Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 1, pp. 363–373
DOI : 10.47974/JDMSC-2609

All the cardinal numbers of ideals G-part of Q-algebras

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Abstract

A Q-algebra $(\xi; *, 0)$ is a generalization of the concepts of *BCK, BCI*-algebras, introduced by Neggers et al. [1]. As for *BCK, BCI*-algebras, the concept ideal is defined for Q-algebras [1]. In this paper, we show that for a given integer n, and for all $k \in \{0, 1, \ldots, \lfloor \log_2 n \rfloor - 1\}$, there is a Q-algebra ξ of cardinality n such that the G-part $G(\xi) = \{v \in \xi : 0 * v = v\}$ is an ideal of ξ with $|G(\xi)| = 2^k$. Moreover, we show that if $G(\xi)$ is an ideal of a Q-algebra ξ and $|\xi| = n$, then $|G(\xi)| = 2^k$, for some $k \in \{0, 1, 2, \ldots, \lfloor \log_2 n \rfloor - 1\}$.

Subject Classification (2020): *Primary 03G25, 03G27, 06F35, 20K01.*

Keywords: *Q-algebra, Ideals, G-part, Cardinality, Abelian group.*

1. Introduction and Preliminaries

Let ξ be a non-empty set which contains an element denoted by 0, together with a binary operation $*$ on ξ satisfying the following three properties: for all $\varsigma, \kappa, \omega \in \xi$

(Q_1) $\varsigma * \varsigma = 0$,
(Q_2) $\varsigma * 0 = \varsigma$,
$(Q_3)(\varsigma * \kappa) * \omega = (\varsigma * \omega) * \kappa$.

The order of the algebra is the cardinality of ξ. Such algebraic structures were called Q-algebras by Negger et al. [1]. For convenient reason, we put

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$x * y = xy$. In [1], the authors studied several properties of Q-algebras. For example, if ξ is a set of sets containing the empty set $\emptyset$ and $*$ is the set operation "\" (set minus) then it is easy to verify that $(\xi;\setminus,\emptyset)$ satisfies the properties $(Q_1), (Q_2)$, and (Q_3). This shows that a Q-algebra is a generalization of BCK, BCI-algebras. Both BCK-algebras and BCI-algebras are widely study since they were introduced by Iseki and Imai in 1966 [2]. For more details of BCK, BCI-algebras we refer to [3], [4], [5] and [6]. So, it is quite natural to study concepts of BCK, BCI-algebras for Q-algebras too. One of the most important concept of BCK, BCI-algebras is the ideal. In 2001, Neggers et al. introduced the concept of an ideal in a Q-algebra [1]. An ideal of a Q-algebra is a non-empty subset Λ of a Q-algebra ξ, satisfying: (I_1) $0 \in \Lambda$ and (I_2) for $\chi, \rho \in \xi$, $\chi\rho \in \Lambda$ and $\rho \in \Lambda$ imply $\chi \in \Lambda$. A proper ideal Λ of a Q-algebra ξ is an ideal Λ such that $\Lambda \neq \{0\}$ and $\Lambda \neq \xi$. Many kinds of subsets of ξ were considered whether they are ideals or not. The authors in [1] showed that the so-called p-radical $B(\xi) = \{\psi \in \xi: 0\psi = 0\}$ is always an ideal of a Q-algebra ξ. The G-part $G(\xi) = \{\eta \in \xi: 0\eta = \eta\}$ is also discussed in [1]. For $|\xi| = 2$, the set $G(\xi)$ is always an ideal of ξ. In the case $|\xi| = 3$, $G(\xi)$ is an ideal implies $|G(\xi)| = 1$ and vice versa. In 2010, Ahn and So [7] studied the decomposition of ideals. They showed that for a non-empty subset Λ of a Q-algebra ξ with $0 \in \Lambda$, the set Λ is an ideal of ξ if and only if $\Lambda = \bigcup_{\varepsilon,\eta\in\Lambda} A(\varepsilon, \eta)$, where $A(\varepsilon, \eta) = \{z \in \xi: (z\varepsilon)\eta = 0\}$. In the same year, Ahn and Kang [8] introduced the concept of a set T-part $\{(0\lambda)\lambda, : \lambda \in \xi\}$ of a Q-algebra ξ. The authors provided that for a Q-algebra ξ with the property: $(xy)(xz) = zy$, for all $x, y, z \in \xi$, the set ξ-part of ξ is an ideal. In 2012, Mostafa, Naby, and Elgendy [9] introduced the notion of a Q-ideal, which is a non-empty subset Λ of a Q-algebra ξ such that $0 \in \Lambda$ and $(x\pi)\nu \in \Lambda, y \in \Lambda$ imply $x\pi \in \Lambda$ for all $x, \pi, \nu \in \xi$. The authors obtained that any Q-ideal is an ideal of ξ. In 2020, Abdullah and Tach [10] proved that every ideal Λ of a Q-algebra ξ with $|\Lambda| = |\xi| - 1$ is a prime ideal. Moreover, they showed that for ideals Λ_1, Λ_2 and a prime ideal A of ξ such that $\Lambda_1 \cap \Lambda_2 \subseteq A$, we have $\Lambda_1 \subseteq A$ or $\Lambda_2 \subseteq A$. In 2024, the authors in [11] considered the connection of $G(\xi)$ and a set of all atoms of a Q-algebra ξ. They showed that ξ dose not contain any strong atom whenever $\{0\} \neq G(\xi)$ is an ideal. Some more properties of $G(\xi)$ are offered in [11] by Anantayasethi and koppitz in 2025. A a necessary condition of $G(\xi)$ to be ideal of ξ is provide. They also proved that $|G(\xi)| \neq |\xi| - 1$ if $G(\xi)$ is an ideal of ξ. Moreover, they obtained that if $G(\xi)$ is an ideal, then $G(\xi)$ is an abelian group. Das et al. [13] combined a concept of a pentapartitioned neutrosophic set and Q-algebra to get the a new notion, so called a pentapartitioned neutrosophic Q-algebra. They discussed and examined some peoperties of pentapartitioned neutrosophic Q-ideals on such algebra. The concept of fuzzy set was considered in Q-algebras. The notion of fuzzy Q-ideal was introduced in 2012 [9]. The authors provided a relation between fuzzy BCK-ideal and fuzzy Q-ideal. They showed that every fuzzy Q-ideal is a fuzzy BCK-ideal. Later, in 2020 Abdullah and Tach [14] discussed fuzzy prime ideals and intuitionistic fuzzy prime ideals in Q-algebras.

