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Fully SP-acts

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Abstract

Purity is one of the important concepts in Algebra, which was studied by many others in Acts, modules, and groups. An *K*-*K*-subact B of *K*-act is SP-sub-act, if every $a \in A$ and $a \notin B$, there is a pure *K*-subact P of A that contains B and $a \notin P$. Now we describe that an SP-sub-act L in K-act A, if and only if, Lj are pure K-subacts of A, for each $j \in J$, and $L = \cap_{j \in J}$ Lj. SP-subact as a generalization of purity. We discuss many Algebraic properties. We found the relationship between SP-sub-act idempotent subact, and multiplication.

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1. Introduction

The set K is said to be a monoid if there is a $K \times K \to K$ written by $(k_1, k_2) \to k_1k_2$ satisfies the conditions (1) $(k_1k_2)\, k_3 = k_1\, (k_2k_3)$ for all $k_1, k_2, k_3 \in K$. (2) $\exists\ 1 \in \mathcal{M} \ni 1k = k1 = k$ for all $k \in K$, [1]. Let K be a monoid. A set $\emptyset \neq \mathcal{A}$ is said to be a right K -Act $\mathcal{A}$, if $\exists \mathcal{A} \times K \to \mathcal{A}$ written by (a, k) is the satisfied to condition $a\,(k_1k_2) = (ak_1)k_2$, for all $k_1k_2 \in K$ and $a \in \mathcal{A}$, [1]. A monoid K is said to be commutative (in short, C-M), if $k_1k_2 = k_2k_1$, for all k_1k_2 in K, [1]. A right K-ideal $[L :_K \mathcal{A}] = \{k \in K \mid a\,k \in L \text{ for all } a \in \mathcal{A}\}$, [2]

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and [3]. A K-sub-act L be said to be pure of K -act A if AL = L∩IM, ∀ I ideal of K. An ideal I of K is pure if it is pure K–sub–act of K-act [4].

Many authors studied these sub-acts,see [5], [6].

2. SP-Sub-Acts.

Definition 2.1: An K-sub-act B of K -act is SP-sub-act, if for every a ∈A and a ∉B, there is a pure K-subact P of A that contains B and a∉ P.

Remarks 2.2:

1. Every pure K -K-subact is SP-sub-act.
2. The intersection of any two SP- K -subacts is SP.

Proof: Let a∈A and a∉ $B_1 \cap B_2$, then either a∉ B_1, since B_1 is an SP -K-subact in A, then P as pure contains B_1 and a∉P. Or a ∉ $B_2 \subseteq$ P.

3. If B is an SP- K -subact of pure K-subact B' in A, then B is an SP in A.

Proof: Let a∈A and a∉B. If a∈B', there is a pure K-subact P that contains B and a∉P. Since purity holds the transitive property, then P is pure in A. Hence, B is an SP in A.

4. If B_1 and B_2 are SP-sub-acts, then $B_1 \cup B_2$ is SP.

Proof: Let a∈ A and a∉ $B_1 \cup B_2$, then a∉ B_1 and a ∉B_2, there are two pure S-subacts P_1 and P_2 that contain B_1 and B_2, respectively a∉ P_1 and a ∉P_2. We have pure $P_1 \cup P_2$ contains $B_1 \cup B_2$, and a ∉$P_1 \cup P_2$. Hence, $B_1 \cup B_2$ is pure K-subact.

5. If B is a SP- K-sub-act of A and B is an K-sub-act of B' & B' is a K-sub-act in A, then B is a SP in B'.

Proof: It follows from pure property.

6. If B be an SP-sub-act in A, then $\frac{B}{\sigma}$ be an SP-sub-act in $\frac{A}{\sigma \cup I_A}$ where σ is a congruence on B, I_A is an equivalent relation on A, and $\varrho = \sigma U I_A$.

Proof: Let [a] $\varrho \in \frac{A}{\varrho}$ and [a] $\notin \frac{B}{\varrho}$. We have a∈A, a∉B, by SP of B, we have P as a pure K-subact that contains B, and a∉P.

7. Let L be an SP-sub-act in K-act A, if and only if, L_j are pure K-subacts of A, for each j∈J, and L=$\cap_{j \in J}$ L_j.

