Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 1, pp. 161–167
DOI : 10.47974/JDMSC-2319

On e*-singular lifting modules

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Abstract

Any R-module G is named e*S-lifting module if each sub-module L of G, there is sub-module V of L such that G = V ⊕ D, where D ⊆ G and $L \cap D \ll_{e^*S} D$. This work proposes to present characteristics and examples of e*S-lifting modules. We list few conditions that must be met for the quotient and direct sums of e*S-lifting modules to be e*S-lifting.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *e*S-Small sub-modules, Lifting modules, e*S-lifting modules, e*S-hollow sub-modules.*

1. Introduction

In this paper G is unitary left R-module, R is the ring with identity. Notationally, a sub-module T of any R-module G is considered small, which is well known. $T \ll G$, if for each sub-module of G, T + L = G, then L = G, [1,2]. A new sub-module type was created by Baanoon in [3], and it is a generalization of essential submodule called e∗-essential as follows. For any non-zero cosingular sub-module B of G, if $V \cap B \neq 0$, we say that V is an e*-essential sub-module in G. Denoted by $V \leq_{e*} G$. This is the definition of the singular submodule: Z(G) = {m in G: ann(m) $\leq_e$ R}[4]. We generalized Z(G) to Z_{e*}(G), by applying e*-essential submodule. Let G be a module define Z_{e*}(G) = {w in G: ann(w) $\leq_{e*}$ R} is e*-singular sub-module. If Z_{e*}(G) = G (Z_{e*}(G) = 0) G is called e*-singular (non e*-singular) modules [5]. The generalization of small sub-module known as e*S-small sub-module is introduced in [5], by Kabban and Khalid. A sub-module T of Gis called e*S-small sub-module of G (signified

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by T $\ll_{e*S}$ G) if whenever G = T + V, with $Z_{e*}(\frac{G}{V}) = \frac{G}{V}$ implies that G = V. A nonzero module is said to e*S-hollow if each proper sub-module of G is e*S-small [6]. if H ⊆ W ⊆ G, if $\frac{W}{H} \ll \frac{G}{H}$, so the sub-module H is called coessential of W in G [7, 8]. A popularization of the coessential sub-module, we present the following as the e*S-coessential sub-module in [6]. Let an R–module G, and H, W ⊆ G such that W ⊆ H ⊆ G, then Wis called e*S-coessential sub-module for H in G (denote as W $\subseteq_{e*S_{ce}}$ H in G) if $\frac{H}{W} \ll_{e*S} \frac{G}{W}$. We say that H is called e*S-coclosed sub-module of G (denoted by H $\subseteq_{e*S_{cc}}$G) if whenever T$\subseteq_{e*S_{ce}}$H, (i.e. $\frac{H}{T} \ll_{e*S} \frac{G}{T}$) implies that T = H [6]. Let V and H be sub-modules of the R-module G. If G = V + H & $V \cap H \ll_{e*S}$ H (V ∩ B $\ll_{e*S}$ G), then V is called e*S-supplement (weak e*S-supplement) of H in G. If each a sub-module of G has (weak) e*S-supplement, then G is called (weakly) e*S-supplemented module [9]. Any R-module G is invited alifting module if for each sub-module L of G there is sub-module V of L such that G = V ⊕ C and L ∩ C ≪ C [7, 10, 11]. Our introduction of the e*S-lifting module is the result of this notion. We provide characteristics and examples.

2. e*-Singular lifting modules

This section defines and introduces some of the basic characteristics of e*S-lifting modules. We have included some new results as well.

Definition 2.1: Any R-module G be e*-singular lifting modules (briefly e*S-lifting) if for each sub-module L of G there is a sub-module V of L such that G = V ⊕ D, where D ⊆ G and L ∩ D $\ll_{e*S}$ D.

The following gives characterizations of e*S-lifting modules.

Remark 2.2: if G is R-module. G is e*S-lifting iff for each sub-module L for G, there exist the sub-module D of L such that G = D ⊕ U, where U⊆ G and L ∩ U $\ll_{e*S}$ G. By (lemma 1 [5]).

Theorem 2.3: *The statements that follow are equivalent if G is the R-module.*

1) *G be e*S-lifting module.*
2) *For all L⊆ G, it possible to writes as L= B ⊕T, where B $\subseteq_{\oplus}$ G and T $\ll_{e*S}$ G.*
3) *For all L ⊆ G, there exists W $\subseteq_{\oplus}$ G such that W $\subseteq_{e*S_{ce}}$ L in G and W ⊆ L. (i.e. $\frac{L}{W} \ll_{e*S} \frac{G}{W}$).*

Proof: 1⇒2) Assume that G is e*S-lifting and let L ⊆ G, there is W ⊆ L, where G = W ⊕ C, C ⊆ G and L ∩ C $\ll_{e*S}$ G, by remark 2.2. Now, L = L ∩ G = L ∩ (W ⊕ C) = W ⊕ (L ∩ C), by modular law. We take B = W and T = L ∩ C. Therefore, L = B ⊕ T with B $\subseteq_{\oplus}$G and T $\ll_{e*S}$ G.