The authors showed a characterization that an intuitionistic fuzzy set μ of a Q-algebra X is an intuitionistic fuzzy ideal if and only if an upper t-level set $U(\mu, t)$ and a lower s-level set $L(\mu, s)$ are either empty or ideal of X for any $t, s \in [0,1]$.

Let us construct a Q-algebra of order four such that the G-part is an ideal with two elements. We will use that Q-algebra to construct new Q-algebras with the property that the G-part is an ideal.

Let $\xi = \{0, \psi, b, \zeta\}$ and let $*$ be a binary operation on ξ defined by the following table 1:

Table 1

$*$	0	ψ	b	ζ
0	0	ψ	ζ	b
ψ	ψ	0	b	ζ
b	b	ζ	0	ψ
ζ	ζ	b	ψ	$0.$

It is a routine to check that $(\xi; *, 0)$ in table 1 is a Q-algebra with $G(\xi) = \{0, \psi\}$. We will denote the Q-algebra in table 1 by Q_4. It is easy to check that $G(\xi)$ is an ideal of Q_4. We also get that $G(\xi)$ is the only one proper ideal of that Q_4.

In this work, we will examine all possible sizes of the set G-part of a Q-algebra which is an ideal. More precisely, for a given natural number n and a positive integer $h \in \{0,1, \dots, \lfloor log_2 n \rfloor - 1\}$, we show that there is a Q-algebra ξ and $G(\xi)$ is an ideal with $|G(\xi)| = 2^h$, $|\xi| = n$. Now we will recall some properties that we will use later.

Proposition 1.1: [12] Let $0 \neq \varpi, \iota, \theta \in G(\xi)$. Then the following conditions are obtained:

(i) If $\varpi \neq \iota$, then $\varpi\iota \notin \{0, \varpi, \iota\}$.

(ii) If $\varpi\iota = \theta$, then $\varpi\theta = \iota$ and $\iota\theta = \omega$.

(iii) $\upsilon\varpi \neq \upsilon$ for all $0 \neq \upsilon \in \xi$.

Proposition 1.2: [12] Let ξ be a Q-algebra. If $G(\xi)$ is an ideal of ξ, then $G(\xi)$ is an abelian group.

Corollary 1.3: [12] *Let ξ be a Q-algebra. Then we get $G(\xi) = \xi$ if and only if ξ is an abelian group.*

Proposition 1.4: [12] Let ξ be a Q-algebra and let $|\xi| \geq 3$. If $|\xi|$ is odd, then $G(\xi) \neq \xi$.

2. The Cardinality of Ideal G-part

In this section, we will discuss the possible cardinality of the set G-part which is an ideal. Let Ω be a group with an identity element 1. It is well-known

that if $\tau^2 = 1$, for any $\tau \in \Omega$, then Ω has the cardinality 2^k for some non-negative integer k. If $G(\xi)$ is an ideal of a Q-algebra ξ, then by Proposition 1.2, the G-part $G(\xi)$ is an abelian group with the identity element 0. Moreover, by (Q_1), we have $\rho\rho = 0$ for all $\rho \in \xi$. The following proposition is obtained from these facts:

Later on we will denote the symbol ξ as a Q-algebra ξ.

Proposition 2.1: If $G(\xi)$ is an ideal of ξ, then $|G(\xi)| = 2^k$ for some non-negative integer k.

From Proposition 2.1, we get immediately Theorem 3.10, Proposition 3.11, and Theorem 3.12 in [1]. Next, when $G(\xi)$ is an ideal of ξ, we will determine all the possible sizes of $G(\xi)$.

