

Central edge metric dimension of joint graph

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Abstract

Assume that w is a non-trivial connected graph, with u and v representing its vertices and $e = uv$ representing its edges. The shortest total length of the pathways connecting u and v in G is the distance between them, represented by $d(u, v)$. The greatest separation between a vertex u and another vertex in graph G is its eccentricity. The diameter of a graph G is the largest eccentricity among its vertices. Otherwise, the radius of G is equal to the smallest eccentricity. In graph G , a central vertex is the vertex with the least eccentricity. It is feasible to create a distribution path that can reach any location using the concept of central set. The

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issue of extreme poverty can be resolved if this course is created. In the meantime, the set in which every element is a central vertex is called the central set. Conversely, $d(e, w) = \min\{d(u, w), d(v, w)\}$ provides the distance between the edge e and the vertex w . In a linked graph G , a set S of the vertex is an edge generating set of G if all the vertices in S distinguish each of two edges in $E(G)$. And then, the least cardinality of an edge generator set of graph G that contains the central set known as the central edge metric dimension, denoted by $edim_{cen}(G)$. In this paper, we investigate $edim_{cen}(G)$ for joint graphs G .

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1. Introduction

The achievement of public welfare is the goal of the state management toward its people. It is only right that various efforts are made to realize it, including involving intellectuals in their research activities. An aspect that can affect public welfare is the fulfillment of basic communal needs of the community, which include education, health, and security. The selection of strategic locations for public services is an interesting topic of discussion, because with it public services can be optimally accessed by the community. Mathematics as a tool can be used to support various aspects of science, to support the achievement of public welfare, one of which is graph theory. As a part of mathematics concept, graph theory presents a relationship between objects called vertex, and the relationship between two vertices is called an edge. By presenting a map of an area in the form of a graph, namely by viewing an area as a vertex and a line as a road connecting two areas, graph theory can be used to determine strategic locations for public services. The concept of studying this is called the central set.

Graph theory is part of the mathematical concept that studies the set of a object namely vertex set and and relationship between two object, namely the set of edges. The concept of graph theory continues to develop until now, according to its needs in answering existing problems. The application of graph theory also continues to develop. The application of graph theory has been carried out so far in various fields, including in the field of communication, where graph theory is used to construct reliable communication networks. The concepts used to solve communication network problems are the concepts of vertex connectivity and edge connectivity. In the problem of project work, the division of work tasks and the division of effective work teams, with certain rules can be solved with the matching concept. The matching concept can also be used to create schedules in various activities. The application of graph theory has also been carried out to regulate the duration of the red, green, and yellow lights on traffic lights. The setting of the duration of the time between light colors affects the length of the queue on one side of the road, if not right. This setting is influenced by the density

of vehicles passing through the road from a lane. For denser lanes, the duration of the green light must be longer than the duration of the green light on a lane with a lower density level. Complex lecture scheduling in universities can also be optimized using graph theory. The complexity level of this scheduling is influenced by space limitations, courses taken by students in one class must not clash, and paying attention to various different study programs, but taking the same courses.

In transportation matters, the graph concept can be used to determine the shortest path between two cities. The use of the shortest path concept is also needed for public services, in terms of sending basic necessities such as rice, cooking oil, vegetables, fruits, fish, meat and others, so that they can reach all places at the lowest possible cost. This efficiency can also save energy, time, and fuel, as well as maintain the quality of the goods sent so that they are not damaged during shipping. This shortest path can also determine the distribution route for basic necessities, so that the country can minimize costs without reducing the objective function in distributing people's basic needs. In this shortest path, there are two possibilities, namely paying attention to the aspect of the length of the road or the aspect of the severity of the terrain. It could be that the shortest path chosen is the path that is further away, but in terms of travel time it is shorter, because the terrain is good. However, sometimes the one chosen as the shortest path is the path with the closest distance. All of the above conditions can be simulated with various existing conditions, so that the best path can be taken while still considering the function of the destination. Furthermore, to determine the route of travel between cities can be made so that each city is only passed once and can cover the entire city, and end in the city of departure. This kind of route is known as the Hamilton cycle. This Hamilton cycle can also be used to solve the traveling salesman problem, so that the sales route does not repeat the places that have been passed, so that the marketing path of a product is effective.

