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# Computing new degree-based topological indices associated with the zero-divisor graphs of finite commutative rings

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**Abstract**

Consider a commutative ring $\mathfrak{R}$. Let $\mathfrak{Z}^*(\mathfrak{R})$ denote the set of zero-divisors of $\mathfrak{R}\backslash\{0\}$. A zero-division graph, denoted by $\Gamma(\mathfrak{R})$, is a graph interrelated with the ring $\mathfrak{R}$, where the vertex set consists of the elements in $\mathfrak{Z}^*(\mathfrak{R})$. In this graph, two vertices $w_1$ and $w_2$ are adjacent if and only if their product satisfies $w_1 w_2 = w_2 w_1 = 0$.

In a recent research paper, we explore various degree-based topological indices related to $\Gamma(R)$. Specifically, we investigate the graph structures for the types of commutative rings including $\mathbb{Z}_{\zeta^2}$, $\mathbb{Z}_{\zeta^3}$, $\mathbb{Z}_{\zeta} \times \mathbb{Z}_{\zeta} \times \mathbb{Z}_{\zeta}$, $\mathbb{Z}_{\zeta_1 \zeta_2}$ and $\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}$ where $\zeta$, $\zeta_1$ and $\zeta_2$ are the primes. This study sheds light on the algebraic properties of these rings and provides insights into their underlying graph structures.

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***Keywords:*** *Topological index, Zero-divisor graph, Commutative ring, Sombor index.*

## 1. Introduction

The notion of a zero-divisor graph was initially presented by Istvan Beck in 1988 [1]. Subsequently, Anderson and Livingstone [2] modified the definition of this graph for a commutative ring $\mathfrak{R}$ and its set of zero-divisors, denoted as $\mathfrak{Z}(\mathfrak{R})$. They established a connection between the commutative ring $\mathfrak{R}$ and a simple graph called $\Gamma(\mathfrak{R})$. In this graph, $V(\mathfrak{Z}^*(\mathfrak{R})) = \mathfrak{Z}(\mathfrak{R})\backslash\{0\}$. Specifically, two vertices $w_1$ and $w_2$ in $\Gamma(\mathfrak{R})$ are adjacent if and only if their product satisfies $w_1 w_2 = w_2 w_1 = 0$.

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In [2], it was proved that the graph $\Gamma(\mathfrak{R})$ is connected if the ring $\mathfrak{R}$ is the commutative ring. Commutative rings have applications in various fields, including cryptography, algorithm analysis, communication theory, engineering, and combinatorics. Therefore, much research has been done on these algebraic structures and their corresponding graphs.

Topological indices are variables associated with a graph and are constant under graph isomorphism. Studies of topological indices in algebraic graph theory can be used to describe the properties of a graph through algebraic structures. Algebraic graph theory has been applied as a helpful tool in fields such as chemistry, neurological systems, and social networks. Many studies have been done on topological indices of algebraic graphs. In their research, Rayer and Jeyaraj [3] calculated several topological indices for the zero-divisor graphs. These indices include the total and connective eccentricity indices, the augmented and Ediz eccentric connectivity indices. Additionally, Movahedi and Akhbari [4] investigated topological indices for zero-divisor graphs. Their investigation covered Zagreb indices, graph energies, and domination parameters. Furthermore, in another study [5], researchers determined topological indices using some polynomials for the zero-divisor graph. In a study by Chokani, Movahedi, and Taheri, [6], various graph energies were calculated for zero-divisor graphs from commutative rings. Additionally, [7] computed topological indices related to two types of graphs including the total graph and the zero-divisor graph using certain polynomials. Furthermore, in another research work [8], linear codes were examined for the Hamming metric based on the zero-divisor graphs.

In this study, we suppose that a simple graph denoted by $G = (V, E)$ of order $|V| = n$ and size $|E| = m$. For any vertex $w_1$, its open neighborhood is defined as the set $N(w_1) = \{w_2 \in V: w_1w_2 \in E\}$. The size of this neighborhood set is called the degree of vertex $w_1$, denoted as $d(w_1)$. Additionally, we consider the maximum degree of the graph as $\Delta$ and the minimum degree as $\delta$ [9].

Gutman [10] introduced a new class of topological indices based on Euclidean metric space, incorporating a geometric interpretation related to the degree radius. These topological indices are called the Sombor index ($SO$), the average Sombor index ($SO_{avg}$), and the reduced Sombor index ($SO_{red}$) such that

$$SO(G) = \sum_{w_1w_2\in E(G)} \sqrt{d(w_1)^2 + d(w_2)^2},$$

$$SO_{avg}(G) = \sum_{w_1w_2\in E(G)} \sqrt{\left(d(w_1) - \frac{2|E|}{|V|}\right)^2 + \left(d(w_2) - \frac{2|E|}{|V|}\right)^2},$$

and

$$SO_{red}(G) = \sum_{w_1w_2\in E(G)} \sqrt{(d(w_1) - 1)^2 + (d(w_2) - 1)^2}.$$

Motivated by the definition of the Sombor indices and well-known topological indices, several topological indices were defined and studied. In [11] the modified Sombor index is defined as follows

In [12], the authors expanded the degree-based topological indices by introducing multiplicative topological indices. Based on extensive research into multiplicative indices and related applications, several indices have been defined. These include the multiplicative version of Sombor index ($SOII$) introduced by Lio [13], and the multiplicative modified Sombor index ($MSOII$), and the multiplicative reduced Sombor index ($RSOII$), as proposed in [14] such that

$$SOII(G) = \prod_{w_1 w_2 \in E(G)} \sqrt{d(w_1)^2 + d(w_2)^2},$$

$$MSOII(G) = \prod_{w_1 w_2 \in E(G)} \frac{1}{\sqrt{d(w_1)^2 + d(w_2)^2}},$$

and

$$RSOII(G) = \prod_{w_1 w_2 \in E(G)} \sqrt{(d(w_1) - 1)^2 + (d(w_2) - 1)^2}.$$

In this study, we compute the Sombor inidices and the multiplicative Sombor indices for the types of commutative rings including $\Re \backsimeq \mathbb{Z}_{\zeta^2}$, $\Re \backsimeq \mathbb{Z}_{\zeta^3}$, $\Re \backsimeq \mathbb{Z}_{\zeta} \times \mathbb{Z}_{\zeta} \times \mathbb{Z}_{\zeta}$, $\Re \backsimeq \mathbb{Z}_{\zeta_1 \zeta_2}$ and $\Re \backsimeq \mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}$ where $\zeta$, $\zeta_1$ and $\zeta_2$ are the primes.