Theorem 2.2: *Let $G(\xi) \neq \xi$. If $G(\xi)$ is an ideal of ξ, $|G(\xi)| \leq |\xi - G(\xi)|$.*

Proof: Let n be a positive integer and let $|\xi| = n$. Let $\xi = \{0 = x_1, x_2, \dots, x_n\}$ and $\xi \neq G(\xi)$. Suppose that $G(\xi)$ is an ideal of ξ and $|G(\xi)| = h$ for some positive integer $h \neq n$. Then $|\xi - G(\xi)| = n - h$. We may assume that $G(\xi) = \{0 = x_1, x_2, \dots, x_h\}$. Assume now that $|G(\xi)| > |\xi - G(\xi)|$. Then $h - 1 > n - h - 1$. Note that $h < n$. Since $G(\xi)$ is ideal, and $x_2, x_3, \dots, x_h \in G(\xi)$, $x_{h+1} \notin G(\xi)$, we can conclude that $x_{h+1}x_2, x_{h+1}x_3, \dots, x_{h+1}x_h \notin G(\xi)$. Since $x_2, x_3, \dots, x_h \in G(\xi)$, then we get $x_{h+1}x_2, x_{h+1}x_3, \dots, x_{h+1}x_h \notin \{x_{h+1}\}$ by Proposition 1.1(iii). Therefore, $x_{h+1}x_2, x_{h+1}x_3, \dots, x_{h+1}x_h \in \{x_{h+2}, x_{h+3}, \dots, x_n\}$. Since $h - 1 > n - h - 1$, then there exist $t, j \in \{2,3, \dots, h\}$ with $t \neq j$ such that $x_{h+1}x_t = x_{h+1}x_j$, i.e. $x_{h+1}x_t = x_{h+1}x_j = x_t$ for some $l \in \{h + 2, h + 3, \dots, n\}$. Then we get

$$x_t = 0x_t = (x_{h+1}x_{h+1})x_t = (x_{h+1}x_t)x_{h+1} = x_l x_{h+1},$$

and

$$x_j = 0x_j = (x_{h+1}x_{h+1})x_j = (x_{h+1}x_j)x_{h+1} = x_l x_{h+1}.$$

Thus, $x_t = x_l x_{h+1} = x_j$, a contradiction. Therefore, $|G(\xi)| \leq |\xi - G(\xi)|$.

Theorem 2.2 gives us an information of size of $G(\xi)$ which is an ideal. We can determine that the order of $G(\xi)$ is less or equal to half of the cardinality of ξ when $G(\xi)$ is an ideal.

Corollary 2.3: *Let ξ be a Q-algebra of order n for some positive integer n with $G(\xi) \neq \xi$. If $G(\xi)$ is an ideal of ξ, then $|G(\xi)| \leq \frac{n}{2}$.*

We can classify the cardinality of all $G(\xi)$ which are ideals of ξ. Let $G(\xi)$ be an ideal of ξ. Then by Proposition 2.1, we know that $|G(\xi)| = 2^k$, for some non-negative integer k. If $G(\xi) \neq \xi$ is an ideal, then by using Corollary 2.3, we can conclude all the possible sizes of $G(\xi)$ as the following theorem.

Theorem 2.4: *If $G(\xi)$ is an ideal of ξ, then $|G(\xi)| = 2^h$ for some integer $h \in \{0,1,2, \dots, \lfloor log_2 n \rfloor - 1\}$ and $|\xi| = n$.*

Proof: Assume that $G(\xi)$ is an ideal of ξ and $G(\xi) \neq \xi$. By Proposition 2.1, we get $|G(\xi)| = 2^h$ for some non-negative integer h. Then we get $2^h = |G(\xi)| \leq \frac{n}{2}$ by Corollary 2.3. Therefore, $2^{h+1} \leq n = 2^{log_2 n}$. It follows that $h + 1 \leq log_2 n$. Since h is a non-negative integer, then $h \leq \lfloor log_2 n \rfloor - 1$. Hence, $h \in \{0,1,2, \dots, \lfloor log_2 n \rfloor - 1\}$.

Example 2.5: Let ξ be a Q-algebra of order 6 such that $G(\xi)$ is an ideal of ξ. Since $G(\xi)$ is an ideal and $|\xi| = 6 \neq 2^h$ for all non-negative integer h, then $G(\xi) \neq \xi$. By Theorem 2.4, we get $|G(\xi)| = 2^h$ for some integer $h \in \{0, \dots, \lfloor log_2 6 \rfloor - 1\} = \{0, \dots, \lfloor 2.5 \rfloor - 1\} = \{0,1\}$. Therefore, $|G(\xi)| \in \{2^0, 2^1\} = \{1,2\}$. Table 2 presents Q-algebras of order 6 with $G(\xi)$ are ideals and $|G(\xi)| = 1$, $|G(\xi)| = 2$.

Table 2

$*$	0	ϑ	b	ζ	z	κ
0	0	0	0	0	0	0
ϑ	ϑ	0	0	0	0	0
b	b	0	0	0	0	0
ζ	c	0	0	0	0	0
z	z	0	0	0	0	0
κ	κ	0	0	0	0	0

(A): $G(\xi) = \{0\}$

$*$	0	ϑ	b	ζ	z	κ
0	0	ϑ	ζ	b	b	b
ϑ	ϑ	0	b	ζ	ζ	ζ
b	b	ζ	0	ϑ	ϑ	ϑ
ζ	ζ	b	ϑ	0	0	0
z	z	b	ϑ	0	0	0
κ	κ	b	ϑ	0	0	0

(B): $G(\xi) = \{0, \vartheta\}$

This example shows that we can find a Q-algebra ξ such that $G(\xi)$ is an ideal with cardinality h and $h \in \{0,1,2, \dots, \lfloor log_2 6 \rfloor - 1\}$. In the next section, we will examine that for any non-negative integer n and for all $h \in \{0,1,2, \dots, \lfloor log_2 n \rfloor - 1\}$, there is a Q-algebra ξ with $|\xi| = n$ such that $G(\xi)$ is an ideal and $|G(\xi)| = 2^h$.