Furthermore, in relation to public services, a map of a region can be viewed as a graph by making cities as points and roads connecting them as sides. Maps can be presented at the country, province, or sub-district level, to villages as needed, with the status of the points adjusting to their level. The higher the level, the more complex the case. The mapping in question is determining the position of optimal places or infrastructure according to needs. Including in terms of public services, determining strategic places to provide services, namely covering the entire region with the closest relative distance to the entire region. This public service includes basic needs services, both individual and communal. To optimize public services, in addition to involving transportation problems as explained above, it also requires other calculations. If the goods sent in the public service process are in large quantities and cannot reach all areas, then it is necessary to determine a strategic location, so that distribution to more remote places can be done. Likewise for the placement of public service points, such as the position of markets, hospitals, schools, or other public service places, the concept of graph theory that can be utilized is the central set and basis. The central set is a set whose members are central points, namely points with the smallest distance to the

furthest point (or known as the radius of a graph). Given that service locations cannot be provided in every location, service points can be placed at central points, meaning that the service can be reached by all other areas with a relatively close distance.

To control and ensure the running of public service activities, monitoring is needed. The monitoring team must ensure that its monitoring reaches all areas. The monitoring team must be able to identify all areas being monitored correctly. In order to be optimal, namely the monitoring team is as minimal as possible, but can still reach monitoring throughout the area, it is necessary to determine the monitoring points. For monitoring public service activities, the placement of strategic positions for monitoring can use the concept of a basis. The way this set of points recognizes points on the graph is with ordered pairs or coordinates that contain the distances of the points recognized with each point on the discriminant or resolving set. The discriminant or resolving set with minimal cardinality is called a basis. Monitoring sites can be placed at points that form a base, so that costs can be minimized, but monitoring can still run and cover all areas. Coordination between monitoring teams is sufficient between areas represented by points on the base. However, the results will be even better if the monitoring points are also used as service locations, because monitors can also understand the real conditions in the field. It's just that a policy like this will require more funds and manpower. This concept is known in graph theory as a central basis.

Sometimes in public services, more detailed assurances are needed, so that at least one area is directly connected to one service location. The concept for studying this problem in graph theory is known as the dominance set. If this dominance set is also a monitoring location, then the concept of dominance basis can be used. If the dominance set is also used as a public service location, then the concept in graph theory can be developed into a central dominance set. Likewise, if the central dominance set is also expected to be able to monitor the entire area, then it can be developed into a central dominance basis set.

For the case of national defense, we all understand that Indonesia is a large country and an archipelago with extraordinary natural wealth. Indonesia is at the crossroads of two continents and two oceans, as well as on a strategic trade route. This condition makes Indonesia need strong defense and security. National defense is all efforts made to maintain the sovereignty and integrity of the country from threats. For this national defense issue, we are talking about areas that are the borders of a country with other countries. In general, border areas are the outermost areas of a country. The outermost region on a map can be studied in graph theory as the point with the furthest distance to other points is the largest (or known as the diameter of a graph).

Border areas are areas that must receive great attention, considering that this area is vulnerable to the entry of foreign powers that can disrupt the sovereignty and security of the country. Treatment of border areas must be different from the aspect of the existence of sovereignty guarding equipment or defense equipment. It may be that not all border areas require defense equipment, but the optimization of its existence must also be ensured. The optimal placement of these defense

guarding equipment can utilize the concept of diameter in graph theory. For the process of monitoring the condition of the border area, the concept of basis in the graph can be used. The concept of basis in this graph can also be combined with the concept of diameter to determine the locations of defense guarding equipment, as well as being a monitor for the entire region, whose existence must be monitored by the defense center. This means that it is necessary to study strategic locations at the outermost points of a country, so that the country's defense equipment from foreign interference can be held as effectively as possible and connected to the defense center, but the goals of the country's defense that have been designed are still met. The combination of the concept of basis with diameter has not been done until now. The development theory related to diameter is still wide open. In theory, findings related to the metric dimension of a graph or the graph operations have been done, but if the facts of the graph change, especially if the order and size of the graph are very large, then finding a discriminant set with minimum cardinality is not an easy matter, especially if done manually. The problem of finding a discriminant set with minimum cardinality is a problem with a high level of complexity. For this reason, collaboration with the field of computing is something interesting to do. Until today, programming related to this is still rarely done.

