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A study of a variant of neighborhood version of degree-based indices for trees

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Abstract

Recently, Mondal et al. introduced several new indices based on neighborhood degree sum of vertices, which received increasing research attention. In this paper, we consider a variant of these indices named sixth neighborhood degree index. We give the minimum value of the sixth neighborhood degree index within the set of trees with ν vertices and maximum degree Δ and the corresponding minimal trees are introduced.

Subject Classification (2020): *05C07.*

Keywords: *Sixth neighborhood degree index, Trees, Lower bound.*

1. Introduction

Consider simple connected graph $\mathcal{S} = (V(\mathcal{S}), E(\mathcal{S}))$. The open neighborhood of the vertex τ of $\mathcal{S}$, $N_{\mathcal{S}}(\tau)$, is the set of all vertices of $\mathcal{S}$ adjacent to τ. The degree of a vertex τ in $\mathcal{S}$, $\deg_{\mathcal{S}}(\tau)$, is the cardinality of $N_{\mathcal{S}}(\tau)$ and let Δ be the maximum degree of $\mathcal{S}$. The distance between $\tau, \varsigma \in V(\mathcal{S})$ is the number of edges in any shortest τ, ς-path in $\mathcal{S}$.

A rooted tree is a tree in which a distinguished vertex is selected as the root of the tree. A vertex of degree 1 is said to be a leaf. A branching vertex of a tree is any vertex of degree more than two. A starlike is a tree with exactly one branching vertex. The vertex with the maximum degree of a starlike is said to be the center of starlike. A leg of a starlike is a path from its center to a leaf. A star can be considered as a starlike with all legs of length 1 and a path can be a starlike with 1 or 2 legs.

Zagreb indices are among the first and foremost vertex degree based graph invariants in mathematical chemistry. These indices were introduced in the 1970s

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[9, 10]. For comprehensive literature on these indices, see the review papers [3, 8] and the references cited therein. After the introduction of Zagreb indices, various modification of these indices have been suggested among which include Zagreb coindices [7, 26], reformulated Zagreb indices [12, 15], multiplicative Zagreb indices [11, 13, 28], Lanzhou index [5, 29], entire Zagreb indices [2, 14], leap Zagreb indices [4, 23, 25, 27], connection degree index [6], etc.

Recently, Mondal et al. introduced the neighborhood version of vertex-degree-based graph invariants [16,17]. These indices based on neighborhood degree sum of vertices. In this paper, we consider a variant of these indices named sixth neighborhood degree index. The sixth neighborhood degree index defined in [18] as:

$$\mathcal{ND}_6(\mathcal{S}) = \sum_{\tau\varsigma\in E(\mathcal{S})} (\deg_{\mathcal{S}}(\tau)\delta_{\mathcal{S}}(\tau) + \deg_{\mathcal{S}}(\varsigma)\delta_{\mathcal{S}}(\varsigma)),$$

where $\delta_{\mathcal{S}}(\tau) = \sum_{z\in N_{\mathcal{S}}(\tau)} \deg_{\mathcal{S}}(z)$. For further results, see [1, 19, 20, 21, 22, 24].

In this paper, we present a new lower bound on the sixth neighborhood degree index of trees with ν vertices and given maximum degree. Also, we obtain the trees achieve this bound.

2. Trees

We let T be a rooted tree with root ϑ where $\deg_T(\vartheta) = \Delta$ and $N_T(\vartheta) = \{\vartheta_1, \vartheta_2, \dots, \vartheta_\Delta\}$. Let $T_{\nu,\Delta}$ be the collection of all trees with ν vertices and maximum degree Δ.

Lemma 1: *Let $T \in T_{\nu,\Delta}$ and let θ be a branching vertex of T with maximum distance to ϑ. If θ is adjacent to at least two leaves, then there exists $T' \in T_{\nu,\Delta}$ with $\mathcal{ND}_6(T') < \mathcal{ND}_6(T)$.*