3. Construction of Q-algebras

Let $(\xi; *, 0)$ be a Q-algebra of order n for some positive integer n. We construct a new Q-algebra from the Q-algebra ξ as follows:

Let Λ be a non-empty set such that $\xi \cap \Lambda = \emptyset$. Let $a \in \xi$. We define a binary operation $*_a$ on $\xi_\Lambda = \xi \cup \Lambda$ by an extension from $*$ as the following: for $\upsilon, \varphi \in \xi_\Lambda$ let

$$\upsilon *_a \varphi = \begin{cases} 0 & \text{if } \upsilon, \varphi \in \Lambda; \\ \upsilon & \text{if } \upsilon \in \xi \cup \Lambda, \varphi = 0; \\ a\varphi & \text{if } \upsilon \in \Lambda, \varphi \in \xi \backslash \{0\}; \\ \upsilon a & \text{if } \upsilon \in \xi, \varphi \in \Lambda; \\ \upsilon\varphi & \text{if } \upsilon \in \xi, \varphi \in \xi \backslash \{0\}. \end{cases}$$

Lemma 3.1: *Let $\Lambda \neq \emptyset$ with $\xi \cap \Lambda = \emptyset$, and let $a \in \xi$. Then $(\xi_\Lambda; *_a, 0)$ is a Q-algebra.*

Proof: It is easy to see that $(\xi_\Lambda;*_a,0)$ satisfies both (Q_1) and (Q_2). Next, we will show the condition (Q_3). Let $\vartheta,\varphi,\rho\in\xi_\Lambda$. We want to show that $(\vartheta *_a \varphi)*_a \rho = (\vartheta *_a \rho)*_a \varphi$. It is sufficient to prove only the case that at least one of ϑ,φ,ρ does not belong to T, i.e. at least one of ϑ,φ,ρ belong to Λ. We consider in two cases.

Case 1: $\varphi = 0$ or $\rho = 0$:

If $\varphi = 0$, then $(\vartheta *_a \varphi)*_a \rho = (\vartheta *_a 0)*_a \rho = \vartheta *_a \rho = (\vartheta *_a \rho)*_a 0 = (\vartheta *_a \rho)*_a \varphi$. If $\rho = 0$, then $(\vartheta *_a \varphi)*_a \rho = (\vartheta *_a \varphi)*_a 0 = \vartheta *_a \varphi = (\vartheta *_a 0)*_a \varphi = (\vartheta *_a \rho)*_a \varphi$.

Caes 2; $\varphi \neq 0$ and $z \neq 0$: Then $\varphi\in\Lambda\cup\xi\backslash\{0\}$ and $\rho\in\Lambda\cup\xi\backslash\{0\}$.

If $\varphi,\rho\in\Lambda$ and $\vartheta\in\Lambda$, then $(\vartheta *_a \varphi)*_a \rho = 0 *_a \rho = 0a = 0 *_a \varphi = (\vartheta *_a \rho)*_a \varphi$.

If $\varphi,\rho\in\Lambda$ and $\vartheta\in\xi$, then $(\vartheta *_a \varphi)*_a \rho = (\vartheta a)*_a \rho = (\vartheta a)a = (\vartheta a)*_a \varphi = (\vartheta *_a \rho)*_a \varphi$.

If $\varphi\in\Lambda$, $\rho\in\xi\backslash\{0\}$ and $\vartheta\in\Lambda$, then $(\vartheta *_a \varphi)*_a \rho = 0 *_a \rho = 0\rho = (aa)\rho = (a\rho)a = (a\rho)*_a \varphi = (\vartheta *_a \rho)*_a \varphi$.

If $\varphi\in\Lambda$, $\rho\in\xi\backslash\{0\}$ and $\vartheta\in\xi$, then $(\vartheta *_a \varphi)*_a \rho = (\vartheta a)*_a \rho = (\vartheta a)\rho = (\vartheta\rho)a = (\vartheta\rho)*_a \varphi = (\vartheta *_a \rho)*_a \varphi$.

If $y\in\xi\backslash\{0\}$, $\rho\in\Lambda$ and $\vartheta\in\Lambda$, then $(\vartheta *_a \varphi)*_a \rho = (ay)*_a \rho = (a\varphi)a = (aa)\varphi = 0\varphi = 0 *_a \varphi = (\vartheta *_a \rho)*_a y$.

If $\varphi\in\xi\backslash\{0\}$, $\rho\in\Lambda$ and $\vartheta\in\xi$, then $(\vartheta *_a \varphi)*_a \rho = (\vartheta\varphi)*_a \rho = (a\varphi)\rho = (\vartheta\varphi)a = (\vartheta a)\varphi = (\vartheta a)*_a \varphi = (\vartheta *_a \rho)*_a \varphi$.