The role of graph theory in more detail in public services is for the management of transportation infrastructure, including designing efficient public transportation networks, such as bus or train routes, optimizing passenger flow, and minimizing travel time. Optimizing transportation routes and schedules can use graph algorithms such as Dijkstra or Floyd-Warshall to minimize travel time. The algorithm must be followed up by programmers or informatics experts to present it in a computing program at the design level. To enter the development planning program, economics is needed for cost analysis and resource allocation and statisticians for data flow modeling and network analysis. Political scientists are needed to analyze and optimize state resource distribution policies. Experts in economics and public management are needed in managing state resources more efficiently and effectively. For example, cost or benefit analysis to help the government determine the right budget allocation for the sectors that need it most. In addition, a good public administration system can speed up the process of serving the community, thereby reducing complicated bureaucracy. Information and communication technology (ICT) experts enable faster and easier public services through the e-government system. Research in the field of ICT helps create a more integrated and transparent system, allowing people to access services more easily and efficiently. For example, e-government applications that can facilitate administrative processes such as making ID cards. From the aspect of physical development, Geophysics and Geology experts are needed to ensure that road construction is carried out in a safe position from potential disasters. From the aspect of transportation equipment, mechanical experts, shipping experts, aircraft experts and navigation experts are needed to ensure the quality of the engine against the capacity and strength of the vehicle in delivering goods, and predicting the weather so that the delivery of goods safely reaches its

destination. Likewise, for service providers such as hospitals, clinics, and health facilities, in addition to requiring doctors, pharmacists, nurses, medical technology experts, nutritionists, and management experts, they also need graph theory for doctor scheduling, distribution of medical resources, and queuing systems in patient flow. Using path finding algorithms to determine patient evacuation routes or efficient distribution of drugs. All of this also requires information system experts and IT experts who are specifically tasked with managing hospital technology infrastructure including hardware and software maintenance, managing networks, and ensuring data security. Equally important are environmental health experts who ensure that hospital waste is disposed of in an environmentally safe manner.

One of the problems that the world faces today is extreme poverty, one of the causes of which is the suboptimal distribution of basic human needs. In graph theory, there is a concept of central metric dimension which has the prospect of solving distribution optimization problems. Metric dimension is a concept in graph theory that was first introduced in [1] and then continued in [2]. This concept refers to the minimum cardinality of an ordering vertices set that makes all vertices in the graph have different coordinates. The ordering set is called locating set or resolving set. From a minimality perspective, many problems in life can be solved using the concept of metric dimensions. In [3] use metric dimension in robot navigation problems. A robot can determine its position in an environment using distance measurements from fixed beacons. A resolving set serves as these beacons. From network security, in [4] uses metric dimensions in the installation of fire sensors. In the other side, the concept of metric dimensions is also used in problems solving chemical structures [5].

Viewed from the perspective of concept development, in [6] modified the metric dimension's concept and named the modified results as strong metric dimension. Then, in [7] present the concept of local metric dimension. Furthermore, in [8] developed the metric dimension's concept by building the concept of neighborhood metric dimensions followed by the concept of neighborhood local metric dimension. Both of them built this new definition by looking at the distance between two points in a graph, there are only three possibilities, namely 0, 1, or 2. In [9] developed the concept of metric dimension for a family of graphs named simultaneous metric dimensions. In this research, Ramirez examines the simultaneous metric dimensions of families of graphs that have the same set of points.

By taking into account the distance of points to sides rather than to other points, in [10] created the idea of metric dimension. By defining distance vertex to an edge is equal to the minimum of its distances to the two end points of that side, concept of edge metric dimension is established. A set of vertices whose order has been determined and causes all edges have different coordinates is called an edge metric generator. Next, the edge basis is the collection of edge distinctions with the lowest cardinality, and the edge metric dimension is the cardinality of this set. As with metric dimension, edge metric dimension also applied in solving life problems. Prabhu and Janany [11] uses edge metric dimension to solve

problem in silicate networks and Sharma, et al [12] in chemical structure problems. In the field of graph theory, in [13] provided edge metric dimension for Petersen Graph and its generalizations. Furthermore, in [14] provided results of edge metric dimension for several graphs resulting from operations. Regarding the relationship between metric dimension and edge metric dimensions, In [15] provided a graph construction so that the $\dim(G)$ are greater than the $\text{edim}(G)$.

The ideas of eccentricity, radius, and diameter on graphs were presented in [16]. The eccentricity of a graph G is the farthest distance between any two vertices of its vertex. The diameter of G is the largest eccentricity between its vertices. The radius is the lowest eccentricity. On a graph G , the center vertex of G is a vertex with the least eccentricity. The set whose entries are all center vertices is known as the center set of a graph G . The concept of eccentricity can be viewed as vertex that have a relatively close distance to all vertex on the graph. This allows its application to optimization problems. In [17] is coined the term "Central Metric Dimension" to refer to the metric dimension notion combined with center sets. Furthermore, in [18] is coined the term "Local Central Metric Dimension" by fusing the ideas of local metric dimension and the center set. The local central metric dimension for complete graphs, complete bipartite graphs and star graphs has been given.