## 2. Main Results

In this section, we explore various Sombor indices of $\Gamma(\Re)$ for specific commutative rings.

First, we consider the graph $\Gamma(\mathbb{Z}_{\zeta^2})$ for the prime number $\zeta$. In this graph, the set of zero divisors $\mathfrak{Z}^*(\mathbb{Z}_{p^2})$ consists of elements like $\zeta, 2\zeta, \dots, (\zeta-1)\zeta$. Notably, for any pair of elements $w_1$ and $w_2$ in $Z^*(\mathbb{Z}_{\zeta^2})$, their product is zero. Consequently, $\Gamma(\mathbb{Z}_{\zeta^2})$ is a complete graph with $\zeta - 1$ vertices and $\frac{(\zeta-1)(\zeta-2)}{2}$ edges. Note that

$$\frac{2|E(\Gamma(\mathbb{Z}_{\zeta^2}))|}{|V(\Gamma(\mathbb{Z}_{\zeta^2}))|} = \zeta - 2.$$

**Theorem 2.1:** *Let $\Gamma(\mathbb{Z}_{\zeta^2})$ for prime number $\zeta$ be the zero-divisor graph. Then*

(i) $SO(\Gamma(\mathbb{Z}_{\zeta^2})) = \frac{\sqrt{2}}{2}(\zeta-1)(\zeta-2)^2$,

(ii) $SO_{red}(\Gamma(\mathbb{Z}_{\zeta^2})) = \frac{\sqrt{2}}{2}(\zeta-1)(\zeta-2)(\zeta-3)$,

(iii) ${}^{m}SO(\Gamma(\mathbb{Z}_{\zeta^2})) = \frac{\sqrt{2}}{4}(\zeta-1)$,

(iv) $SO_{avg}(\Gamma(\mathbb{Z}_{\zeta^2})) = 0$,

(v) $SOII(\Gamma(\mathbb{Z}_{\zeta^2})) = \left(\sqrt{2}(\zeta-2)\right)^{\frac{(\zeta-1)(\zeta-2)}{2}}$,

(vi) $MSOII(\Gamma(\mathbb{Z}_{\zeta^2})) = \left(\sqrt{2}(\zeta-2)\right)^{-\frac{(\zeta-1)(\zeta-2)}{2}}$,

(vii) $RSOII(\Gamma(\mathbb{Z}_{\zeta^2})) = \left(\sqrt{2}(\zeta-3)\right)^{\frac{(\zeta-1)(\zeta-2)}{2}}$.

We investigate the structure of the graph $\Gamma(\mathbb{Z}_{\zeta^3})$ to compute given topological indices. Note that that this graph is the complete split graph with $\zeta^2-1$ vertices, having the independent set of cardinality $\zeta^2-\zeta$ and the induced subgraph $K_{\zeta-1}$ [15].

**Theorem 2.2:** *Let* $\Gamma(\mathbb{Z}_{\zeta^3})$ *for prime number* $\zeta$ *be the zero-divisor graph. Then*

(i) $SO(\Gamma(\mathbb{Z}_{\zeta^3})) = \zeta(\zeta-1)^2\sqrt{\alpha} + \frac{\sqrt{2}}{2}((\alpha-1)-(\zeta-1)(3\zeta^2-8))$,

(ii) $SO_{red}(\Gamma(\mathbb{Z}_{\zeta^3})) = \zeta(\zeta-1)^2\sqrt{\beta} + \frac{\sqrt{2}}{2}(\zeta-1)(\zeta-2)(\zeta^2-3)$,

(iii) ${}^{m}SO(\Gamma(\mathbb{Z}_{\zeta^3})) = \frac{\zeta-1}{4}\left(\frac{\sqrt{2}(\zeta-2)}{\zeta^2-2} + \frac{4\zeta(\zeta-1)}{\sqrt{\alpha}}\right)$,

(iv) $SO_{avg}(\Gamma(\mathbb{Z}_{\zeta^3})) = \frac{\zeta(\zeta^3-2\zeta^2+1)}{\zeta+1}\left(\frac{\sqrt{2}}{2}(\zeta-2) + (\zeta-1)\sqrt{\zeta^2+1}\right)$,

(v) $SOII(\Gamma(\mathbb{Z}_{\zeta^3})) = 2^{\frac{(\zeta-1)(\zeta-2)}{4}} \times (\zeta^2-2)^{\frac{(\zeta-1)(\zeta-2)}{2}} \times (\alpha)^{\frac{\zeta(\zeta-1)^2}{2}}$,

(vi) $MSOII(\Gamma(\mathbb{Z}_{\zeta^3})) = 2^{\frac{-(\zeta-1)(\zeta-2)}{4}} \times (\zeta^2-2)^{\frac{-(\zeta-1)(\zeta-2)}{2}} \times (\alpha)^{\frac{-\zeta(\zeta-1)^2}{2}}$,

(vii) $RSOII(\Gamma(\mathbb{Z}_{\zeta^3})) = 2^{\frac{(\zeta-1)(\zeta-2)}{4}} \times (\zeta^2-3)^{\frac{(\zeta-1)(\zeta-2)}{2}} \times (\beta)^{\frac{\zeta(\zeta-1)^2}{2}}$.

*where* $\alpha = \zeta^4-3\zeta^2-2\zeta+5$ *and* $\beta = \zeta^4-5\zeta^2-4\zeta+13$.

***Proof:*** From the graph $\Gamma(\mathbb{Z}_{\zeta^3})$ with $\zeta^2-1$ vertices, independence number $\zeta^2-\zeta$, and a clique of size $\zeta-1$, we have $m = E(\Gamma(\mathbb{Z}_{\zeta^3})) = \frac{(\zeta-1)(2\zeta^2-\zeta-2)}{2}$.