Proof. Assume that $\deg_T(\theta) = \pi$ and $N_T(\theta) = \{\theta_1, \theta_2, \dots, \theta_\pi\}$, where θ_π lies on the ϑ, θ-path. Therefore, for $1 \le i \le \pi - 1$, $\deg_T(\theta_i) = 1$ or $\deg_T(\theta_i) = 2$. Since θ adjacent to at least two leaves, we let θ_1 and θ_2 be leaves. Let $T' = (T - \{\theta\theta_1\}) \cup \{\theta_1\theta_2\}$. Since $\pi \ge 3$ and $\delta_T(\theta) \ge 4$, we have

$$\begin{aligned}\beta = &\sum_{i=3}^{\pi} [\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_i)\delta_T(\theta_i)] \\ &- \sum_{i=3}^{\pi} [\deg_{T'}(\theta)\delta_{T'}(\theta) + \deg_{T'}(\theta_i)\delta_{T'}(\theta_i)] \\ = &\sum_{i=3}^{\pi} [\pi\delta_T(\theta) + \deg_T(\theta_i)\delta_T(\theta_i) - (\pi-1)\delta_T(\theta) \\ &-\deg_T(\theta_i)(\delta_T(\theta_i) - 1)] \\ = &\sum_{i=3}^{\pi} (\delta_T(\theta) + \deg_T(\theta_i)) > 0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{ND}_6(T) - \mathcal{ND}_6(T') \ge\ &\beta + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_1)\delta_T(\theta_1)] \\ &+[\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_2)\delta_T(\theta_2)] \\ &-[\deg_{T'}(\theta_1)\delta_{T'}(\theta_1) + \deg_{T'}(\theta_2)\delta_{T'}(\theta_2)] \\ &-[\deg_{T'}(\theta)\delta_{T'}(\theta) + \deg_{T'}(\theta_2)\delta_{T'}(\theta_2)] \\ =\ &\beta + 2(\pi + \pi\delta_T(\theta)) - (2\pi + 2) \\ &-((\pi - 1)\delta_T(\theta) + 2\pi) \\ >\ &\pi\delta_T(\theta) + \delta_T(\theta) - 2\pi - 2 \ge 2\pi + 2 > 0.\end{aligned}$$

Lemma 2: *Let $T \in T_{\nu,\Delta}$ and let θ be a branching vertex of T with maximum distance to ϑ. If θ adjacent to a leaf, then there exists $T' \in T_{\nu,\Delta}$ with $\mathcal{ND}_6(T') < \mathcal{ND}_6(T)$.*

Proof: Assume that $\deg_T(\theta) = \pi$ and $N_T(\theta) = \{\theta_1, \theta_2, \dots, \theta_\pi\}$, where θ_π lies on the ϑ, θ-path. Therefore, for $1 \le i \le \pi - 1$, $\deg_T(\theta_i) = 1$ or $\deg_T(\theta_i) = 2$. Since θ is adjacent to a leaf, we let θ_1 be a leaf and $\deg_T(\theta_i) = 2$ for $2 \le i \le \pi - 1$. Let $\theta\,\eta_1\eta_2 \dots \eta_l$ be a path in T with $\theta_2 = \eta_1$ and $l \ge 2$. Let $T' = (T - \{\theta\theta_1\}) \cup \{\theta_1\eta_l\}$. Since $\pi \ge 3$ and $\delta_T(\theta) \ge 5$, we have

$$\begin{aligned}
\beta = {} & \sum_{i=3}^{\pi} [\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_i)\delta_T(\theta_i)] \\
& - \sum_{i=3}^{\pi} [\deg_{T'}(\theta)\delta_{T'}(\theta) + \deg_{T'}(\theta_i)\delta_{T'}(\theta_i)] \\
= {} & \sum_{i=3}^{\pi} [\pi\delta_T(\theta) + \deg_T(\theta_i)\delta_T(\theta_i) - (\pi - 1)(\delta_T(\theta) - 1) \\
& - \deg_T(\theta_i)(\delta_T(\theta_i) - 1)] \\
= {} & \sum_{i=3}^{\pi} (\delta_T(\theta) + \deg_T(\theta_i) + \pi - 1) > 0.
\end{aligned}$$

If $l = 2$, then

$$\begin{aligned}
\mathcal{ND}_6(T) - \mathcal{ND}_6(T') \ge {} & \beta + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_1)\delta_T(\theta_1)] \\
& + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\eta_1)\delta_T(\eta_1)] \\
& + [\deg_T(\eta_1)\delta_T(\eta_1) + \deg_T(\eta_2)\delta_T(\eta_2)] \\
& - [\deg_{T'}(\theta_1)\delta_{T'}(\theta_1) + \deg_{T'}(\eta_2)\delta_{T'}(\eta_2)] \\
& - [\deg_{T'}(\eta_2)\delta_{T'}(\eta_2) + \deg_{T'}(\eta_1)\delta_{T'}(\eta_1)] \\
& - [\deg_{T'}(\eta_1)\delta_{T'}(\eta_1) + \deg_{T'}(\theta)\delta_{T'}(\theta)] \\
= {} & \beta + (\pi + \pi\delta_T(\theta)) + (2(\pi + 1) + \pi\delta_T(\theta)) \\
& + (2(\pi + 1) + 2) - (2 + 6) - (2(\pi + 1) + 6) \\
& - ((\pi - 1)(\delta_T(\theta) - 1) + 2(\pi + 1)) \\
> {} & \pi\delta_T(\theta) + \delta_T(\theta) + 2\pi - 13 \ge 7\pi - 8 > 0.
\end{aligned}$$

