

***H*-Pu-supplemented modules**

Halal Qahtan Hamdi *

Nuhad Salim Al-Mothafar †

Department of Mathematics

College of Science

University of Baghdad

Baghdad

Iraq

Abstract

In this work, we introduce the concept of a purely supplement submodule, also consider some properties of this idea, and investigate connections between purely supplement submodules and related ideas. Moreover, we give a large part of this work to introduce and train a generalization of some, H-supplemented, H-purely supplemented.

Subject Classification: 16D80, 16D40.

Keywords: Supplement sub modules, Pure sub modules, Purely-supplement sub modules, Weak supplement sub modules, Ample supplement sub modules H-supplemented, H-Pu-supplemented.

1. Introduction

R is commutative and has an identity in this study. The left R -modules be M . That unitary. a sub-module N for M is pure (for short $N \leq_p M$) if $IM \cap N = IN$ [1] for all I ideal of R . A proper submodule for M named the small submodule for short ($N \ll M$), which means, if the submodule K for M has $N + K = M$, then $K = M$ [2]. Ahmed, Dhari, and Atiya [3] developed a different sub-module that is a generalization of a small one and is named a purely small sub-module, where each proper pure submodule K in M , a sub-module N for modules M named purely small submodules $N \ll_{pu} M$, where $M \neq N + K$. We will use purely small submodules to present a new generalization of modules using H - supplements, called H -Purely supplemented modules. We will prove the main properties of all these concepts.

* E-mail: hala.hamdi@sc.uobaghdad.edu.iq (Corresponding Author)

† E-mail: nuhad.salim@sc.uobaghdad.edu.iq

2. H-Pu-Supplemented Modules

We have a few new H-purely-supplemented module characterizations.

Remember that, an R-modules N namely H-supplemented, where for submodules S for N , where exist direct summand D for N , with $Y + S = N$ iff $Y + D = N$ for each $Y \leq N$, See [4], [5], [6] and [7].

Remember that: The submodules A for R-module M named a weak-supplement to the sub-module B in M , where $M = A + B$ also $A \cap B$ be small for M . See [8].

Recall that M 's submodule N has an ample supplement in M if, for each submodule L , & $L \subset M$ by $L + N = M$, L' for N be the supplement to $L' \subset L$. See [8].

Definition 2.1 the R-module M named H-purely supplemented, used for short (H- Pu-supplemented) if the direct summand exists for every submodule A for R-module M . A_1 for M with, for each $X <_p M$, $X + A = M$ iff $X + A_1 = M$.

Examples and Remarks 2.2

1. Every H-supplemented be H- Pu-supplemented modules. if M is H-supplemented modules also B is submodule in M , hence there is directsummand D in module M , with $B + F = M$ iff $D + F = M \quad \forall F \leq M$. If $B + U = M \quad \forall U$ proper pure submodules for module M , so $D + U = M$. If $D + U_1 = M \quad \forall U_1$ proper pure for M , so $B + U_1 = M$. Thus, M be H- Pu-supplemented modules.
2. If M is F-regular R-modules implies M be H- Pu-supplemented modules coincide. The Z -module Z_{10} is F-regular which is H- supplemented [5]. Thus, Z_{10} as a Z -modules be H- Pu-supplemented.
3. Z -module Z not H- Pu-supplemented. Since only direct summand of Z are $\langle 0 \rangle$ and Z . For the sub module $2Z$, if the direct summand is Z , when $Z + 4Z = Z$ but $2Z + 4Z = 2Z$ and if the direct summand is $\langle 0 \rangle$, $2Z + 3Z = Z$ but $\langle 0 \rangle + 3Z = 3Z$.
4. Modules with H- Pu-supplemented are closed under isomorphism.
5. The Z -module Z_{12} is H- Pu-supplemented since the proper pure submodules of Z_{12} are $\langle \bar{0} \rangle, \langle \bar{3} \rangle, \langle \bar{4} \rangle$ and Z_{12} and the direct summand of Z_{12} $\langle \bar{0} \rangle, \langle \bar{3} \rangle, \langle \bar{4} \rangle$ and Z_{12} .
6. Every semi simple module is an H- Pu-supplemented module. The Z_6 as Z -module and Z_6 as Z_6 -modules are H- Pu-supplemented modules.
7. The weakly Pu-supplemented not H- Pu-supplemented. For example, Z_{12} as Z -module. See examples and remarks ((3.2),1) ,[9].
8. The H- Pu-supplemented must not be amply Pu-supplemented module. For example: Z_{24} as a Z -module. See examples and remarks ((3.2),4) ,[9].
9. The H- Pu-supplemented need not be Pu-hollow. For example, Z_6 as a Z_6 -module.