The degree of any vertex of the independent set in graph $\Gamma(\mathbb{Z}_{\zeta^3})$ is equal to $\zeta-1$. The degree of any vertex in the induced subgraph $K_{\zeta-1}$ of $\Gamma(\mathbb{Z}_{\zeta^3})$ is $\zeta^2-2$. Therefore, we divide the edge set as follows

$$E_1 = \{w_1w_2 \in E : d(w_1) = \zeta^2-2, d(w_2) = \zeta^2-2\},$$

and

$$E_2 = \{w_1w_2 \in E : d(w_1) = \zeta^2-2, d(w_2) = \zeta-1\}.$$

**Table 1**
**The size of the edge subsets in $\Gamma(\mathbb{Z}_{\zeta^3})$.**

| **Edge-partition sets** | $E_1$ | $E_2$ |
|---|---|---|
| **Number of edges** | $\frac{(\zeta-1)(\zeta-2)}{2}$ | $\zeta(\zeta-1)^2$ |

Since the size of edge subsets in $\Gamma(\mathbb{Z}_{\zeta^3})$ is shown in Table 1, we get

(i)

$$\begin{aligned} SO(\Gamma(\mathbb{Z}_{\zeta^3})) &= |E_1|\sqrt{(\zeta^2-2)^2+(\zeta^2-2)^2} + |E_2|\sqrt{(\zeta^2-2)^2+(\zeta-1)^2} \\ &= \frac{\sqrt{2}}{2}(\zeta-1)(\zeta-2)(\zeta^2-2) + \zeta(\zeta-1)^2\sqrt{(\zeta^2-2)^2+(\zeta-1)^2}. \end{aligned}$$

With simplifications of the above relation and putting $\alpha = \zeta^4 - 3\zeta^2 - 2\zeta + 5$, the result follows.

$$SO_{red}\left(\Gamma(\mathbb{Z}_{\zeta^3})\right) = |E_1|\sqrt{(\zeta^2-3)^2+(\zeta^2-3)^2} + |E_2|\sqrt{(\zeta^2-3)^2+(\zeta-2)^2}$$
$$= \frac{\sqrt{2}}{2}(\zeta-1)(\zeta-2)(\zeta^2-3) + \zeta(\zeta-1)^2\sqrt{(\zeta^2-3)^2+(\zeta-2)^2}.$$

With simplifications of the above relation and putting $\beta = \zeta^4 - 5\zeta^2 - 4\zeta + 13$, the result follows.

(ii)
$$^{m}SO\left(\Gamma(\mathbb{Z}_{\zeta^3})\right) = |E_1|\left(\frac{1}{\sqrt{(\zeta^2-2)^2+(\zeta^2-2)^2}}\right) + |E_2|\left(\frac{1}{\sqrt{(\zeta^2-2)^2+(\zeta-1)^2}}\right)$$
$$= \frac{\zeta-2}{4}\left(\frac{\sqrt{2}(\zeta-1)}{\zeta^2-2} + \frac{4\zeta(\zeta-1)}{\sqrt{\zeta^4-3\zeta^2-2\zeta+5}}\right)$$

(iv) Since $|V| = \zeta^2 - 1$ and $|E| = \frac{(\zeta-1)(2\zeta^2-\zeta-2)}{2}$, we have $\frac{2|E|}{|V|} = \frac{2\zeta^2-\zeta-2}{\zeta+1}$. Thus,

$$SO_{avg}\left(\Gamma(\mathbb{Z}_{\zeta^3})\right) = |E_1|\sqrt{(\zeta^2-2-\frac{2\zeta^2-\zeta-2}{\zeta+1})^2+(\zeta^2-2-\frac{2\zeta^2-\zeta-2}{\zeta+1})^2}$$
$$+|E_2|\sqrt{(\zeta^2-2-\frac{2\zeta^2-\zeta-2}{\zeta+1})^2+(\zeta-1-\frac{2\zeta^2-\zeta-2}{\zeta+1})^2}$$
$$= \frac{\zeta(\zeta^3-2\zeta^2+1)}{\zeta+1}\left(\frac{\sqrt{2}}{2}(\zeta-2) + (\zeta-1)\sqrt{\zeta^2+1}\right).$$

(v)
$$SOII\left(\Gamma(\mathbb{Z}_{\zeta^3})\right) = \left(\sqrt{(\zeta^2-2)^2+(\zeta^2-2)^2}\right)^{|E_1|} \times \left(\sqrt{(\zeta^2-2)^2+(\zeta-1)^2}\right)^{|E_2|}$$
$$= 2^{\frac{(\zeta-1)(\zeta-2)}{4}} \times (\zeta^2-2)^{\frac{(\zeta-1)(\zeta-2)}{2}} \times (\zeta^4-3\zeta^2-2\zeta+5)^{\frac{\zeta(\zeta-1)^2}{2}}.$$

(vi)
$$MSOII\left(\Gamma(\mathbb{Z}_{\zeta^3})\right) = \left(\frac{1}{\sqrt{(\zeta^2-2)^2+(\zeta^2-2)^2}}\right)^{|E_1|} \times \left(\frac{1}{\sqrt{(\zeta^2-2)^2+(\zeta-1)^2}}\right)^{|E_2|}$$
$$= 2^{\frac{-(\zeta-1)(\zeta-2)}{4}} \times (\zeta^2-2)^{\frac{-(\zeta-1)(\zeta-2)}{2}} \times (\zeta^4-3\zeta^2-2\zeta+5)^{\frac{-\zeta(\zeta-1)^2}{2}}.$$

(vii)
$$RSOII\left(\Gamma(\mathbb{Z}_{\zeta^3})\right) = \left(\sqrt{(\zeta^2-3)^2+(\zeta^2-3)^2}\right)^{|E_1|} \times \left(\sqrt{(\zeta^2-3)^2+(\zeta-2)^2}\right)^{|E_2|}$$
$$= 2^{\frac{(\zeta-1)(\zeta-2)}{4}} \times (\zeta^2-3)^{\frac{(\zeta-1)(\zeta-2)}{2}} \times (\zeta^4-5\zeta^2-4\zeta+13)^{\frac{\zeta(\zeta-1)^2}{2}}.$$