If $l = 3$, then

$$\begin{aligned}
\mathcal{ND}_6(T) - \mathcal{ND}_6(T') \ge {} & \beta + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_1)\delta_T(\theta_1)] \\
& + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\eta_1)\delta_T(\eta_1)] \\
& + [\deg_T(\eta_1)\delta_T(\eta_1) + \deg_T(\eta_2)\delta_T(\eta_2)] \\
& + [\deg_T(\eta_2)\delta_T(\eta_2) + \deg_T(\eta_3)\delta_T(\eta_3)] \\
& - [\deg_{T'}(\eta_1)\delta_{T'}(\eta_1) + \deg_{T'}(\theta)\delta_{T'}(\theta)] \\
& - [\deg_{T'}(\theta_1)\delta_{T'}(\theta_1) + \deg_{T'}(\eta_3)\delta_{T'}(\eta_3)] \\
& - [\deg_{T'}(\eta_3)\delta_{T'}(\eta_3) + \deg_{T'}(\eta_2)\delta_{T'}(\eta_2)] \\
& - [\deg_{T'}(\eta_2)\delta_{T'}(\eta_2) + \deg_{T'}(\eta_1)\delta_{T'}(\eta_1)] \\
= {} & \beta + (\pi + \pi\delta_T(\theta)) + (2(\pi + 2) + \pi\delta_T(\theta)) \\
& + (2(\pi + 2) + 6) + (2 + 6) \\
& - (2 + 6) - (6 + 8) - (2(\pi + 1) + 8) \\
& - ((\pi - 1)(\delta_T(\theta) - 1) + 2(\pi + 1)) \\
> {} & \pi\delta_T(\theta) + \delta_T(\theta) + 2\pi - 13 \ge 7\pi - 8 > 0.
\end{aligned}$$

Now if $l > 3$, then

$$\begin{aligned}\mathcal{ND}_6(T) - \mathcal{ND}_6(T') \geq\ & \beta + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_1)\delta_T(\theta_1)]\\ & +[\deg_T(\eta_l)\delta_T(\eta_l) + \deg_T(\eta_{l-1})\delta_T(\eta_{l-1})]\\ & +[\deg_T(\eta_{l-1})\delta_T(\eta_{l-1}) + \deg_T(\eta_{l-2})\delta_T(\eta_{l-2})]\\ & -[\deg_{T'}(\theta_1)\delta_{T'}(\theta_1) + \deg_{T'}(\eta_l)\delta_{T'}(\eta_l)]\\ & -[\deg_{T'}(\eta_l)\delta_{T'}(\eta_l) + \deg_{T'}(\eta_{l-1})\delta_{T'}(\eta_{l-1})]\\ & -[\deg_{T'}(\eta_{l-1})\delta_{T'}(\eta_{l-1}) + \deg_{T'}(\eta_{l-2})\delta_{T'}(\eta_{l-2})]\\ =\ & \beta + (\pi + \pi\delta_T(\theta)) + (2+6) + (6+8)\\ & -(2+6) - (6+8) - (8+8))\\ >\ & \pi\delta_T(\theta) + \pi - 16 \geq 5\pi - 13 > 0.\end{aligned}$$

Lemma 3: *Let $T \in T_{\nu,\Delta}$ and let θ be a branching vertex of T with maximum distance to ϑ. Then there exists $T' \in T_{\nu,\Delta}$ with $\mathcal{ND}_6(T') < \mathcal{ND}_6(T)$.*