If $\varphi,\rho\in\xi\backslash\{0\}$ and $\vartheta\in\Lambda$, then $(\vartheta *_a \varphi)*_a \rho = (a\varphi)*_a \rho = (a\varphi)\rho = (a\rho)\varphi = (a\rho)*_a \varphi = (\vartheta *_a \rho)*_a \varphi$.

Altogether, $(\xi_\Lambda;*_a,0)$ is satisfied (Q_3). Therefore, $(\xi_\Lambda;*_a,0)$ is a Q-algebra.

We denote the Q-algebra $(\xi_\Lambda;*_a,0)$ by $\xi_\Lambda^{(a)}$. In the case $\Lambda=\{b\}$, we write $\xi_b^{(a)}$ instead of $\xi_{\{b\}}^{(a)}$.

Proposition 3.2: Let $a\in\xi$ and let $\Lambda\neq\emptyset$ such that $\xi\cap\Lambda=\emptyset$. Then $\varpi\colon\xi_\Lambda^{(a)}\to\xi$ defined by

$$\varpi(\vartheta)=\begin{cases}\vartheta & \text{if } \vartheta\in\xi;\\ a & \text{if } \vartheta\in\Lambda,\end{cases}$$

for all $a\in\xi$, ϖ is a homomorphism.

Proof: Let $a\in\xi$ and let $\vartheta,y\in\xi_\Lambda^{(a)}$. If $\vartheta,y\in\Lambda$, then $\varpi(\vartheta y)=\vartheta y=\varpi(\vartheta)\varpi(y)$. If $\vartheta\in\Lambda$ and $y\in X\backslash\{0\}$, then $\varpi(\vartheta y)=\varpi(ay)=ay=a\varpi(y)=\varpi(\vartheta)\phi(y)$. If $\vartheta\in\Lambda$ and $y=0$, then $\varpi(\vartheta y)=\varpi(\vartheta 0)=\varpi(\vartheta)=a=a0=\varpi(\vartheta)\varpi(0)=\varpi(\vartheta)\varpi(y)$. If $\vartheta\in\xi$ and $y\in\Lambda$, then $\varpi(\vartheta y)=\varpi(\vartheta a)=\vartheta a=\varpi(\vartheta)\varpi(y)$. If $\vartheta,y\in\Lambda$, then $\varpi(\vartheta y)=\varpi(0)=0=aa=\varpi(\vartheta)\varpi(y)$. Altogether, ϖ is a homomorphism.

Let us denote by $\varpi_{\Lambda}^{(a)}$ the homomorphism defined in Proposition 3.2. Now we consider the kernel of $\varpi_{\Lambda}^{(a)}$:

$$ker(\varpi_{\Lambda}^{(a)}) \quad = \{(\vartheta, y) \ : \ \varpi_{\Lambda}^{(a)}(\vartheta) = \varpi_{\Lambda}^{(a)}(y), \qquad \vartheta, y \in \xi \cup \Lambda\}.$$

Proposition 3.3: Let D and M be non-empty sets and let $D \subseteq M$. Let $T \cap D = T \cap M = \emptyset$ and let $\nu, \iota \in T$. Then we get:

(i) $ker(\varpi_{D}^{(\nu)}) = ker(\varpi_{D}^{(\iota)})$.

(ii) $ker(\varpi_{D}^{(\nu)}) \subseteq ker(\varpi_{M}^{(\iota)})$.

Proof: It is easy to verify that $ker(\varpi_{D}^{(\nu)}) = ker(\varpi_{D}^{(\iota)}) = \{(\xi, \xi) | \xi \in T\} \cup D^2 \subseteq \{(\xi, \xi) \ : \ \xi \in T\} \cup M^2 = ker(\varpi_{M}^{(\iota)})$.

Proposition 3.4: Let $\emptyset \neq \Lambda$ and let $\emptyset \neq \Gamma \subseteq \xi$ such that $\Lambda \cap \xi = \emptyset$. For any $a \in \xi$, Γ is an ideal of ξ with $a \in \Gamma$ if and only if $\Gamma \cup \Lambda$ is an ideal of $\xi_{\Lambda}^{(a)}$.

Proof: ($\Rightarrow$) Assume that Γ is an ideal of ξ and $a \in \Gamma$. Let $\Gamma^* = \Gamma \cup \Lambda$. We will prove that Γ^* is an ideal of $\xi_{\Lambda}^{(a)}$. Let $\varsigma, y \in \xi \cup \Lambda$. Suppose that $\varsigma y \in \Gamma^*$ and $y \in \Gamma^*$. If $\varsigma \in \Lambda$, then $\varsigma \in \Gamma^*$. If $\varsigma, y \in \xi$, then $y, \varsigma y \in \xi \cap (\Gamma \cup \Lambda)$, i.e. $\varsigma y \in \Gamma$ and $y \in \Gamma$. Since Γ is an ideal of ξ, there follows $\varsigma \in \Gamma \subseteq \Gamma^*$. If $\varsigma \in \xi$ and $y \in A$, then $\varsigma y = \varsigma a$. Therefore, $\varsigma a = \varsigma y \in \Gamma^* \cap \xi = \Gamma$. Since $\varsigma a \in \Gamma$, $a \in \Gamma$, and Γ is an ideal of ξ, then $\varsigma \in \Gamma \subseteq \Gamma^*$. Hence, Γ^* is an ideal of $\xi_{\Lambda}^{(a)}$.