Here we consider the graph $\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})$ where $\zeta_1$ and $\zeta_2$ are prime numbers. The vertex set of $\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})$ can divide as

$$V_1 = \{k\zeta_1 | k = 1,2,\dots,\zeta_2 - 1\},$$

and

$$V_2 = \{k\zeta_2 | k = 1,2,\dots,\zeta_1 - 1\}.$$

Thus, $Z^*(\mathbb{Z}_{\zeta_1\zeta_2}) = V_1 \cup V_2$. According to the structure of graph $\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})$, we have for any $w_1, w_2 \in \mathbb{Z}_{\zeta_1\zeta_2}$, $w_1w_2 = 0$ if and only if $w_1 \in V_1$ and $w_2 \in V_2$ or $w_1 \in V_2$ and $w_2 \in V_1$. Therefore, the graph $\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})$ is the complete bipartite graph $K_{\zeta_1-1,\zeta_2-1}$. In the following theorems, we investigate the Sombor-type topological indices of graph $\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})$ for different cases of $\zeta_1$ and $\zeta_2$.

**Theorem 2.3:** *Let $\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})$ be a zero-divisor graph for prime numbers $\zeta_1$ and $\zeta_2$ of order $n$ and size $m$. Then*

(i) $SO(\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})) = m\sqrt{n^2 - 2m}$,

(ii) $SO_{red}(\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})) = m\sqrt{n^2 - 4m + 2}$,

(iii) ${}^{m}SO(\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})) = \frac{m}{n^2-2m}$,

(iv) $SO_{avg}(\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})) = \frac{m(\zeta_1-\zeta_2)}{n}\sqrt{n^2 - 2m}$,

(v) $SOII(\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})) = (n^2 - 2m)^{\frac{m}{2}}$,

(vi) $MSOII(\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})) = (n^2 - 2m)^{-\frac{m}{2}}$,

(vii) $RSOII(\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})) = (n^2 - 4m + 2)^{\frac{m}{2}}$.

***Proof:*** Note that $\Gamma(\mathbb{Z}_{\zeta_1\zeta_2})$ is the complete bipartite graph $K_{\zeta_1-1,\zeta_2-1}$ with $n = \zeta_1 + \zeta_2 - 2$ vertices and $m = (\zeta_1 - 1)(\zeta_2 - 1)$ edges. Therefore, for $u \in V_1$ and $v \in V_2$, we have $d(u) = \zeta_2 - 1$ and $d(v) = \zeta_1 - 1$. Following a similar approach to the proof of Theorem 2.2 and the definition of given Sombor indices, the results are obtained.

Next, we get these topological indices of $\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2})$ which is defined as follows.

From the graph $\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2})$, the vertices of this graph are characterized as follows [16].

$$\begin{aligned}
\mathcal{A} &= \{(0,b): b \in \mathbb{Z}_{\zeta_2}\backslash\{0\}\}, |\mathcal{A}| = \zeta_2 - 1,\\
\mathcal{B} &= \{(a,b): a = \zeta_1, 2\zeta_1, \dots, (\zeta_1 - 1)\zeta_1 \text{ and}\\
&\qquad b \in \mathbb{Z}_{\zeta_2}\backslash\{0\}\}, |\mathcal{B}| = (\zeta_1 - 1)(\zeta_2 - 1), \qquad (1)\\
\mathcal{C} &= \{(a,0): a \in \mathbb{Z}_{\zeta_1^2}\backslash\{0, \zeta_1, 2\zeta_1, \dots, (\zeta_1 - 1)\zeta_1\}\}, |\mathcal{C}| = \zeta_1(\zeta_1 - 1),\\
\mathcal{D} &= \{(a,0): a = \zeta_1, 2\zeta_1, \dots, (\zeta_1 - 1)\zeta_1\}, |\mathcal{D}| = \zeta_1 - 1.
\end{aligned}$$

Therefore, $|V(\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}))| = \zeta_1^2 + \zeta_1\zeta_2 - \zeta_1 - 1$. Based on the above vertex-partitions, the degree of vertices is shown in Table 2 [16].

**Table 2**
**The degree of vertex-partition sets (1) of $\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2})$.**

| $\mathbf{w} \in$ | $\mathcal{A}$ | $\mathcal{B}$ | $\mathcal{C}$ | $\mathcal{D}$ |
|---|---|---|---|---|
| $\mathbf{d}(\mathbf{w})$ | $\zeta_1^2 - 1$ | $\zeta_1 - 1$ | $\zeta_2 - 1$ | $\zeta_1\zeta_2 - 2$ |

**Theorem 2.4:** *Let $\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)$ be the zero-divisor graph for $\zeta_1, \zeta_2$ primes. Then*

(i) $SO\left(\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)\right) = \frac{\alpha}{2(\zeta_2-1)}[2\zeta_1\sqrt{\alpha^2(\zeta_1+1)^2+(\zeta_2-1)^4}$
$+2(\zeta_1-1)\sqrt{\alpha^2+\beta^2}+2\sqrt{\alpha^2(\zeta_1+1)^2+\beta^2}+\frac{\sqrt{2}(\zeta_1-2)\beta}{\zeta_2-1}]$,

(ii) $SO_{red}\left(\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)\right) = \frac{\alpha}{2}[2\zeta_1\sqrt{(\zeta_1^2-2)^2+(\zeta_2-2)^2}$
$+2(\zeta_1-1)\sqrt{(\zeta_1-2)^2+\gamma^2}+2\sqrt{(\zeta_1^2-2)^2+\gamma^2}+\frac{\sqrt{2}(\zeta_1-2)\gamma}{\zeta_2-1}]$,