2⇒3) Let L ⊆ G. By (2), L = B ⊕ T, where $B \subseteq_{\oplus} G$ and $T \ll_{e*S} G$. We are done if we show that $\frac{L}{B} \ll_{e*S} \frac{G}{B}$. To see this, suppose $\frac{G}{B} = \frac{L}{B} + \frac{H}{B}$ with $Z_{e*}(\frac{G}{H}) = \frac{G}{H}$. Then $\frac{B+T}{B} + \frac{H}{B} = \frac{G}{B}$, hence G = B + T + H = T + H. But $T \ll_{e*S} G$ and $Z_{e*}(\frac{G}{H}) = \frac{G}{H}$, therefore G = H, thus $\frac{G}{B} = \frac{H}{B}$.

3⇒1) Let L ⊆ G. By (3), there is $W \subseteq_{\oplus} G$, where W ⊆ L, G = W ⊕ C, and $\frac{L}{W} \ll_{e*S} \frac{G}{W}$. We want to show that $L \cap C \ll_{e*S} C$. Let C = (L ∩ C) + A with $Z_{e*}(\frac{C}{A}) = \frac{C}{A}$, where A ⊆ C. Since G = W + C = W + (L ∩ C) + A, thus $\frac{G}{W} = \frac{W + (L \cap C) + A}{W} = \frac{W + (L \cap C)}{W} + \frac{A+W}{W}$, since W ⊆ W + (L ∩ C) ⊆ L and $W \subseteq_{e*S_{ce}} L$. Then $W \subseteq_{e*S_{ce}} W + (L \cap C)$ in G, by ([6], proposition 3.5) and $\frac{G}{W+A} = \frac{W+C}{W+A} = \frac{C+(W+A)}{W+A} \cong \frac{C}{C \cap (W+A)} = \frac{C}{A}$, by Second Isomorphism Theorem. Since $Z_{e*}(\frac{C}{A}) = \frac{C}{A}$, then $Z_{e*}(\frac{G}{W+A}) = \frac{G}{W+A}$. Because $\frac{W + (L \cap C)}{W} \ll_{e*S} \frac{G}{W}$, then $\frac{G}{W} = \frac{A+W}{W}$, which implies that G = A + W. But A ⊆ C and W ∩ C = 0, then W ∩ A = 0, and G = W ⊕ A, that is C = A. Therefore, G is e*S-lifting module.

Remark 2.4: Each e*S-hollow module is e*S-lifting.

Proof: Let V ⊆ G, if V is proper sub-module of G, then $V \ll_{e*S} G$, V = 0 ⊕ V and by theorem 2.3, G is e*S-lifting.

Examples and remarks 2.5:

1) Each lifting module is e*S-lifting. However, this is not always the case. As an example, let $G = Z_2 \oplus Z_2$ as Z_2–module, take L = G, $V = \{\bar{0}\} \oplus Z_2$, $D = Z_2 \oplus \{\bar{0}\}$ s.t G = L + D & $L \cap D = Z_2 \oplus \{\bar{0}\}$ is e*S-small but not small sub-modules in G, see ([5], examples and remarks 1). So, G is e*S-lifting but not lifting modules.
2) Lifting and e*S-lifting modules coincide if G is a uniform cosingular module. In particular, the Q as Z-module is uniform cosingular, which isn't lifting [7]. So, in Z-module Q isn't e*S-lifting.
3) Every semi simple module ise*S-lifting. However, this is not always the case. As an example, Z_4 as Z-module.
4) In general, the reverse remark 2.4 isn't true. For example, Z_6 as Z-module. See ([6]. example and remark 2.4).
5) Z as Z-module isn't e*S-lifting.
6) Each e*S-lifting is e*S-supplemented module.

Proposition 2.6: Let a module G is indecomposable. G is e*S-lifting module iff is e*S-hollow.

Proof: ⇒) Assume G is e*S-lifting indecomposable module & V is $a$proper sub-module of G. Since Gis e*S-lifting, there is T ⊆ V such that G = T ⊕ U, where

U ⊆ G and V ∩ U $\ll_{e*S}$ U. But G is indecomposable, therefore either T = G or T = {0}. If T = G, then V = G which is contradictions, then T = {0} and hence G = U, which implies that V ∩ U = V $\ll_{e*S}$ U = G. Therefore, G is e*S-hollow.

⇐) Clear by remark 2.4.