Proof: Suppose L isn't an SP-sub-act in K-subact. By hypothesis, there are Lj pure K-subacts in A, for each j is an element in J and L$\subseteq \cap_{j\in J}$ L_j.Suppose $\ell \in \cap_{j\in J}$ L_j, then $\ell \in L_j$ and assume $\ell \notin$L. Since L is an SP, that implies that $\ell \notin$ L_j, Contradiction. Now suppose L=$\cap_{j\in J}$ L_j where pure K-subacts L_j in A. Suppose e ∈ A and e∉L. Since L=$\cap_{j\in J}$ L_j, j in J L subset or equals Lj and e is an such that e ∉ L_j. Thus L⊆ L_j and e ∈ L_j. L is an SP-sub-act.

8. Let K={1, 2, 3 } and A=K. Define a binary operation in the table 1:

Table 1

Binary operation defined on the set $K = \{1, 2, 3\}$.

0	1	2	3
1	1	1	1
2	1	1	1
3	1	2	3

Then, A is an K-act with A× K →K written by (a, b) = ab for all a,b ∈ M. Here, N_1={1, 2} and N_2={1,3} are subacts in A. N_1 is a SP-sub-act, but N_2 isn't a SP-sub-act in A.

$\mathcal{A}$ is called a multiplication K-act, if there is an ideal $\mathcal{J}$ of $\mathcal{M}$, $\mathcal{A}$ ' = $\mathcal{A}\mathcal{J}$, for each K-subact $\mathcal{A}'$ of $\mathcal{A}$. L is an idempotent sub-act in K-act A if L=[L:A] L, [7].

Proposition 2.3:

1. Pure K-subacts of multiplication K-act are multiplication and idempotent.

Proof: Suppose A is a multiplication & L is a pure K-sub-act in A and N is a K-sub-act in L. Then N=IA, ∃ I ideal in K . By purity of L, then N=N∩ L= IA ∩ L= IL. We have L is a multiplication. Since L=[L: A] A and L is pure, then [L: A] L= L∩ [L: A] A = L. Then, L is idempotent in A.

2. Every SP- K-subact of multiplication K-act, there is idemptent K- subact contains SP.

Proof: It follows from pro(2.3),(1).

Lemma 1.4: *Let L be a multiplication K-sub-act in K-act A, iff F∩L =[F: L] L for F sub-act in A.*

Proof: Suppose w ∈ F∩L. Then w ∈ L, by multiplication of L, ∃ ideal I in K, such that Kw=IL. So, IL⊆ F We have I⊆ [F:L] and IL⊆ [F:L] L. Therefore, w ∈ [F: L] L. Suppose c ∈ [F:L] L. Then $c = s\ell$ for some s in [F: L]& ℓ ∈ L. So, sL

$\subseteq$ F thus $c = s\ell \in$ F.Since [F:L] L $\subseteq$ KL$\subseteq$ L then $c \in L$. We get c $\in$F$\cup$L. That is F$\cap$L = [F: L] L.

Lemma 2.5: *If F is a faithful multiplication K-act,then_I=[IF: F], for the ideal I in K.*

Proof: Suppose i $\in$ I.Then, iF$\subseteq$ IF. Hence, i $\in$ [IF: F]. Conversely, let s $\in$ [IF:F] then sF$\subseteq$ IF and hence $\forall f \in$ F, $\exists$ i $\in$ I, & $f_\circ \in$ F such that $sf = if_\circ$ that is, $sf = if_\circ$,for all $if \in$ F. In particular $sf_\circ = if_\circ$. By the faithfulness of M, we have $s = i$. So, s $\in$ I. Therefore, I=[IF:F].

Theorem 2.6: *Suppose K is a C-M & A a faithful K-act. Then, A is a multiplication if and only if 1- $\cap_{j\in J}$ (LjA)= ($\cap_{j\in J}$ L_j)A for a nonempty collection of ideals L_j, $\forall j \in J$ in K and 2- any K-subact B in A & ideal L in K such that B $\subset$LA, $\exists$ N an ideal with N $\subset$ L & L$\subseteq$NA.*

Proof: Let 1 & 2 be verified. Suppose B is a K-subact in A& Ψ = {L: L is an ideal in K & B$\subseteq$ LA}. K$\in$ Ψ. Let $\emptyset \neq$$L_j$ $(j \in J)$ be any set of ideals in Ψ. Having (1) $\cap_{j\in J}$ L_j $\in$ Ψ.Zorn's lemma, Ψ has a minimal element C (say). Then B $\subseteq$ CA. Suppose that B$\neq$ CA. By statement (2), there exists an ideal D of K with D$\subset$C and B$\subseteq$ DA. In this case, D $\in$ Ψ contradicts the choice of C. Thus, B = CA. It follows that A is a multiplication K-act.