($\Leftarrow$) Suppose that $\Gamma \cap \Lambda$ is and ideal of $\xi_{\Lambda}^{(a)}$. We set $\Gamma \cup \Lambda = \Gamma^*$. Assume that $a \notin \Gamma$. Then $a \notin \Gamma^*$. Let $w \in \Lambda \subseteq \Gamma^*$. Because of Γ^* is an ideal of $\xi_{A}^{(a)}$ and $a \notin \Gamma^*$, then $aw \notin \Gamma^*$. But $aw = aa = 0 \in \Gamma^*$. This gives a contradiction. Hence, $a \in \Gamma$. Next, we will to show that Γ is an ideal of ξ. Let $\mu, \tau \in \xi$ and assume that $\mu\tau \in \Gamma$ and $\eta \in \Gamma$. Then $\mu\tau \in \Gamma^*$ and $\tau \in \Gamma^*$. There follows that $\mu \in \Gamma^*$ since Γ^* is an ideal. Now, we have $\mu \in \xi$ and $\mu \in \Gamma^*$, i.e. $\mu \in \Gamma$. Therefore, Γ is an ideal of ξ.

Let n be a positive integer. Let us now fix a Q-algebra ξ of order n.

Proposition 3.5: If ξ contains a proper ideal $G(\xi)$, then for any positive integer $k > n$, there is a Q-algebra W of order k such that $G(W) = G(\xi)$ is a proper ideal of W.

Proof: Assume $G(\xi)$ is a proper ideal of ξ. Then there is an element $a \in \xi$ and $a \notin G(\xi)$. Let $\emptyset \neq \Lambda$, $\Lambda \cap \xi = \emptyset$ and let$|\Lambda \cup \xi| = k$ for some positive integer k.

Let $x \in G(\xi_{\Lambda}^{(a)})$. Then $0x = x$. Assume that $x \in \Lambda$. Then $x = 0x = 0a \in \xi$, a contradiction. Hence, $x \in \xi$, i.e. $x \in G(\xi)$. This gives $G(\xi_{\Lambda}^{(a)}) \subseteq G(\xi)$. The converse inclusion is clear since $\xi \subseteq \xi \cup \Lambda$. Hence, $G(\xi_{\Lambda}^{(a)}) = G(\xi)$. Next, we

want to show that $G(\xi_\Lambda^{(a)})$ is an ideal of $\xi_\Lambda^{(a)}$. Let $x, y \in \xi \cup \Lambda$. Suppose that $xy \in G(\xi_\Lambda^{(a)})$ and $y \in G(\xi_\Lambda^{(a)}) = G(\xi)$. If $y = 0$, then $x = x0 = xy \in G(\xi_\Lambda^{(a)})$. Suppose now $y \neq 0$. Assume that $x \notin \xi$, i.e. $x \in \Lambda$. Then we get $xy = ay$, where $ay \notin G(\xi)$ since $y \in G(\xi)$ but $a \notin G(\xi)$. Hence, $xy \notin G(\xi)$, a contradiction. Therefore, $x \in \xi$. It follows that $x \in G(\xi_\Lambda^{(a)})$ since $G(\xi_\Lambda^{(a)}) = G(\xi)$ and $G(\xi)$ is an ideal of ξ. Thus, $G(\xi_\Lambda^{(a)}) = G(\xi)$ is an ideal of $\xi_\Lambda^{(a)}$. Take $W = \xi_\Lambda^{(a)}$, the proof is finished.

Corollary 3.6: *Let $\xi = \{0 = \varpi_1, \varpi_2, \dots, \varpi_n\}$ and let Λ be a non-empty set with $\Lambda \cap \xi = \emptyset$. If $G(\xi) = \{0 = \varpi_1, \varpi_2, \dots, \varpi_k\}$ for some $k \in \{1,2, \dots, n-1\}$ is a proper ideal of ξ, then $G(\xi) = G(\xi_\Lambda^{(\varpi_i)})$ and $G(\xi)$ is a proper ideal of $\xi_\Lambda^{(x_i)}$ for all $i \in \{k+1, \dots, n\}$.*

Since $\xi_\Lambda^{(a)}$ is an extension of the Q-algebra ξ, we obtain immediately:

Proposition 3.7: Let Λ be a set with $\Lambda \cap \xi = \emptyset$, let $a \in \xi \backslash \{0\}$ and let $\emptyset \neq \Gamma \subseteq \xi$. If Γ is an ideal of $\xi_\Lambda^{(a)}$, then Γ is an ideal of ξ.

The converse of Proposition 3.7 is not true as seen in the following counterexample.