Proposition 2.7: Any directsummand of an e*S-lifting modules is e*S-lifting.

Proof: Let G be an e*S-lifting module. Suppose that G = $G_1 \oplus G_2$, to show G_1 is e*S-lifting. Let V ⊆ G_1 , so V ⊆ G. By theorem 2.3, V = B ⊕ U, where B $\subseteq_{\oplus}$ G and U $\ll_{e*S}$ G. Since B $\subseteq_{\oplus}$ G contained in G_1, then B $\subseteq_{\oplus}$ G_1 and U $\ll_{e*S}$ G, U⊆ G_1 and $G_1 \subseteq_{\oplus}$ G, then U $\ll_{e*S}$ G_1, by ([5], Proposition 13). Therefore, G_1 is e*S-lifting. Similarly, G_2 is e*S-lifting.

Theorem 2.8: *The statements that follow are equivalent if G is an R-module.*

1) *G is e*S-lifting module.*
2) *Each sub-module V of G has an e*S-supplement K in G, where K ⊆ G such that (K ∩ V) $\subseteq_{\oplus}$V.*

Proof: 1⇒2) Let G e*S-lifting module and V ⊆ G. By theorem 2.3, there exists A $\subseteq_{\oplus}$ G such that A ⊆ V, G = A ⊕ L and A $\subseteq_{e*S_{ce}}$V in G and V ∩ L $\ll_{e*S}$ L. Now, V = V ∩ G = V ∩ (A ⊕ L) = A ⊕ (V ∩ L), by modular law. Since A ⊆ V, then G = V + L and V ∩ L $\ll_{e*S}$ L. Hence, L is e*S-supplement of Vin G and V ∩ L $\subseteq_{\oplus}$V.

2⇒1) Let V be a sub-module of G. By (2) V has e*S-supplement K in G s.t K ∩ V $\subseteq_{\oplus}$V. Then G = K + V, K ∩ V $\ll_{e*S}$ K, & V = (K ∩ V) ⊕ X, where X ⊆ V. Since G = K + V = K + (K ∩ V) + X = K + X and {0} = K ∩ V ∩ X = K ∩ X. Then G = K ⊕ X and K ∩ V $\ll_{e*S}$ K. Therefore, G is e*S-lifting.

Another description of the e*S-lifting module is given below:

Proposition 2.9: Let Gbe any R-module, G is e*S-lifting if and only if for each sub-module L of G, there is anidempotent φ ∈ End (G) such that φ(G) ⊆ L and (I – φ) (L) $\ll_{e*S}$ (I – φ) (G).

Proof: ⇒) Suppose that G is e*S-lifting modules and L ⊆ G. By theorem 2.8, L has e*S-supplement Tin G such that T ∩ L $\subseteq_{\oplus}$ L, then G = T + L, T ∩ L $\ll_{e*S}$ T and L = (L ∩ T) ⊕ C, where C ⊆ L. Now G = L + T = (L ∩ T) + C + T = C + T, & L ∩ T ∩ C = T ∩ C = {0}, so G = T ⊕ C. Consider the projection map φ: G ⟶ C, clear that $\varphi$is anidempotent and φ(G) ⊆ C ⊆ L. It is sufficient to prove that (I – φ) (L) $\ll_{e*S}$ (I – φ) (G). One can easily show that (I – φ) (L) = L ∩ (I – φ) (G) = L ∩ T $\ll_{e*S}$ T = (I – φ) (G).

⇐) Let L ⊆ G. By our assumption there is an idempotentφ ∈ End(G) where φ(G) ⊆ L and (I – φ) (L) $\ll_{e*S}$ (I – φ) (G), clearly that G = φ(G) ⊕ (I –

φ) (G) and L ∩ (I – φ) (G) = (I – φ) (L) $\ll_{e*S}$ (I – φ) (G). Therefore, G is e*S-lifting.

Proposition 2.10: Let G e*S-lifting module and K ⊆ G. Then $\frac{G}{K}$ is e*S-lifting in every of the following cases.

1) For each direct summands W of G, $\frac{W+K}{K}$ is a directsummand of $\frac{G}{K}$.
2) G is distributive module.
3) K is afully invariant submodule of G.

