*SS} X$.
- 2) Suppose $B \subseteq_{e^*S} N$, and $B + f(K) = N$. Then $f^{-1}(B) \subseteq_{e^*S} X$, [9].
 And $f^{-1}(B) + K = X$, since $K \ll_{e^*SS} X$, so $f^{-1}(B) = X$. Thus $B = N$, so $f(K) \ll_{e^*SS} N$.

3) Direct from (1) and (2).

Corollary 2.6: Suppose X be an R -module such that $K_1 \ll_{e^*SS} X$ and $K_2 \ll_{e^*SS} X$, then $K_1 \cap K_2 \ll_{e^*SS} X$.

Proposition 2.7: Assume that X be module and T, K, J are submodules of X with $T \subseteq K \subseteq J \subseteq X$, if $K \ll_{e^*SS} J$, then $T \ll_{e^*SS} X$.

Proof: Suppose $T + N = X$, where $N \subseteq_{e^*S} X$. Thus $J \cap (T + N) = X \cap J = J$, by modular law, since $T \subseteq K$, we get $J = T + (N \cap J)$, now $J = K + (N \cap J)$, since $N \subseteq_{e^*S} X$ and $J \subseteq_{e^*S} J$, then by [10], $N \cap J \subseteq_{e^*S} J$, and $K \ll_{e^*SS} X$, hence $N \cap J = J$, so $J \subseteq N$, but $T \subseteq J$, thus $T \subseteq N$, but $T + N = X$, then $N = X$, hence $T \ll_{e^*SS} X$.

Proposition 2.8: Let $T_1, T_2, \dots, T_n \subseteq X$. Then $\sum_{i=1}^n T_i \ll_{e^*SS} X$, if and only if $T_i \ll_{e^*SS} X, \forall i = 1, 2, \dots, n$.

Proof: By indication, special case $T_1 + T_2 \ll_{e^*SS} X$ if and only if $T_1 \ll_{e^*SS} X$ and $T_2 \ll_{e^*SS} X$. (prop.(2.5)). Assuming the result is right for $k < n$, where k is the number of submodules in X . To prove the outcome is correct for $k = n$, let $T_1, T_2, \dots, T_n$ be a submodules of X , such that $\sum_{i=1}^n T_i \ll_{e^*SS} X$, to prove $T_i \ll_{e^*SS} X, \forall i = 1, 2, \dots, n$,

$T_1 + (T_2 + \dots + T_n) \ll_{e^*SS} X$ if and only if $T_1 \ll_{e^*SS} X$, and $T_2 + T_3 + \dots + T_n \ll_{e^*SS} X$, by special case and $T_2 + T_3 + \dots + T_n \ll_{e^*SS} X$ if and only if $T_i \ll_{e^*SS} X, \forall i = 2, \dots, n$, we continue to get $T_i \ll_{e^*SS} X, \forall i = 1, 2, \dots, n$.

3. e^*SS -Hollow Modules

Remember that a module $X(X \neq 0)$ is named a hollow if each submodule $L(L \neq X)$ is small in X . See [12,13,14]. In this part we insert e^*SS -hollow module as a popularization to a hollow module and examine features this notion.

Definition 3.1: the module $X(X \neq 0)$ named e^* -singular essential-hollow module (used for brief e^*SS -hollow) when all proper submodules for X is e^* -singular essential small sub-module of M .

Examples and Remarks 3.2:

1. Clearly all hollow module is an e^*SS -hollow module. The opposite is not correct, for example Z_6 as Z -module is e^*SS -hollow module but non hollow because $\langle \bar{2} \rangle, \langle \bar{3} \rangle$ are non small in Z_6 .
2. Every simple module is e^*SS -hollow. For example, Z_p as Z -module where p is a prime number.
3. The Z_4 as Z -module is an e^*SS -hollow module.

4. Since $\langle \bar{2} \rangle, \langle \bar{3} \rangle$ are not e^* SS-small in Z_6 as Z_6 -module, because $\langle \bar{2} \rangle, \langle \bar{3} \rangle$ are e^* -singular essential submodules and $\langle \bar{2} \rangle + \langle \bar{3} \rangle = Z_6$, while $\langle \bar{2} \rangle, \langle \bar{3} \rangle \neq Z_6$. Therefore Z_6 as Z_6 -module is not e^* SS-hollow.
5. In Z -module Z is not e^* SS-hollow.
6. If X is e^* -singular module. Then X is e^* SS-hollow iff all submodule N of X is small, ($N \neq X$).