Example 3.8: Let consider a Q-algebras $\xi = \{0, a, b, c, d, f\}$ in Table 3(A) and a Q-algebra $\xi_\Lambda^{(c)}$ in Table 3(B), where $\Lambda = \{w\}$ as the followings:

Table 3

$*$	0	a	b	c	d	f
0	0	a	a	0	a	0
a	a	0	0	a	0	a
b	b	c	0	a	0	a
c	c	d	a	0	a	0
d	d	f	0	a	0	a
f	f	b	a	0	a	0

(A)

$*$	0	a	b	c	d	f	w
0	0	a	a	0	a	0	0
a	a	0	0	a	0	a	a
b	b	c	0	a	0	a	a
c	c	d	a	0	a	0	0
d	d	f	0	a	0	a	a
f	f	b	a	0	a	0	0
w	w	d	a	0	a	0	0

(B)

It is a routine to check that $\Gamma = \{0, c, f\}$ is an ideal of ξ. We get that Γ is not an ideal of $\xi_\Lambda^{(c)}$ since $wc = 0 \in \Gamma$ and $c \in \Gamma$ but $w \notin \Gamma$. Now, it appears the question for which $a \in \xi$, the ideals of ξ are also ideals of $\xi_\Lambda^{(a)}$.

Proposition 3.9: Let Λ be a non-empty set with $\Lambda \cap \xi = \emptyset$, let $a \in \xi \backslash \{0\}$ and let $\emptyset \neq \Gamma \subseteq \xi$. If Γ is an ideal of ξ, then Γ is an ideal of $\xi_\Lambda^{(a)}$ if and only if $a \in \xi \backslash \Gamma$.

Proof: Suppose that Γ is an ideal of ξ. Suppose that $a \in \Gamma$. Then there is $w \in \Lambda$ with $wa = aa = 0 \in \Gamma$. Since $a \in \Gamma$ and $w \notin \Gamma$, we conclude that Γ is not an ideal of $\xi_\Lambda^{(a)}$. Let $a \in \xi \backslash \Gamma$. We want to show that Γ is an ideal of $\xi_\Lambda^{(a)}$. Let

$x, y \in \xi_{\Lambda}^{(a)}$ such that $xy \in \Gamma$ and $y \in \Gamma \subseteq \xi$. If $y = 0$, then $x = x0 = xy \in \Gamma$. So, we suppose $y \neq 0$. Since $x \in \xi_{\Lambda}^{(a)} = \xi \cup \Lambda$, then $x \in \xi$ or $x \in \Lambda$. Assume that $x \in \Lambda$. Since $y \in \xi$, then $xy = ay \in \Gamma$. Since $a \in \xi$, $ay \in \Gamma, y \in \Gamma$, and Γ is an ideal of ξ, we get $a \in \Gamma$. This gives a contradiction. Hence, $x \in \xi$. Therefore, $x, y \in \xi_{\Lambda}^{(a)}$, $xy \in \Gamma, y \in \Gamma$ imply $x \in \xi$. There follows that $x \in \Gamma$ since Γ is an ideal of ξ. Thus, Γ is an ideal of $\xi_{\Lambda}^{(a)}$.

Propositions 3.7 and 3.9 give the following theorem:

Theorem 3.10: *Let Λ be a non-empty set with $\Lambda \cap \xi = \emptyset$ and let $\emptyset \neq \Gamma \subseteq \xi$. Then Γ is an ideal of ξ if and only if Γ is an ideal of $X_{\Lambda}^{(a)}$ for all $a \in \xi \backslash \Gamma$.*

Now we construct a Q -algebra of order 2^h for a given $h \in \{0,1, \dots, \lfloor log_2 n \rfloor - 1\}$.

Let $h \in \{0,1, \dots, \lfloor log_2 n \rfloor - 1\}$. If $h = 0$, then let $Y = \{0,1, \dots, n-1\}$ and let $*$ be the binary operation on Y defined by:
for $\varpi, y \in Y$, we put

$$\varpi * y = \begin{cases} 0 & \text{if } y \neq 0; \\ \varpi & \text{if } y = 0. \end{cases}$$

It is easy to verify that $(Y; *, 0)$ is a Q-algebra with $G(Y) = \{0\}$ and $|G(Y)| = 1 = 2^0$. Suppose now that $h \geq 1$. We see that the G-part of the Q-algebra Q_4 is an ideal with the cardinality 2. We will construct a Q-algebra Y of order 2^{h+1} from the Q-algebra $\xi = Q_4$, where $G(Y)$ is an ideal of Y with $|G(Y)| = 2^h$, as follows:

Let $$Y = \underbrace{G(\xi) \times G(\xi) \times \cdots \times G(\xi)}_{h-1 \text{ copy}} \times \xi.$$

We define a binary operation on Y by componentwise multiplication. For $\varpi_1, \varpi_2, \dots, \varpi_{h-1}$,

$$y_1, y_2, \dots, y_{h-1} \in G(\xi) \text{ and } \varpi_h, y_h \in \xi \text{ let:}$$

$$(\varpi_1, \varpi_2, \dots, \varpi_h) * (y_1, y_2, \dots, y_h) = (\varpi_1 y_1, \varpi_2 y_2, \dots, \varpi_h y_h).$$

It is easy to see that with $0 = (0, \dots, 0) \in Y$ both conditions (Q_1), (Q_2) are satisfied. Moreover, for any $\varpi_i, y_i, z_i \in Y$ and $i \in \{1, \dots, h\}$, we get

$$\begin{aligned} &((\varpi_1, \varpi_2, \dots, \varpi_h)(y_1, y_2, \dots, y_h))(z_1, z_2, \dots, z_h) \\ &= (\varpi_1 y_1, \varpi_2 y_2, \dots, \varpi_h y_h)(z_1, z_2, \dots, z_h) \\ &= ((\varpi_1 y_1) z_1, (\varpi_2 y_2) z_2, \dots, (\varpi_h y_h) z_h) \\ &= ((\varpi_1 z_1) y_1, (\varpi_2 z_2) y_2, \dots, (\varpi_h z_h) y_h) \\ &= ((\varpi_1, \dots, \varpi h)(z_1, \dots, z_h))(y_1, \dots, y_h). \end{aligned}$$

Therefore, (Q_3) is satisfied. Hence, $(Y; *, (0, \dots, 0))$ is a Q-algebra.