Theorem 2.7: *Suppose L_j are a K-subacts of faithful multiplication K-act A. The next conditions are equal:*

1. *L is an SP-subact in K-act A*
2. *L_j are pure K-subacts of A, for each j$\in$J and L=$\cap_{j\in J}$ L_j.*
3. *L_j is multiplication and idempotent in A.*
4. *I [L_j: M] = I$\cap$ [L_j: M] for ideal I in K.*

Proof: (1) $\Leftrightarrow$ (2) Remark 2.2(6). $\Longrightarrow$ Pro (2.3). $\Longrightarrow$ (Suppose D is a K-subact of A. Then [D: L_j] L_j =[D: L_j] ([L_j: A] L_j) =([D: L_j] [L_j:A]) L_j $\subseteq$ [D: A] L_j. Also, [D:A] L_j $\subseteq$ [D: L_j] L_j. So, [D: L_j] L_j = [D:A] L_j .Since L_j is multiplication, then by Lemma(2.4) and (2.5), for every ideal I of K, IA$\cap$ L_j = [IA: L_j] L_j = [IA: A] L_j =I L_j. Thus, I L_j =IA$\cap$ L_j. So, (I [L_j:A]) A=I L_j =IA$\cap$ L_j =IA$\cap$ [L_j:A] A. Therefore, IA$\cap$ [L_j: A] A=(I [L_j: A]) A, and by Theorem(2.6), (I$\cap$ [L_j: A]) A = (I [L_j: A]) A. By faithfulness in A, I [L_j: A] =I$\cap$ [L_j: A]. (4) $\Longrightarrow$ (2) Suppose I is ideal in K. Theom (2.6), and (3) $L_j$$\cap$IA=[L_j: A] A $\cap$ IA=([L_j: A] $\cap$I) A=([L_j: A] I) A = (I [L_j: A]) A=I ([L_j: A] A)= IL_j. Therefore, L_j is a pure K-subact of A.

Theorem 2.8:

1. *If L is an SP sub K-act in faithful multiplication in A, then [L: A] is an SP-ideal in K.*

Proof: Let s isn't in [L: A], we have sA isn't in L, then sA isn't in L, for some, by 1, there is pure N in A such that L⊆N and sa isn't in N, then s isn't in [N: A], then it is a pure ideal of K. Hence [L: A] is SP- ideal of K.

2. *If L is a sub K-act in faithful multiplication in A & [L: A] is SP-ideal in K, then L is an SP sub K-act in A*

Proof: Let ka$\notin L$, then k∉[L: A] is SP-ideal, thus there is a pure ideal [P: A] such that k∉[P: A] with [L: A]⊆ [P: A]. We have ka$\notin P$ where P is a pure sub-act that contains L. Hence, L is an SP sub K-act.

Many authors studied these acts [2],

3. Fully SP-Acts

Definition 3.1: A K-act A is named fully SP-acts if each K-subact B of A is K-isomorphic to SP.

Remarks 3.2: The Z as Z-act is Fully SP-act.

Proposition 3.3: Every K-sub-act B of Fully SP-act D is Fully SP.

Proof: Suppose D is fully SP-act, & K-inclusion i: B→D and L is K-subact of B. By inclusion, there is SP-K-sub act H in D such that H≅ L is K-sub act of D since D is fully SP. We have SP-K-sub act H in B, by Remarks 2.2(5). Thus, B is Fully SP-act.

Proposition 3.4: The Intersection of two fully SP-acts is a fully SP-act.

Proof: Suppose L is a sub-K-act of F∩G. Then L is a sub-K-act of F. By full SP of F, we have L is SP in F. By Remak (2.2):5, L is SP in F∩G.

Proposition 3.5: Any disjoint union of two fully SP-acts is fully SP.

Proof: Suppose L is a sub-K-act of F$\dot{\cup}$G. Then either L is a sub-K-act of F or L is a sub-K-act of G. By full SP of F or G, we have L is SP in F or G. By Remak (2.2):3, L is SP in F$\dot{\cup}$G.

4. Conclusion

We introduced fully SP-Acts that are very important in the Act theory. We have the field to study more notions, SP-stability, and SP-injectivity.

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