(iii) ${}^{m}SO\left(\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)\right) = \frac{\alpha(\zeta_2-1)}{2}[\frac{2\zeta_1}{\sqrt{\alpha^2(\zeta_1+1)^2+(\zeta_2-1)^4}}+\frac{2(\zeta_1-1)}{\sqrt{\alpha^2+\beta^2}}$
$+\frac{2}{\sqrt{\alpha^2(\zeta_1+1)^2+\beta^2}}+\frac{\sqrt{2}(\zeta_1-2)}{2(\zeta_2-1)\beta}]$,

(iv) $SO_{avg}\left(\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)\right) = \frac{\alpha}{2}[2\zeta_1\sqrt{(\zeta_1^2+\lambda)^2+(\zeta_2+\lambda)^2}$
$+2(\zeta_1-1)\sqrt{(\zeta_1+\lambda)^2+(\zeta_1\zeta_2+\lambda-1)^2}$
$+2\sqrt{(\zeta_1^2+\lambda)^2+(\zeta_1\zeta_2+\lambda-1)^2}+\frac{\sqrt{2}(\zeta_1-2)(\zeta_1\zeta_2+\lambda-1)}{\zeta_2-1}]$,

(v) $SOII\left(\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)\right) = (\zeta_2-1)^{\frac{\alpha(2\zeta_2+\zeta_1-4)}{1-\zeta_2}}[(\alpha^2(\zeta_1+1)^2+(\zeta_2-1)^4)^{\frac{\zeta_1\alpha}{2}}$
$\times(\alpha^2+\beta^2)^{\frac{\alpha(\zeta_1-1)}{2}}\times(\alpha^2(\zeta_1+1)^2+\beta^2)^{\frac{\alpha}{2}}\times(2)^{\frac{\alpha(\zeta_1-2)}{4(\zeta_2-1)}}\times(\beta)^{\frac{\alpha(\zeta_1-2)}{2(\zeta_2-1)}}]$,

(vi) $MSOII\left(\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)\right) = (\zeta_2-1)^{\frac{\alpha(4\zeta_2+\zeta_1-6)}{2(\zeta_2-1)}}$
$[(\alpha^2(\zeta_1+1)^2+(\zeta_2-1)^4)^{\frac{-\zeta_1\alpha}{2}}\times(\alpha^2+\beta^2)^{\frac{\alpha(1-\zeta_1)}{2}}$
$\times(\alpha^2(\zeta_1+1)^2+\beta^2)^{\frac{-\alpha}{2}}\times(\sqrt{2})^{\frac{\alpha(\zeta_1-2)}{2(1-\zeta_2)}}\times(\beta)^{\frac{\alpha(\zeta_1-2)}{2(1-\zeta_2)}}]$,

(vii) $RSOII\left(\Gamma\left(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}\right)\right) = ((\zeta_1^2-2)^2+(\zeta_2-2)^2)^{\frac{\alpha\zeta_1}{2}}\times((\zeta_1-2)^2$
$+\lambda^2)^{\frac{\alpha(\zeta_1-1)}{2}}\times((\zeta_1^2-2)^2+\lambda^2)^{\frac{\alpha}{2}}\times(2)^{\frac{\alpha(\zeta_1-2)}{4(\zeta_2-1)}}\times(\lambda)^{\frac{\alpha(\zeta_1-2)}{2(\zeta_2-1)}}$,

where $\alpha = (\zeta_1-1)(\zeta_2-1)$, $\beta = (\zeta_1\zeta_2-2)(\zeta_2-1)$, $\gamma = \zeta_1\zeta_2-3$ and $\lambda = -1-\frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}$.

***Proof:*** Let $G$ be the graph $\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2})$. Thus, $G$ contains $\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1$ vertices. The partition of edge set in this graph is as follows [16].

$E_1 = \{w_1w_2 \in E: w_1 \in \mathcal{A}, w_2 \in \mathcal{C}\}, E_2 = \{w_1w_2 \in E: w_1 \in \mathcal{B}, w_2 \in \mathcal{D}\},$
$E_3 = \{w_1w_2 \in E: w_1 \in \mathcal{A}, w_2 \in \mathcal{D}\}, E_4 = \{w_1w_2 \in E: w_1 \in \mathcal{D}, w_2 \in \mathcal{D}\}.$

Therefore, $|E(\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2}))| = \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{2}$. Table 3 is shown the cardinality of edge-partition sets of $\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2})$.

**Table 3**
**The size of the edge subsets in $\Gamma(\mathbb{Z}_{\zeta_1^2} \times \mathbb{Z}_{\zeta_2})$.**

| Edge-partition sets | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|
| Number of edges | $(\zeta_1^2 - \zeta_1)(\zeta_2 - 1)$ | $(\zeta_1 - 1)^2(\zeta_2 - 1)$ | $(\zeta_1 - 1)(\zeta_2 - 1)$ | $\frac{(\zeta_1 - 1)(\zeta_1 - 2)}{2}$ |

By referring to Table 3, we get

(i) $SO(G) = ((\zeta_1^2 - \zeta_1)(\zeta_2 - 1))\sqrt{(\zeta_1^2 - 1)^2 + (\zeta_2 - 1)^2}$
$+((\zeta_1 - 1)^2(\zeta_2 - 1))\sqrt{(\zeta_1 - 1)^2 + (\zeta_1\zeta_2 - 2)^2}$
$+((\zeta_1 - 1)(\zeta_2 - 1))\sqrt{(\zeta_1^2 - 1)^2 + (\zeta_1\zeta_2 - 2)^2}$
$+\frac{\sqrt{2}}{2}(\zeta_1 - 1)(\zeta_1 - 2)(\zeta_1\zeta_2 - 2).$

We put $\alpha = (\zeta_1 - 1)(\zeta_2 - 1)$ and $\beta = (\zeta_1\zeta_2 - 2)(\zeta_2 - 1)$. With simplifications of the above relation and since $\zeta_1 - 1 = \frac{\alpha}{\zeta_2-1}$ and $(\zeta_1^2 - 1)^2 = (\zeta_1 - 1)^2(\zeta_1 + 1)^2 = \frac{\alpha^2(\zeta_1+1)^2}{(\zeta_2-1)^2}$, we have