Results and Discussion

The focus of this research is to find the central edge metric dimension of the graph resulting from the join operation of any two connected graphs. Let's say the two graphs are connected graph G and connected graph H . The first thing to do is to find the metric dimensions of the graph resulting from the joining operation of graphs G and H where G is complete graph with one vertex and H is a special graph such as a circle graph and a path graph. First, we determine the central edge metric dimension of the wheel graph which is the result of the operation of joining graph K_1 with circle graph C_n . Second, we determine the central edge metric dimension of the fan graph which is the result of the operation of joining graph K_1 with the path graph P_n . Third, we determine the central edge metric dimension of the friendship graph which is the result of the operation of joining graph K_1 with the combined path graph P_2 . Fourth, we determine the central edge metric dimension of the operation of joining graph K_1 with arbitrary connected graph H . Finally, we determine the metric dimension of the central edge of the joined graph G and H for any connected graph G and H .

In this section we will discuss the combination of the concepts of central set and edge metric dimensions. Let G be a connected graph with $W = \{w_1, w_2, \dots, w_k\} \subseteq V(G)$. For $e = uv \in E(G)$, coordinate e respect to W is denoted by k -tuple $r(e|W)$, that is

$$r(e|W) = (d(e, w_1), d(e, w_2), d(e, w_3), \dots, d(e, w_k))$$

with $d(e, w_i) = \min\{d(u, w_i), d(v, w_i)\}$ for $i = 1, 2, \dots, k$. For $t = 1, 2, 3, \dots, k$, we define $r(e|W)_t = d(v, w_t)$ and $|r(e|W)|_s = |\{i: d(v, w_i) = s\}|$. If $r(e|W) \neq r(f|W)$ for all $e, f \in E(G)$, we called W as edge generator

for G . A central edge generator for G is a edge generator for G that contains the center set of G . The central edge metric dimension, represented as $edim_{cen}(G)$, is the minimum cardinality of central edge generator for G .

Visually, the meaning of the set W as an edge generator is presented as follows. The representation of edge e to the vertex set W states the perspective of the set W to edge e . The set W can identify each edge in a graph G if each edge has a distinct representation from the set W . The perspective of W to distinguish each edge is by using ordered pairs, whose elements are the distances of the edge to each ordered vertex in the set W . Since the set $V(G)$ is a edge generator for G , then, the interesting thing to study is the edge generator for G with minimum cardinality.

The following presents the properties of the edge coordinate with respect to the set that contains it.

Lemma 1: *Let G be a connected graph and $S = \{v_1, v_2, \dots, v_k\}$ be a subset of $V(G)$. If F is a subset of $E(G)$ with $F = \{e = v_i v_j \in E(G) : v_i, v_j \in S\}$, then $r(e|S) \neq r(f|S)$ for every $e, f \in F$ where $e \neq f$.*

Proof: Let for $e = v_i v_j \in F$, $r(e|S) = (a_1, a_2, \dots, a_k)$. Then, $a_t = 0$ for $t = i, j$. Suppose there are $e_m, e_n \in F$ with $e_m \neq e_n$ such that $r(e_m|S) = r(e_n|S)$, then $r(e_m|S)$ and $r(e_n|S)$ have two entry 0 in same position. Therefore, we have $e_m = e_n$.

Lemma 1 say that every different edges whose endpoints are both points on the edge generator have different representations from each other. Then, from Lemma 1 above we get corollary below.

Lemma 2: *For connected graph G , $V(G)$ is edge generator for G .*

Proof: If $S = V(G)$, then every edge have end point in S because G connected. Therefore, $V(G)$ is edge generator for G by Lemma 1.

Lemma 2 say that set of all vertex of a graph is a edge generator for itself. Apart from Lemma 2, we have another corollary from Lemma 1.

Lemma 3: *Assume that G is a connected graph and that $C(G)$ is its center set. If $C(G)$ contains all the vertices of G , then $edim_{cen}(G)$ is the order of G .*

Proof: If $C(G) = V(G)$, then $S = V(G)$ because S must be contain $C(G)$. So, S is central edge generator for G by Lemma 1. In the other word, $edim_{cen}(G) = |V(G)|$.