Proposition 3.3: A non zero epimorphic image for the e^* SS-hollow modules be e^* SS-hollow modules.

Proof: Suppose $f: X \rightarrow W$ epimorphism and X is e^* SS-hollow, with $K \subsetneq W$, to prove $K \ll_{e^*SS} W$, because $K \subsetneq W$, this lead to $f^{-1}(K) \subsetneq X$. If $f^{-1}(K) = X$, then $K = f(X) = W$, hence $K = X$, this is a contradiction, and since X is e^* SS-hollow, therefore $f^{-1}(K) \ll_{e^*SS} X$, and by (prop.(2.6)) $f(f^{-1}(K)) \ll_{e^*SS} f(X)$. Thus $K \ll_{e^*SS} W$.

Corollary 3.4: Suppose X be a module & $N \subseteq X$, if X be e^* SS-hollow modules, so $\frac{X}{N}$ be e^* SS-hollow module.

Recall that a submodule K of X is a fully invariant if: $f(K) \subseteq K$, $\forall f \in \text{End}(X)$ and X is well known duo module if each submodule of X is fully invariant. See [15,16].

Proposition 3.5: if X is duo module & $X = X_1 \oplus X_2$, so X be e^* SS-hollow if and only if X_1 and X_2 , are e^* SS-hollow.

Proof: $\Rightarrow$) Suppose $X = X_1 \oplus X_2$ is an e^* SS-hollow and $N_1 \subsetneq X_1$. So $N_1 \oplus \{0\} \subsetneq X_1 \oplus X_2$, since $X_1 \oplus X_2$ is an e^* SS-hollow, then $N_1 \oplus \{0\}$ is e^* SS-small in $X_1 \oplus X_2$. There exists an epimorphism $\rho: X_1 \oplus X_2 \rightarrow X_1$ which define by $\rho(x_1, x_2) = x_1$, for all $(x_1, x_2) \in X_1 \oplus X_2$, with $\rho(N_1 \oplus \{0\}) = N_1$, then by (prop.(2.6)(2)), $N_1 \ll_{e^*SS} X_1$. Similarly, for X_2 .

$\Leftarrow$) Let X_1 and X_2 are e^* SS-hollow, since $N_1 \ll_{e^*SS} X_1 \subseteq X$ and $N_2 \ll_{e^*SS} X_2 \subseteq X$, by (prop. (2.6)) again, $N_1 \oplus N_2 \ll_{e^*SS} X_1 \oplus X_2 = X$.

Proposition 3.6: Suppose X be an e^* SS-hollow module, and D is e^* SS-Small proper submodule, such that $\frac{X}{D}$ is a finitely generated (f.g) e^* S-Singular, subsequently X is (f.g).

Proof: We have $\frac{X}{D}$ is (f.g) that mean there exists $t_1, t_2, \dots, t_n \in X$ so that $\frac{X}{D} = \langle t_1 + D, t_2 + D, \dots, t_n + D \rangle$. We believe that $X = \langle t_1, t_2, \dots, t_n \rangle$, let $t \in X$, then $t + D \in \frac{X}{D}$, with $t + D = (s_1 t_1 + s_2 t_2 + \dots + s_n t_n) + D$, for some $s_1, s_2, \dots, s_n \in R$, so, $t - (s_1 t_1 + s_2 t_2 + \dots + s_n t_n) \in D$, let $n = t - (s_1 t_1 + s_2 t_2 + \dots + s_n t_n)$, where, $n \in D$. Hence, $t = (s_1 t_1 + s_2 t_2 + \dots + s_n t_n) +$

n , thus, $X = \langle t_1, t_2, \dots, t_n \rangle + D$. If $X \neq \langle t_1, t_2, \dots, t_n \rangle$, since X is e^*SS -hollow module, then $\langle t_1, t_2, \dots, t_n \rangle$ is e^*SS -Small in X , and since D is an e^*SS -Small submodule. Hence X is e^*SS -Small submodule, which is antinomy. There for $X = \langle t_1, t_2, \dots, t_n \rangle$.

Conclusions

We consent the following results:

- 1) Small submodules and e^*SS -small submodules coincide if X is e^* -singular module.
- 2) Every hollow module is e^*SS -hollow.
- 3) Every simple module is e^*SS -hollow.
- 4) If X is e^* -singular module. Then X is e^*SS -hollow if and only if every submodule $K (K \neq X)$ of X is small.

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