$SO(G) = \frac{\alpha}{2(\zeta_2-1)}[2\zeta_1\sqrt{\alpha^2(\zeta_1 + 1)^2 + (\zeta_2 - 1)^4}$
$+2(\zeta_1 - 1)\sqrt{\alpha^2 + \beta^2} + 2\sqrt{\alpha^2(\zeta_1 + 1)^2 + \beta^2} + \frac{\sqrt{2}(\zeta_1-2)\beta}{\zeta_2-1}].$

(ii) $SO_{red}(G) = ((\zeta_1^2 - \zeta_1)(\zeta_2 - 1))\sqrt{(\zeta_1^2 - 2)^2 + (\zeta_2 - 2)^2}$
$+((\zeta_1 - 1)^2(\zeta_2 - 1))\sqrt{(\zeta_1 - 2)^2 + (\zeta_1\zeta_2 - 3)^2}$
$+((\zeta_1 - 1)(\zeta_2 - 1))\sqrt{(\zeta_1^2 - 2)^2 + (\zeta_1\zeta_2 - 3)^2}$
$+\frac{\sqrt{2}}{2}(\zeta_1 - 1)(\zeta_1 - 2)(\zeta_1\zeta_2 - 3).$

With simplifications of the above relation and putting $\alpha = (\zeta_1 - 1)(\zeta_2 - 1)$ and $\gamma = \zeta_1\zeta_2 - 3$, the result follows.

(iii) ${}^{m}SO(G) = \frac{(\zeta_1^2-\zeta_1)(\zeta_2-1)}{\sqrt{(\zeta_1^2-1)^2+(\zeta_2-1)^2}} + \frac{(\zeta_1-1)^2(\zeta_2-1)}{\sqrt{(\zeta_1-1)^2(\zeta_1\zeta_2-2)^2}}$
$+\frac{(\zeta_1-1)(\zeta_2-1)}{\sqrt{(\zeta_1^2-1)^2+(\zeta_1\zeta_2-2)^2}} + \frac{\sqrt{2}(\zeta_2-1)(\zeta_2-1)}{\sqrt{4(\zeta_1\zeta_2-2)}}$

With simplifications of the above relation and putting $\alpha = (\zeta_1 - 1)(\zeta_2 - 1)$ and $\beta = (\zeta_1\zeta_2 - 2)(\zeta_2 - 1)$, we have

$$^{m}SO(G) = \frac{\alpha(\zeta_2-1)}{2}\Big[\frac{2\zeta_1}{\sqrt{\alpha^2(\zeta_1-1)^2+(\zeta_2-1)^4}} + \frac{2(\zeta_1-1)}{\sqrt{\alpha^2+\beta^2}} + \frac{2}{\sqrt{\alpha^2(\zeta_1+1)^2+\beta^2}} + \frac{\sqrt{2}(\zeta_1-2)}{2(\zeta_2-1)\beta}\Big]$$

(iv) Since $|V| = \zeta_1^2 + \zeta_1\zeta_2 - \zeta_1 - 1$ and $|E| = \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{2}$, we have $\frac{2|E|}{|V|} = \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}$. Thus,

$$SO_{avg}(G) = |E_1|\sqrt{\left(\zeta_1^2 - 1 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2 + \left(\zeta_2 - 1 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2}$$
$$+|E_2|\sqrt{\left(\zeta_1 - 1 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2 + \left(\zeta_1\zeta_2 - 2 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2}$$
$$+|E_3|\sqrt{\left(\zeta_1^2 - 1 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2 + \left(\zeta_1\zeta_2 - 2 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2}$$
$$+|E_4|\sqrt{\left(\zeta_1\zeta_2 - 2 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2 + \left(\zeta_1\zeta_2 - 2 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}\right)^2}$$

With simplifications of the above relation and putting $\alpha = (\zeta_1 - 1)(\zeta_2 - 1)$ and $\lambda = -1 - \frac{(\zeta_1-1)(4\zeta_1\zeta_2-3\zeta_1-2)}{\zeta_1^2+\zeta_1\zeta_2-\zeta_1-1}$, we have

$$SO_{avg}(G) = \frac{\alpha}{2}[2\zeta_1\sqrt{(\zeta_1^2 + \lambda)^2 + (\zeta_2 + \lambda)^2}$$
$$+2(\zeta_1 - 1)\sqrt{(\zeta_1 + \lambda)^2 + (\zeta_1\zeta_2 + \lambda - 1)^2}$$
$$+2\sqrt{(\zeta_1^2 + \lambda)^2 + (\zeta_1\zeta_2 + \lambda - 1)^2} + \frac{\sqrt{2}(\zeta_1-2)(\zeta_1\zeta_2+\lambda-1)}{\zeta_2-1}].$$

(v) $$SOII(G) = ((\zeta_1^2 - 1)^2 + (\zeta_2 - 1)^2)^{\frac{(\zeta_1^2-\zeta_1)(\zeta_2-1)}{2}}$$
$$\times ((\zeta_1 - 1)^2 + (\zeta_1\zeta_2 - 2)^2)^{\frac{(\zeta_1-1)^2(\zeta_2-1)}{2}}$$
$$\times ((\zeta_1^2 - 1)^2 + (\zeta_1\zeta_2 - 2)^2)^{\frac{(\zeta_1-1)(\zeta_2-1)}{2}} \times \left(\sqrt{2}(\zeta_1\zeta_2 - 2)\right)^{\frac{(\zeta_1-1)(\zeta_1-2)}{2}}.$$

With simplifications of the above relation and putting $\alpha = (\zeta_1 - 1)(\zeta_2 - 1)$ and $\beta = (\zeta_1\zeta_2 - 2)(\zeta_2 - 1)$, we have

$$SOII(G) = (\zeta_2 - 1)^{\frac{\alpha(2\zeta_2+\zeta_1-4)}{1-\zeta_2}}[(\alpha^2(\zeta_1 + 1)^2 + (\zeta_2 - 1)^4)^{\frac{\zeta_1\alpha}{2}}$$
$$\times (\alpha^2 + \beta^2)^{\frac{\alpha(\zeta_1-1)}{2}} \times (\alpha^2(\zeta_1 + 1)^2 + \beta^2)^{\frac{\alpha}{2}}$$
$$\times (2)^{\frac{\alpha(\zeta_1-2)}{4(\zeta_2-1)}} \times (\beta)^{\frac{\alpha(\zeta_1-2)}{2(\zeta_2-1)}}].$$