Lemma 3 say that if all vertices of G are center set, then the central edge metric dimension of G is the order of G . Next, For a connected graph G with order n , we denoted set $P(G)$ as set of all vertex which degree $n - 1$ and $C(G)$

as the center set of G . Next, we investigate existence the central set of joint graph $G + H$.

Let G and H are connected graphs with order m and n , respectively. Suppose $V(G) = \{v_1, v_2, v_3, v_4, \dots, v_m\}$ and $V(H) = \{u_1, u_2, u_3, u_4, \dots, u_n\}$, then joint graph of G and H is denoted by $G + H$, that is graph with set of all vertices is $\{g_{v_1}, g_{v_2}, g_{v_3}, \dots, g_{v_n}, h_{u_1}, h_{u_2}, h_{u_3}, \dots, h_{u_m}\}$, and set of all edges is $\{g_{v_i}g_{v_j} : v_i v_j \in E(G)\} \cup \{h_{u_i}h_{u_j} : u_i u_j \in E(H)\} \cup \{g_{v_i}h_{u_j} : v_i \in V(G), u_j \in V(H)\}$. It's easy to see that order of graph $G + H$ is $m + n$ and size of graph $G + H$ is $|E(G)| + |E(H)| + m \times n$.

Theorem 1: Let G and H are connected graph with $C(G)$ is the center set of G and $C(H)$ is the center set of H . If $P(G) = \{v \in V(G) : d(v) = |V(G)| - 1\}$, then

$$C(G + H) = \begin{cases} P(G) \cup P(H) & ; P(G) \neq \emptyset \text{ or } P(H) \neq \emptyset \\ V(G + H) & ; \text{the others} \end{cases}$$

Proof: Let $P(G) \neq \emptyset$ or $P(H) \neq \emptyset$. For $u \in P(G)$ we have $d(u, t) = 1$ for all $t \in G + H$. So, $e(u) = 1$ for $u \in P(G)$. For $u \in V(G) \setminus P(G)$ we have $d(u, t) = 1$ for $t \in V(H)$ or $ut \in E(G)$. In the other side, for $u \in V(G) \setminus P(G)$ we have $d(u, t) = 2$ for $ut \notin E(G)$. So, $e(u) = 2$ for u in $V(G)$ but not in $P(G)$. For vertex v in $P(H)$ we have the distance of v and t is equal to 1 for all $t \in V(G + H)$. So, $e(v) = 1$ for $v \in P(H)$. For $u \in V(H) \setminus P(H)$ we have $d(v, t) = 1$ for $t \in V(G)$ or $vt \in E(H)$. In the other side, for $v \in V(H) \setminus P(H)$ we have $d(u, t) = 2$ for $ut \notin E(H)$. So, $e(u) = 2$ for $u \in V(H) \setminus P(H)$. Therefore, we have the center set of $G + H$, $C(G + H)$, is equal to $P(G) \cup P(H)$. Therefore, $P(G) \cup P(H) = C(G) \cup C(H)$. So, we have $C(G + H) = C(G) \cup C(H)$.

For $P(G) = \emptyset$ and $P(H) = \emptyset$, $e(u) = 2$ for $u \in G + H$ because for $K = G, H$ and $u \in V(K)$ we have $d(u, t) = 1$ for $t \in V(G + H) \setminus V(K)$ or $ut \in E(K)$; and we have $d(u, t) = 2$ for $ut \notin E(K)$. Therefore, $rad(G + H) = diam(G + H)$. So, based Lemma 1 we have $C(G + H) = V(G + H)$.

Theorem 1 say that to find the existence of a center set of a graph $G + H$, we can consider the set of vertex in G that have degree $|V(G)| - 1$ and the set of vertex in H that have degree $|V(H)| - 1$. If G has no vertices with degree $|V(G)| - 1$ and H also has no vertices with degree $|V(H)| - 1$ then the center vertex set of $G + H$ is all vertices of $G + H$. However, if this is not the case then the center set of $G + H$ is the set whose elements are all vertices in $V(G)$ that have degree $|V(G)| - 1$ and all vertices in $V(H)$ that have degree $|V(H)| - 1$. Next, After becoming aware that a joint graph's center set exists, we are ready to find the central edge metric dimension of a joint graph. We begin for wheel graph $K_1 + C_n$.

Theorem 2: For $G = K_1 + C_n$, then $edim_{cen}(G) = n$.