The following shows the basic properties of H- Pu-supplemented modules.

Recall that, if the sum of any of two pure sub-modules is again pure, it is said R-modules have pure sum property (PSP) [10].

Proposition 2.3 if M is H- Pu-supplemented modules using (PSP) property. so for every pure sub-module G for M, exist direct summand R for M, with $\frac{G+R}{R} \ll_{Pu} R$ also $\frac{G+R}{G} \ll_{Pu} \frac{M}{G}$.

Proof: Suppose G pure sub-module for M. Thus R directsummand for M, with $X + G = M$ iff each $X <_p M$, $R + X = M$. assume $\frac{T}{G} <_p \frac{M}{G}$, with $\frac{M}{G} = \frac{G+R}{G} + \frac{T}{G} = \frac{G+R+T}{G}$, since M have (PSP) property & $G <_p M$ then $G + T <_p M$ and $G \leq T$ implies that $G + T = T$, so $M = R + G + T = G + T = T$, hence $\frac{T}{G} = \frac{M}{G}$. $G \leq G + R$ implies $G + R = G$. Thus, $\frac{G+R}{G} \ll_{Pu} \frac{M}{G}$. Similarly for $\frac{G+R}{R} \ll_{Pu} \frac{M}{R}$.

Recall that If a frame of sub modules in the module M is distributive, then amodule named distributive, means, $N \cap (K + S) = (N \cap K) + (N \cap S)$ for submodule N, K & C for M. See [11].

The submodule A for N named fully invariant sub modules, where $f(A) \subseteq A$ for each $f \in S$, when $S = \text{End}R(N)$. the R-modules N named duo, if each submodule for N be fully invariant. [12].

The following suggestion gives an instance where the submodules for H- Pu -supplemented is H- Pu -supplemented.

Proposition 2.4 if M is the duo (distributive) modules with (PSP) property. If M is H- Pu-supplemented implies that all direct summands are H- Pu-supplemented.

Proof: If M be H- Pu –supplemented & let direct summand A for M, hence $M = A \oplus A_1$, $A_1 \leq M$ and $B \leq A$. Then B sub module for M. There is a direct summand D for M, $M = D \oplus D_1$, $D_1 \leq M$, with $B+X = M$ iff $D+X=M$, for each $X <_p M$. So direct summand for A. If $B + Y = A$ for the proper pure sub module Y of A. Since $M = A \oplus A_1$, for each sub-module A_1 for M, so $M = B + Y + A_1$. Since $Y <_p A <_p M$, so $Y <_p M$ [13, Remark (3.1.5)] and since M has (PSP) property and Y, A are pure submodules hence $Y + A <_p M$ by [10]. Hence $M = D + Y + A_1$ and $A = A \cap D + Y$. Again if $A = A \cap D + T$ for a proper pure submodule T for A, hence $M = A \cap D + T + A_1 = D + T + A_1$, so $M = B + T + A_1$. so, $A = B + T$. Therefore, A is H- Pu -supplemented.

Lemma 2.5 [14, Lemma 5.4] if M is modules. where $M = M_1 \oplus M_2$, so $\frac{M}{A} = \frac{A+M_1}{A} \oplus \frac{A+M_2}{A}$. if each fully invariant sub-module A for M.

Proposition 2.6 if M be H -Pu-supplemented modules using (PSP) Property also a proper submodule A is any pure for M . so $\frac{M}{A}$ be H -Pu-supplemented if each directsummand D for M , $\frac{D+A}{A}$ be a direct summand for $\frac{M}{A}$.