Proof:

1) Assume that G is e*S-lifting module and let $\frac{Y}{K}$ be sub-module of $\frac{G}{K}$, then there existsW ⊆ Y such that G = W ⊕ U, where U ⊆ G and W $\subseteq_{e*S_{ce}}$ Y in G. By hypothesis, $\frac{W+K}{K} \subseteq_{\oplus} \frac{G}{K}$. By ([6], proposition 3.4) $\frac{W+K}{K} \subseteq_{e*S_{ce}} \frac{Y}{K}$ in $\frac{G}{K}$. Therefore, $\frac{G}{K}$ is e*S-lifting.
2) Assume that G is distributive module, let W $\subseteq_{\oplus}$ G and G = W ⊕ U, where U ⊆ G, then $\frac{G}{K} = \frac{W+U}{K} = \frac{W+K}{K} + \frac{U+K}{K}$. Since G is distributive module, then $\frac{W+K}{K} \cap \frac{U+K}{K} = \frac{(W+K)\cap U+[(W+K)\cap K]}{K} = \frac{(K\cap U)+(W\cap K)+K}{K} =$ K. Hence, $\frac{W+K}{K} \subseteq_{\oplus} \frac{G}{K}$. So, by (1) G is e*S-lifting module.
3) Let $\frac{Y}{K}$ be sub-module of $\frac{G}{K}$. Since G is e*S-lifting there is the sub-module W of Y s.t W $\subseteq_{e*S_{ce}}$ Y in G &G = W ⊕ U, U ⊆ G. By ([12], lemma 5.4), we have $\frac{G}{K} = \frac{W\oplus K}{K} \oplus \frac{U\oplus K}{K}$, let φ: $\frac{G}{W} \to \frac{G}{W+K}$ be a map defined by φ(g + W) = g + W + K, for all g ∈ G, it is clear that φ is an epimorphosis. Now, since W $\subseteq_{e*S_{ce}}$Y in G, $\frac{Y}{W} \ll_{e*S} \frac{G}{W}$ and $\varphi(\frac{Y}{W}) \ll_{e*S} \varphi(\frac{G}{W})$ by ([5], proposition 12), which implies that $\frac{Y}{W+K} \ll_{e*S} \frac{G}{W+K}$, then W + K $\subseteq_{e*S_{ce}}$Y in G, and hence $\frac{W+K}{K} \subseteq_{e*S_{ce}} \frac{Y}{K}$ in $\frac{G}{K}$, by ([6], proposition 3.4). Therefore, $\frac{G}{K}$ is e*S-lifting.

Theorem 2.11: *Let G = N + K be an e*S-lifting module, if G is e*-singular module, then there is L $\subseteq_{\oplus}$ G, where G = N + L and L $\subseteq_{e*S_{ce}}$ K in G.*

Proof: Let N + K = G e*S-lifting and suppose that $Z_{e*}(\frac{G}{N}) = \frac{G}{N}$. Since K ⊆ G and G is e*S-lifting, there is L $\subseteq_{\oplus}$ G s.t L $\subseteq_{e*S_{ce}}$K in G. Then $\frac{G}{L} = \frac{N+K}{L} = \frac{N+L}{L} + \frac{K}{L}$. Since $Z_{e*}(\frac{G}{N}) = \frac{G}{N}$ and by ([5], corollary 3), $Z_{e*}(\frac{G}{N+L}) = \frac{G}{N+L}$. But $\frac{K}{L} \ll_{e*S} \frac{G}{L}$, then $\frac{G}{L} = \frac{N+L}{L}$, so G = N + L. So, we get the results.

Remark 2.12: Adirect sum of e*S-lifting modules needn't be e*S-lifting module, shows as the following example. Assume G = $Z_8 \oplus Z_2$ as Z-module. Since Z_8 and Z_2 are cosingular, then Z_8 and Z_2 are e*S-lifting modules. But G is not e*S-lifting module.

Proposition 2.13: Let G = $G_1 \oplus G_2$ be duo (distributive) module such that G_1 and G_2 are e*S-lifting modules, then G is e*S-lifting module.

Proof: Let G = $G_1 \oplus G_2$ duo (distributive) module and let $L \subseteq G$, then L is afully invariant. Thus, L = L ∩ G = L ∩ $(G_1 \oplus G_2)$ = $(L \cap G_1) \oplus (L \cap G_2)$. Since G_1 and G_2 are e*S-lifting modules, then $L \cap G_1 = L_1 \oplus L_2$ and $L \cap G_2 = L_3 \oplus L_4$, where L_1 and L_3 are direct summands of G_1 and G_2 respectively and L_2, L_4 are e*S-small sub-modules of G_1 and G_2 respectively, by theorem 2.3. It is clear that $L_1 \oplus L_3 \subseteq_{\oplus} G$ and $L_2 \oplus L_4$ is e*S-small sub-module of G. Therefore, G is e*S-lifting module.

Conclusion

We confirm the following outcomes:

1) Every e*S-hollow module is e*S-lifting.
2) Every lifting module is e*S-lifting.
3) Lifting and e*S-lifting modules coincide if G is a uniform cosingular module.
4) Every e*S-lifting is e*S-supplemented module.
5) Any directsummand of an e*S-lifting module is e*S-lifting.

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Received March, 2025