Proof: Let $V(K_1) = \{v\}$ and $V(K_1 + C_n) = \{u, u_1, u_2, \dots, u_n\}$ with $d(u) = n$. Since $P(K_1) = \{v\}$ and $P(C_n) = \emptyset$, then according by Theorem 1, we have $C(K_1 + C_n) = \{u\}$. Next, let $W = \{u, u_1, u_2, \dots, u_{n-1}\}$, will be proved that W is a central edge generator set for G . First, clearly that $C(K_1 + C_n) \subset W$. And then:

1. $r(uu_i|W)$ and $r(u_j u_{j+1}|W)$ all are different for $i = 1, 2, 3, \dots, n-1$ and $j = 1, 2, 3, \dots, n-2$ by Lemma 1 with two entry 0.
2. The representation of uu_n respect to W is equal to $(0, 1, 1, 1, \dots, 1)$
3. For $r(u_1 u_n|W)$ and $r(u_{n-1} u_n|W)$
 - For $n = 3$, the representation of $v_1 v_3$ respect to W is equal to $(1, 0, 1)$ and the representation of $v_2 v_3$ respect to W is equal to $(1, 1, 0)$
 - For $n = 4$, the representation of $v_1 v_4$ respect to W is equal to $(1, 0, 1, 1)$ and the representation of $v_3 v_4$ respect to W is equal to $(1, 1, 1, 0)$
 - $n \geq 5$: $r(v_1 v_n|W) = (1, 0, 1, 2, 2, \dots, 2, 2, 1)$ and $r(v_{n-1} v_n|W) = (1, 1, 2, 2, \dots, 2, 1, 0)$

So, $\text{edim}_{\text{cen}}(K_1 + C_n) \leq n$. Finally, Let $W' = \{u, u_{i_1}, u_{i_2}, \dots, u_{i_m}\}$ with $u_{i_1}, u_{i_2}, \dots, u_{i_m} \in \{u_1, u_2, \dots, u_n\}$ and $|W'| \leq n-1$, then W' contains central set of $K_1 + C_n$ but W' not central edge generator set for G because there exists $u_j, u_k \in V(K_1 + C_n)$ but $u_j, u_k \notin W'$ with $j \neq k$ such that $r(uu_j|W') = (0, 1, 1, 1, \dots, 1) = r(uu_k|W')$. Therefore, $\text{edim}_{\text{cen}}(K_1 + C_n) = n$.

Next, we state central edge metric dimension of fan graph $K_1 + P_n$ in the following theorem.

Theorem 3: For $G = K_1 + P_n$, then $\text{edim}_{\text{cen}}(G) = \begin{cases} n+1; & n=1,2 \\ n; & \text{the others} \end{cases}$

Proof: For $n = 1$ then $G \cong P_2$. So $C(G) = V(G)$, consequently $\text{edim}_{\text{cen}}(G) = n+1$. For $n = 2$, then $G \cong C_3$. So $C(G) = V(G)$, consequently $\text{edim}_{\text{cen}}(G) = n+1$. For $n \geq 3$, let $V(K_1) = \{v\}$ and $V(K_1 + P_n) = \{u, u_1, u_2, \dots, u_n\}$ with $d(u) = n$. Since $P(K_1) = \{v\}$ and $P(P_n) = \emptyset$, then $C(K_1 + P_n) = \{u\}$ by Theorem 1. Next, let $W = \{u, u_1, u_2, \dots, u_{n-1}\}$, will be proved that W is central edge metric generator set for G . First, clearly that because $C(K_1 + P_n) \subset W$. And then:

1. $r(uu_i|W)$ and $r(u_j u_{j+1}|W)$ all different for $i \in \{1, 2, 3, \dots, n-1\}$ and $j \in \{1, 2, 3, \dots, n-2\}$ by Lemma 1 with two entry 0
2. $r(uu_n|W) = (0, 1, 1, 1, \dots, 1)$
3. for $r(u_{n-1} u_n|W)$
 - for $n = 3$: $r(u_2 u_3|W) = (1, 1, 0)$
 - for $n \geq 4$: $r(u_{n-1} u_n|W) = (1, 2, 2, \dots, 2, 1, 0)$

So, $\text{edim}_{\text{cen}}(K_1 + P_n) \leq n$. Finally, Let $W' = \{u, u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$ with $u_{i_1}, u_{i_2}, \dots, u_{i_k} \in \{u_1, u_2, \dots, u_n\}$ and $|W'| \leq n - 1$, then contains central set of $K_1 + P_n$ but W not central edge generator set for G because there exists $v_i, v_j \in V(K_1 + P_n) \setminus W'$ with $i \neq j$ such that $r(vv_i|W') = (0, 1, 1, 1, \dots, 1) = r(vv_j|W')$. Therefore,

$$\text{edim}_{\text{cen}}(K_1 + P_n) = \begin{cases} n+1; & n=1,2 \\ n; & \text{the others} \end{cases}$$

For next, we state central edge metric dimension of friendship graph $K_1 + \bigcup_{i=1}^n K_2$ in the following theorem.