Proof: Suppose $\frac{X}{A}$ is sub modules for $\frac{M}{A}$. Hence $X \leq M \exists$ direct summand D for M , with $X + L = M$ iff $D + L = M$ for all $L <_P M$. From the hypotheses, there exists the direct summand. $\frac{D+A}{A}$ for $\frac{M}{A}$. If $\frac{X}{A} + \frac{K}{A} = \frac{M}{A}$, wherever $\frac{K}{A}$ is the proper pure submodules for $\frac{M}{A}$, because A pure in M implies the sub module K be proper pure for M by [13, remark (3.1.5)] also $X + K = M$, hence $D + K = M$ also $\frac{D+A}{A} + \frac{K}{A} = \frac{M}{A}$. If $\frac{D+A}{A} + \frac{T}{A} = \frac{M}{A}$ where $\frac{T}{A}$ is the proper pure sub module of $\frac{M}{A}$, and A pure in M , so T be proper pure sub-module of M by [13, remark (3.1.5)] and $D + T = M$, hence $X + T = M$ & $\frac{X}{A} + \frac{T}{A} = \frac{M}{A}$ therefore, $\frac{M}{A}$ is H -Pu-supplemented.

Corollary 2.7 if M is distributive and H -Pu-supplemented modules with (PSP) property and K be every pure sub module for M . Now $\frac{M}{K}$ is H -Pu-supplemented [15, 16].

Proof: Assume P is directsummand for M , $M = P \oplus P_1$ when $P_1 \leq M$.

$\frac{M}{K} = \frac{P \oplus P_1}{K} = \frac{P+K}{K} = \frac{P_1+K}{K}$ and sine M is distributive, $(P + K) \cap (P_1 + K) = P \cap P_1 + K = K$. Thus $\frac{M}{K} = \frac{P+K}{K} \oplus \frac{P_1+K}{K}$, hence by proposition (2.6) $\frac{M}{K}$ is H -Pu-supplemented.

Corollary 2.8 if M is H -Pu-supplemented modules with (PSP) property & A is any pure and M 's fully invariant sub module implies that $\frac{M}{A}$ is H -Pu-supplemented.

Proof: Assume that D is directsummand for M , $M = D \oplus D_1$ when $D_1 \leq M$. Since A is fully invariant, by Lemma (2.5), $\frac{A+D}{A}$ is the directsummand for $\frac{M}{A}$, thus, from (2.6) $\frac{M}{A}$ be H -Pu-supplemented.

Proposition 2.9 if $N = U_1 \oplus U_2$ be duo (distributive) modules, with U_1 & U_2 is H -Pu-supplemented modules implies that N be H -Pu-supplemented modules.

Proof: If K be submodules for N . Because N a duo (distributive) module, $K = K \cap (U_1 \oplus U_2) = (K \cap U_1) \oplus (K \cap U_2)$. $K \cap U_1$ is the pure sub module of U_1 by [13, Remark(3.1.5)] and $K \cap U_2$ is a pure submodule for U_2 s.t $K \cap U_i + X_i = U_i$ if and only if $D_i + X_i = U_i$, for all $X_i <_P U_i$ and $i = 1, 2$. As a result, N has a direct summand $D_1 \oplus D_2$. Now if $X+K=N$, for the sub module X a

proper pure of N . Then $(K \cap U_1) \oplus (K \cap U_2) \oplus (X \cap U_1) \oplus (X \cap U) = U_1 \oplus U_2$. $X \cap U_1$ is a proper pure submodule for U_1 & $X \cap U_2$ be proper pure submodule for U_2 . $(K \cap U_1 + X \cap U_1) \oplus (K \cap U_2 + X \cap U_2) = U_1 \oplus U_2 = U_1 \oplus U_2$, and $K \cap U_2 + X \cap U_2 = U_2$. Hence $D_1 + X \cap U_1 = U_1$ and $D_2 + X \cap U_2 = U_2$, thus $D_1 \oplus D_2 + X = U_1 \oplus U_2 = N$. Also, if $D_1 \oplus D_2 + Y = N$ for $Y <_p N$, then $K + Y = N$. Therefore, N is H- Pu - supplemented.