Next, we want to show that $G(Y) = \underbrace{G(\xi) \times G(\xi) \times \cdots \times G(\xi)}_{h \text{ copy}}$.

Let $(\varpi_1, \varpi_2, \dots, \varpi_h) \in G(Y)$. Then $(\varpi_1, \varpi_2, \dots, \varpi_h) = (0,0, \dots, 0)(\varpi_1, \varpi_2, \dots, \varpi_h) = (0\varpi_1, 0\varpi_2, \dots, 0\varpi_h)$. Therefore, $\varpi_1 = 0\varpi_1, \varpi_2 = 0\varpi_2, \dots, \varpi_h =$

$0\varpi_h$. Thus, we get $\varpi_1, \varpi_2, \dots, \varpi_h \in G(\xi)$. Hence, we get that $(\varpi_1, \varpi_2, \dots, \varpi_h) \in G(\xi) \times \dots \times G(\xi)$. Now let $\varpi_1, \varpi_2, \dots, \varpi_h \in G(\xi)$. Then $0\varpi_1 = \varpi_1, 0\varpi_2 = \varpi_2, \dots, 0\varpi_h = \varpi_h$. Then, we get $(0, \dots, 0)(\varpi_1, \varpi_2 \dots, \varpi_h) = (\varpi_1, \varpi_2, \dots, \varpi_h)$. Therefore, $(\varpi_1, \varpi_2, \dots, \varpi_h) \in G(Y)$. Altogether, we get $G(Y) = G(\xi) \times G(\xi) \times \cdots G(\xi)$. Next, we want to show that $G(Y)$ is an ideal of Y. Let $(\varpi_1, \varpi_2, \dots, \varpi_h)$, $(y_1, y_2, \dots, y_h) \in Y$. Assume that $(\varpi_1, \varpi_2, \dots, \varpi_h)(y_1, y_2, \dots, y_h) \in G(Y)$ and $(y_1, y_2, \dots, y_h) \in G(Y)$. Then we get $(\varpi_1 y_1, \varpi_2 y_2, \dots, \varpi_h y_h) \in G(Y)$ and $y_1, y_2, \dots, y_h \in G(\xi)$. Since $\varpi_1 y_1, \varpi_2 y_2, \dots, \varpi_h y_h \in G(\xi)$and $G(\xi)$ is an ideal of ξ, then $\varpi_1, \varpi_2, \dots, \varpi_h \in G(\xi)$. Therefore, $(\varpi_1, \varpi_2, \dots, \varpi_k) \in G(\xi) \times G(\xi) \times \cdots \times G(\xi) = G(Y)$. Thus, $G(Y)$ is an ideal of Y. So, we have proved the following proposition.

Proposition 3.11: Let $h \in \{0,1,2, \dots, \lfloor log_2 n \rfloor - 1\}$. Then there is a Q-algebra Y of order 2^{h+1} such that $G(Y)$ is an ideal of Y with $|G(Y)| = 2^h$.

Now, we can prove the main results of this section.

Theorem 3.12: *Let $h \in \{0,1,2, \dots, \lfloor log_2 n \rfloor - 1\}$. Then there is a Q-algebra Y of order n such that $G(Y)$ is an ideal of Y with $|G(Y)| = 2^h$.*

Proof: By Proposition 3.11, there is a Q-algebra $Y$of order 2^{h+1} such that $G(Y)$ is an ideal of Y with $|G(Y)| = 2^h$. If $n = 2^{h+1}$, then all is done. Suppose now that $n > 2^{h+1}$. Then let $a \in Y \backslash \{0\}$ and let Λ be a set with $\Lambda \cap Y = \emptyset$ and $|\Lambda| = n - 2^{h+1}$. Then $Y_\Lambda^{(a)}$ is a Q-algebra of order n such that $G(Y)$ is an ideal of $Y_\Lambda^{(a)}$ of the cardinality 2^h by Theorem 3.10.

4. Conclusion

The present paper gives us information about the G-part $G(\xi)$ which is an ideal of a Q-algebra ξ. The size of an ideal $G(\xi)$ is even and not greater than a half of a size of ξ. For more precisely, $|G(\xi)| = 2^h$ for some $h \in \{0,1,2, \dots, \lfloor log_2 |\xi| \rfloor - 1\}$. The construction of Q-algebras from a fixed Q-algebra ξ and some of their properties are provided. The further study, one can examine the connections between these constructed algebras.

Acknowledments

This research project was financially supported by Thailand Science Research and Innovation(TSRI).

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Received May, 2025