Theorem 4: For $G = K_1 + \bigcup_{i=1}^n K_2$, then $\text{edim}_{\text{cen}}(G) = \begin{cases} 2n+1; & n=1 \\ 2n; & \text{the others} \end{cases}$

Proof: For $n = 1$ then $G \cong C_3$. So, $C(G) = V(G)$ such that $\text{edim}_{\text{cen}}(G) = 2n + 1$. For $n \geq 2$, let $V(K_1) = \{v\}$ and $V(K_1 \bigcup_{i=1}^n K_2) = \{u, v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, \dots, v_{n,1}, v_{n,2}\}$ with $d(u) = 2n$. Since $P(K_1) = \{v\}$ and $P(\bigcup_{i=1}^n K_2) = \emptyset$, then $C(K_1 + \bigcup_{i=1}^n K_2) = \{u\}$ by Theorem 1. Next, let $W = \{v, v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, \dots, v_{n-1,1}, v_{n-1,2}, v_{n,1}\}$, will be proved that W central edge generator set for G . Clearly that $C(K_1 + \bigcup_{i=1}^n K_2) \in W$, and then:

1. $r(uv_i|W)$ and $r(v_{j,1}v_{j,2}|W)$ all different for $i \in \{1, 2, 3, \dots, n\}$ and $j \in \{1, 2, 3, \dots, n-1\}$ by Lemma 1 with two entry 0
2. $r(uv_{n,2}|W) = (0, 1, 1, 1, \dots, 1)$
3. $r(v_{n,1}v_{n,2}|W) = (1, 2, 2, \dots, 2, 2, 0)$

So, $\text{edim}_{\text{cen}}(K_1 + \bigcup_{i=1}^n K_2) \leq n$. Finally, let $W' = \{u\} \cup S$ with $S \subseteq V(K_1 + \bigcup_{i=1}^n K_2) \setminus \{u\}$ and $|W'| \leq n - 1$, then W' not central edge generator set for G because there exists $v_i, v_j \in V(K_1 + \bigcup_{i=1}^n K_2) \setminus W'$ with $i \neq j$ such that $r(uv_i|W') = (0, 1, 1, 1, \dots, 1) = r(uv_j|W')$. Therefore,

$$\text{edim}_{\text{cen}}(V(K_1 \bigcup_{i=1}^n K_2)) = \begin{cases} 2n+1; & n=1 \\ 2n; & \text{the others} \end{cases}.$$

Then, we state central edge metric dimension of joint graph $K_1 + H$ for a connected graph H in the following theorem.

Theorem 5: Let H be a connected graph with ordo n and $P(H) = \emptyset$. If $G = K_1 + H$, then $\text{edim}_{\text{cen}}(G) = \begin{cases} n+1; & n=1,2 \\ n; & \text{the others} \end{cases}$

Proof: For $n = 1$, then $G \cong P_2$. So, $C(G) = V(G)$ such that $\text{edim}_{\text{cen}}(G) = n + 1$. For $n = 2$, then $G \cong C_3$. So, $C(G) = V(G)$ such that $\text{edim}_{\text{cen}}(G) = n + 1$. For $n \geq 3$, let $V(K_1) = \{v\}$ and $V(K_1 + H) = \{u, v_1, v_2, v_3, \dots, v_n\}$ with $d(u) = n$. Since $P(K_1) = \{v\}$ and $P(H) = \emptyset$, then $C(K_1 + H) = \{v\}$ by Theorem 1. Next, let $W = \{u, v_1, v_2, \dots, v_{n-1}\}$, will be proved that W is central edge generator for G . Clearly that $C(K_1 + H) \subset W$, and then:

1. $r(uv_i|W)$ and $r(v_jv_{j+1}|W)$ all different for $i \in \{1, 2, 3, \dots, n-1\}$ and $j \in \{1, 2, 3, \dots, n-2\}$ by Lemma 1 with two entry 0.