(vi) $$MSOII(G) = \left(\frac{1}{(\zeta_1^2-1)^2+(\zeta_2-1)^2}\right)^{\frac{(\zeta_1^2-\zeta_1)(\zeta_2-1)}{2}}$$

$$\times \left(\frac{1}{(\zeta_1-1)^2+(\zeta_1\zeta_2-2)^2}\right)^{\frac{(\zeta_1-1)^2(\zeta_2-1)}{2}} \times \left(\frac{1}{(\zeta_1^2-1)^2+(\zeta_1\zeta_2-2)^2}\right)^{\frac{(\zeta_1-1)(\zeta_2-1)}{2}}$$

$$\times \left(\frac{\sqrt{2}}{2(\zeta_1\zeta_2-2)}\right)^{\left(\frac{(\zeta_1-1)(\zeta_1-2)}{2}\right)}.$$

With simplifications of the above relation and putting $\alpha = (\zeta_1 - 1)(\zeta_2 - 1)$ and $\beta = (\zeta_1\zeta_2 - 2)(\zeta_2 - 1)$, we have

$$SOII(G) = (\zeta_2 - 1)^{\frac{\alpha(4\zeta_2+\zeta_1-6)}{2(\zeta_2-1)}} [(\alpha^2(\zeta_1 + 1)^2 + (\zeta_2 - 1)^4)^{\frac{-\zeta_1\alpha}{2}}$$

$$\times (\alpha^2 + \beta^2)^{\frac{\alpha(1-\zeta_1)}{2}} \times (\alpha^2(\zeta_1 + 1)^2 + \beta^2)^{\frac{-\alpha}{2}}$$

$$\times (\sqrt{2})^{\frac{\alpha(\zeta_1-2)}{2(1-\zeta_2)}} \times (\beta)^{\frac{\alpha(\zeta_1-2)}{2(1-\zeta_2)}}].$$

(vii) $$RSOII(G) = ((\zeta_1^2 - 2)^2 + (\zeta_2 - 2)^2)^{\frac{(\zeta_1^2-\zeta_1)(\zeta_2-1)}{2}}$$

$$\times ((\zeta_1 - 2)^2 + (\zeta_1\zeta_2 - 3)^2)^{\frac{(\zeta_1-1)^2(\zeta_2-1)}{2}}$$

$$\times ((\zeta_1^2 - 2)^2 + (\zeta_1\zeta_2 - 3)^2)^{\frac{(\zeta_1-1)(\zeta_2-1)}{2}}$$

$$\times \left(\sqrt{2}(\zeta_1\zeta_2 - 3)\right)^{\frac{(\zeta_1-1)(\zeta_1-2)}{2}}.$$

With simplifications of the above relation and putting $\alpha = (\zeta_1 - 1)(\zeta_2 - 1)$ and $\gamma = \zeta_1\zeta_2 - 3$, the result follows.

Finally, we compute the Sombor-type topological indices of $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$ [17]. The vertices of graph $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$ are characterized as follows.

$$\begin{aligned}
V_1 &= \{(a, 0,0): a \in \mathbb{Z}_\zeta\backslash\{0\}\}, |V_1| = \zeta - 1,\\
V_2 &= \{(0, b, 0): b \in \mathbb{Z}_\zeta\backslash\{0\}\}, |V_2| = \zeta - 1,\\
V_3 &= \{(0,0, c): c \in \mathbb{Z}_\zeta\backslash\{0\}\}, |V_3| = \zeta - 1,\\
V_4 &= \{(0, b, c): b, c \in \mathbb{Z}_\zeta\backslash\{0\}\}, |V_4| = (\zeta - 1)^2,\\
V_5 &= \{(a, 0, c): a, c \in \mathbb{Z}_\zeta\backslash\{0\}\}, |V_5| = (\zeta - 1)^2,\\
V_6 &= \{(a, b, 0): a, b \in \mathbb{Z}_\zeta\backslash\{0\}\}, |V_6| = (\zeta - 1)^2.
\end{aligned} \tag{2}$$

Therefore, $|V(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta))| = \sum_{i=1}^{6} |V_i| = 3\zeta(\zeta - 1)$. Based on the above vertex-partitions, the degree of vertices is shown in Table 4.

**Table 4**
**The degree of vertex-partition sets (2) of $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$.**

| $\mathbf{w} \in$ | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ |
|---|---|---|---|---|---|---|
| $\mathbf{d}(\mathbf{w})$ | $\zeta^2 - 1$ | $\zeta^2 - 1$ | $\zeta^2 - 1$ | $\zeta - 1$ | $\zeta - 1$ | $\zeta - 1$ |

According to the structure of $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$, the partition of edge set is as follows.

$E_1 = \{w_1w_2 \in E: w_1 \in V_1, w_2 \in V_2\}$, $E_2 = \{w_1w_2 \in E: w_1 \in V_1, w_2 \in V_3\}$,
$E_3 = \{w_1w_2 \in E: w_1 \in V_2, w_2 \in V_3\}$, $E_4 = \{w_1w_2 \in E: w_1 \in V_1, w_2 \in V_4\}$,
$E_5 = \{w_1w_2 \in E: w_1 \in V_2, w_2 \in V_5\}$, $E_6 = \{w_1w_2 \in E: w_1 \in V_3, w_2 \in V_6\}$.

Therefore,

$$|E(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta))| = \sum_{i=1}^{6} |E_i| = 3\zeta(\zeta-1)^2.$$

The size of edge subsets of $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$ is shown in Table 5.