3. Conclusions

In this work, purely-supplement sub modules and weak supplement sub modules, ample supplement sub modules, are facts of H -supplemented, H-Pu-supplemented. We also show some of the following results:

- If M is H - Pu-supplemented module using (PSP) property. Then for each pure submodule B for M , there is the direct summand D for M .
- If M is H - Pu-supplemented module with (PSP) Property, also the proper sub-module A is any pure for M . hence $\frac{M}{A}$ be H- Pu-supplemented.
- If M be a distributive and H - Pu-supplemented module with (PSP) property and A be every pure submodule for M . so $\frac{M}{A}$ be H - Pu-supplemented.
- A distributive module $N = U_1 \oplus U_2$, with U_1 & U_2 be H- Pu-supplemented module, so N be H- Pu - supplemented modules.

References

- [1] Friedrich Kasch, Modules and Rings, vol. 17. New York, NY, USA: Academic Press (1982).
- [2] Ahmed Kabban and Waleed Khalid, "On Jacobson-small submodules," *Iraqi Journal of Science*, vol. 60, no. 7, pp. 1584–1591 (2019), doi: 10.24996/ij. 2019.60.7.18.
- [3] Mohammed Abbas Ahmed, Ismail Abbas Dhari, and Zainab Abdul Atiya, "Purely small submodules and purely hollow modules," *Iraqi Journal of Science*, vol. 63, no. 12, pp. 5487–5495 (2022), doi: 10.24996/ij. 2022.63.12.34.
- [4] Salah Hassan Mohamed and Bruno J. Müller, Continuous and Discrete Modules, London Mathematical Society Lecture Note Series, vol. 147. Cambridge, U.K.: Cambridge University Press (1990).
- [5] Dongsheng Wu, "Cofinitely H-supplemented modules," *Southeast Asian Bulletin of Mathematics*, vol. 35, no. 3, pp. 529–535 (2011).

- [6] Hussein Qasim Hamdi and Noor Sabah Al-Mothafar, "Pu-supplement sub-modules," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 4-B, pp. 1343–1353 (2025).
- [7] Yong Wang and Dongsheng Wu, "On H-supplemented modules," *Communications in Algebra*, vol. 40, no. 10, pp. 3679–3689 (2012).
- [8] John Clark, Claudia Lomp, Natarajan Vanaja, and Robert Wisbauer, *Lifting Modules: Supplements and Projectivity in Module Theory*, Frontiers in Mathematics. Basel, Switzerland: Birkhäuser, pp. 1–407 (2006), doi: 10.1007/3-7643-7573-6.
- [9] Kamal Ahmmed and Noor Sabah Al-Mothafar, "Pr-small R-submodules of modules and Pr-radicals," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 7, pp. 1511–1516 (2023).
- [10] Adnan Ghazi Nauom and Bahar Hassan Al-Bahranny, "Modules with pure sum property," *Iraqi Journal of Science*, vol. 43, no. 3, pp. 1–10 (2002).
- [11] Veli Erdoğan, "Distributive modules," *Canadian Mathematical Bulletin*, vol. 30, no. 2, pp. 248–254 (1987).
- [12] Ali Cem Özcan, Abdullah Harmanci, and Patrick F. Smith, "Duo modules," *Glasgow Mathematical Journal*, vol. 48, no. 3, pp. 533–545 (2006).
- [13] I. H. Muslem, "Some types of retractable and compressible modules," M.S. thesis, College of Education for Pure Science, University of Baghdad, Baghdad, Iraq (2016).
- [14] Neslihan Orhan, Dilek K. Tütüncü, and Rachid Tribak, "On hollow-lifting modules," *Taiwanese Journal of Mathematics*, vol. 11, no. 2, pp. 545–568 (2007).
- [15] S. A. Al-Ameedee and A. J. Obaid, "Sandwich results of meromorphic multivalent functions defined by multiplier transform," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 4, pp. 651–658 (2023), doi: 10.47974/JIM-1484.
- [16] A. A. A. Ziyarah and N. S. Al-Mothafar, "On primary radical RF -submodules of RF -modules," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 23, no. 5, pp. 1001–1007 (2020).

Received March, 2025