2. Let $d(v_n) = k < n - 1$ with $v_n v_{i_1}, v_n v_{i_2}, \dots, v_n v_{i_k} \in E(H)$, then
 - $|r(uv_n|W)|_0 = 1$ with $r(uv_n|W)_1 = 0$
 - $|r(v_{i_s} v_n|W)|_0 = 1$ with $r(v_{i_s} v_n|W)_{i_s+1} = 0$

So, $\text{edim}_{\text{cen}}(K_1 + H) \leq n$. Finally, let $W' = \{u, v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ with $v_{i_1}, v_{i_2}, \dots, v_{i_k} \in \{v_1, v_2, v_3, \dots, v_n\}$ and $|W'| \leq n - 1$, then W' not central edge generator for G because there exists $v_i, v_j \in V(K_1 + H) \setminus W'$ with $i \neq j$ such that $r(vv_i|W') = (0, 1, 1, 1, \dots, 1) = r(vv_j|W')$. Therefore,

$$\text{edim}_{\text{cen}}(K_1 + H) = \begin{cases} n+1; & n=1,2 \\ n; & \text{the others} \end{cases}$$

Finally, we are ready to prove central edge metric dimension of joint graph $G + H$ for a connected graph G and H in the following theorem.

Theorem 6: Let G and H are connected graph with ordo m and n , respectively. If $P(H) = \emptyset$, then

$$\text{edim}_{\text{cen}}(G + H) = \begin{cases} |V(G)| + |V(H)|; & P(G) = \emptyset \\ |V(G)| + |V(H)| - 1; & \text{the others} \end{cases}$$

Proof: If $P(G) = \emptyset$, then $\text{edim}_{\text{cen}}(G + H) = |V(G)| + |V(H)|$ by Lemma 3. If $P(G) \neq \emptyset$, then $C(G + H) = P(G)$ by Theorem 1. Suppose that $V(G + H) = \{g_{v_1}, g_{v_2}, g_{v_3}, \dots, g_{v_m}, h_{u_1}, h_{u_2}, h_{u_3}, \dots, h_{u_n}\}$. Let $W = \{g_{v_1}, g_{v_2}, g_{v_3}, \dots, g_{v_m}, h_{u_1}, h_{u_2}, h_{u_3}, \dots, h_{u_{n-1}}\}$. Will be proved that W is a central edge generator for G . First, clearly that $C(G + H) \subset W$. And then:

1. $r(g_{v_i} h_{u_j}|W)$ all different for $i \in \{1, 2, 3, \dots, m\}$ and $j \in \{1, 2, 3, \dots, n - 1\}$ by Lemma 1 with two entry 0.
2. $r(g_{v_i} h_{u_n}|W)$ all different for $i \in \{1, 2, 3, \dots, m\}$ with one's entry 0 in i -th position.
3. $r(e_i|W)$ all different for $i = 1, 2, \dots, m$ with $e_i = g_{v_{i_j}} g_{v_{i_k}} \in E(G + H)$ by Lemma 1 with two entry 0
4. $r(f_j|W)$ all different for $f_j = h_{u_{j_s}} h_{u_{j_t}} \in E(H)$ with $u_{j_s}, u_{j_t} \neq u_n$ by Lemma 1 with two entry 0
5. $r(f_{j,n}|W)$ all different for $f_{j,n} = h_{u_j} h_{u_n} \in E(G + H)$ with $u_j u_n \in E(H)$ by Lemma 1 with one's entry 0 in j -th position.

So, $\text{edim}_{\text{cen}}(G + H) \leq m + n - 1$. Finally, let $W' = \{g_{v_{i_1}}, g_{v_{i_2}}, g_{v_{i_3}}, \dots, g_{v_{i_s}}, h_{u_{j_1}}, h_{u_{j_2}}, h_{u_{j_3}}, \dots, h_{u_{j_t}}\}$ with $P(G) \subset W'$ and $|W'| \leq m + n - 1$, then W' not central edge generator set for G because there exists $x, y \in V(G + H) \setminus W'$ such that $r(wx|W') = r(wy|W')$. Therefore,

$$\text{edim}_{\text{cen}}(G + H) = \begin{cases} |V(G)| + |V(H)|; & P(G) = \emptyset \\ |V(G)| + |V(H)| - 1; & \text{the others} \end{cases}$$

Conclusion

From the results of the explanation above, can be concluded that for any connected graph G and H , $edim_{cen}(G + H) = |V(G + H)|$ if both $P(G)$ and $P(H)$ are nonempty sets, while $edim_{cen}(G + H) = |V(G + H)| - 1$ for others. Furthermore, this research can be applied to graphs resulting from other operations, such as corona graphs, comb graphs and Cartesian product graphs.

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