**Table 5**
**The size of the edge subsets in $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$.**

| **Edge-partition sets** | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ |
|---|---|---|---|---|---|---|
| **Number of edges** | $(\zeta-1)^2$ | $(\zeta-1)^2$ | $(\zeta-1)^2$ | $(\zeta-1)^3$ | $(\zeta-1)^3$ | $(\zeta-1)^3$ |

**Theorem 2.5:** *Let $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$ be the zero-divisor graph for $\zeta$ prime. Then*

(i) $SO\left(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)\right) = 3(\zeta-1)^3\left[\sqrt{2}(\zeta+1) + (\zeta-1)\sqrt{(\zeta+1)^2+1}\right]$,

(ii) $SO_{red}\left(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)\right) =$
$3(\zeta-1)^2\left[\sqrt{2}(\zeta^2-2) + (\zeta-1)\sqrt{\zeta^4-3\zeta^2-4\zeta+8}\right]$,

(iii) ${}^{m}SO\left(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)\right) = \frac{3(\zeta-1)}{2}\left[\frac{\sqrt{2}}{(\zeta^2-1)} + \frac{6}{\sqrt{(\zeta+1)^2+1}}\right]$,

(iv) $SO_{avg}\left(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)\right) = 3(\zeta-1)^4\left[\sqrt{2} + \sqrt{(\zeta+1)^2+1}\right]$,

(v) $SOII\left(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)\right) = (2)^{\frac{3(\zeta-1)^2}{2}} \times (\zeta-1)^{3\zeta(\zeta-1)^2} \times (\zeta+1)^{3(\zeta-1)^2} \times$
$((\zeta+1)^2+1)^{3(\zeta-1)^3}$,

(vi) $MSOII\left(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)\right) = (2)^{\frac{-3(\zeta-1)^2}{2}} \times (\zeta-1)^{-3\zeta(\zeta-1)^2}$
$\times (\zeta+1)^{-3(\zeta-1)^2} \times ((\zeta+1)^2+1)^{-3(\zeta-1)^3}$,

(vii) $RSOII\left(\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)\right) =$
$(2)^{\frac{3(\zeta-1)^2}{2}} \times (\zeta^2-2)^{3(\zeta-1)^2} \times \left(\sqrt{\zeta^4-3\zeta^2-4\zeta+8}\right)^{3(\zeta-1)^3}$.

***Proof:*** Consider $G$ is the zero-divisor graph $\Gamma(\mathbb{Z}_\zeta \times \mathbb{Z}_\zeta \times \mathbb{Z}_\zeta)$. Thus, $G$ contains $3\zeta(\zeta-1)$ vertices and $3\zeta(\zeta-1)^2$ edges. From Table 5, we get

(i) $SO(G) = 3(\zeta-1)^2\sqrt{2(\zeta^2-2)^2} + 3(\zeta-1)^3\sqrt{(\zeta-2)^2+(\zeta^2-2)^2}$
$= 3(\zeta-1)^3\left[\sqrt{2}(\zeta+1) + (\zeta-1)\sqrt{(\zeta+1)^2+1}\right]$.

(ii) $SO_{red}(G) = 3(\zeta-1)^2\sqrt{2(\zeta^2-1)^2} + 3(\zeta-1)^3\sqrt{(\zeta-1)^2+(\zeta^2-1)^2}$
$= 3(\zeta-1)^2\left[\sqrt{2}(\zeta^2-2) + (\zeta-1)\sqrt{\zeta^4-3\zeta^2-4\zeta+8}\right].$

(iii) ${}^{m}SO(G) = \frac{(3(\zeta-1)^2)}{\sqrt{2(\zeta^2-1)^2}} + \frac{3(\zeta-1)^3}{\sqrt{9(\zeta-1)^2+(\zeta^2-1)^2}}$
$= \frac{3(\zeta-1)}{2}\left[\frac{\sqrt{2}}{(\zeta^2-1)} + \frac{6}{\sqrt{(\zeta+1)^2+1}}\right].$

(iv) Since $|V| = 3\zeta(\zeta-1)$ and $|E| = 3\zeta(\zeta-1)^2$, we have $\frac{2|E|}{|V|} = 2(\zeta-1)$.
Thus,
$SO_{avg}(G) = 3(\zeta-1)^2\sqrt{2(\zeta-1)^4} + 3(\zeta-1)^3\sqrt{(\zeta-1)^2+(\zeta-1)^4}$
$= 3(\zeta-1)^4\left[\sqrt{2} + \sqrt{(\zeta+1)^2+1}\right].$

(v) $SOII(G) = (2)^{\frac{3(\zeta-1)^2}{2}} \times (\zeta^2-1)^{3(\zeta-1)^2} \times \left(\sqrt{(\zeta^2-1)^2+(\zeta-1)^2}\right)^{3(\zeta-1)^3}$
$= (2)^{\frac{3(\zeta-1)^2}{2}} \times (\zeta-1)^{3\zeta(\zeta-1)^2} \times (\zeta+1)^{3(\zeta-1)^2} \times ((\zeta+1)^2+1)^{3(\zeta-1)^3}.$

(vi) $MSOII(G) = (2)^{\frac{-3(\zeta-1)^2}{2}} \times (\zeta^2-1)^{-3(\zeta-1)^2} \times \left(\sqrt{(\zeta^2-1)^2+(\zeta-1)^2}\right)^{-3(\zeta-1)^3}$
$= (2)^{\frac{-3(\zeta-1)^2}{2}} \times (\zeta-1)^{-3\zeta(\zeta-1)^2} \times (\zeta+1)^{-3(\zeta-1)^2}$
$\times ((\zeta+1)^2+1)^{-3(\zeta-1)^3}.$

(vii) $RSOII(G) = (2)^{\frac{3(\zeta-1)^2}{2}} \times (\zeta^2-2)^{3(\zeta-1)^2} \times \left(\sqrt{(\zeta^2-2)^2+(\zeta-2)^2}\right)^{3(\zeta-1)^3}$
$= (2)^{\frac{3(\zeta-1)^2}{2}} \times (\zeta^2-2)^{3(\zeta-1)^2} \times \left(\sqrt{\zeta^4-3\zeta^2-4\zeta+8}\right)^{3(\zeta-1)^3}.$

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