Proof: Assume that $\deg_T(\theta) = \pi$ and $N_T(\theta) = \{\theta_1, \theta_2, \ldots, \theta_\pi\}$, where θ_π lies on the ϑ, θ-path. By Lemmas 1 and 2, $\deg_T(\theta_i) = 2$ for $1 \leq i \leq \pi - 1$. Let $\theta\, \eta_1\eta_2 \ldots \eta_l$ and $\theta\, z_1z_2 \ldots z_s$, $l, s \geq 2$, be two paths in T with $\theta_1 = \eta_1$ and $\theta_2 = z_1$. Let $T' = (T - \{\theta\theta_1\}) \cup \{\theta_1 z_s\}$. Since $\pi \geq 3$ and $\delta_T(\theta) \geq 6$, we have

$$\begin{aligned}\beta =\ & \textstyle\sum_{i=3}^{\pi} [\deg_T(\theta)\delta_T(\theta) + \deg_T(\theta_i)\delta_T(\theta_i)]\\ & -\textstyle\sum_{i=3}^{\pi} [\deg_{T'}(\theta)\delta_{T'}(\theta) + \deg_{T'}(\theta_i)\delta_{T'}(\theta_i)]\\ =\ & \textstyle\sum_{i=3}^{\pi} [\pi\delta_T(\theta) + \deg_T(\theta_i)\delta_T(\theta_i) - (\pi-1)(\delta_T(\theta)-2)\\ & -\deg_T(\theta_i)(\delta_T(\theta_i)-1)]\\ =\ & \textstyle\sum_{i=3}^{\pi} (\delta_T(\theta) + \deg_T(\theta_i) + 2\pi - 2) > 0.\end{aligned}$$

If $l = s = 2$, then

$$\begin{aligned}\mathcal{ND}_6(T) - \mathcal{ND}_6(T') \geq\ & \beta + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\eta_1)\delta_T(\eta_1)]\\ & +[\deg_T(\eta_1)\delta_T(\eta_1) + \deg_T(\eta_2)\delta_T(\eta_2)]\\ & +[\deg_T(\theta)\delta_T(\theta) + \deg_T(z_1)\delta_T(z_1)]\\ & +[\deg_T(z_1)\delta_T(z_1) + \deg_T(z_2)\delta_T(z_2)]\\ & -[\deg_{T'}(\eta_2)\delta_{T'}(\eta_2) + \deg_{T'}(\eta_1)\delta_{T'}(\eta_1)]\\ & -[\deg_{T'}(\eta_1)\delta_{T'}(\eta_1) + \deg_{T'}(z_2)\delta_{T'}(z_2)]\\ & -[\deg_{T'}(z_2)\delta_{T'}(z_2) + \deg_{T'}(z_1)\delta_{T'}(z_1)]\\ & -[\deg_{T'}(z_1)\delta_{T'}(z_1) + \deg_{T'}(\theta)\delta_{T'}(\theta)]\\ =\ & \beta + 2(2(\pi+1) + \pi\delta_T(\theta)) + 2(2(\pi+1) + 2)\\ & -(2+6) - (6+8) - (2(\pi+1) + 8)\\ & -((\pi-1)(\delta_T(\theta)-2) + 2(\pi+1))\\ >\ & \pi\delta_T(\theta) + \delta_T(\theta) + 6\pi - 24 \geq 12\pi - 18 > 0.\end{aligned}$$

If $l = 2$ and $s = 3$, then

$$\begin{aligned}
\mathcal{ND}_6(T)-\mathcal{ND}_6(T') \geq\ & \beta+[\deg_T(\theta)\delta_T(\theta)+\deg_T(\eta_1)\delta_T(\eta_1)] \\
& +[\deg_T(\eta_1)\delta_T(\eta_1)+\deg_T(\eta_2)\delta_T(\eta_2)] \\
& +[\deg_T(\theta)\delta_T(\theta)+\deg_T(z_1)\delta_T(z_1)] \\
& +[\deg_T(z_1)\delta_T(z_1)+\deg_T(z_2)\delta_T(z_2)] \\
& +[\deg_T(z_2)\delta_T(z_2)+\deg_T(z_3)\delta_T(z_3)] \\
& -[\deg_{T'}(\eta_2)\delta_{T'}(\eta_2)+\deg_{T'}(\eta_1)\delta_{T'}(\eta_1)] \\
& -[\deg_{T'}(\eta_1)\delta_{T'}(\eta_1)+\deg_{T'}(z_3)\delta_{T'}(z_3)] \\
& -[\deg_{T'}(z_3)\delta_{T'}(z_3)+\deg_{T'}(z_2)\delta_{T'}(z_2)] \\
& -[\deg_{T'}(z_2)\delta_{T'}(z_2)+\deg_{T'}(z_1)\delta_{T'}(z_1)] \\
& -[\deg_{T'}(z_1)\delta_{T'}(z_1)+\deg_{T'}(\theta)\delta_{T'}(\theta)] \\
=\ & \beta+(2(\pi+1)+\pi\delta_T(\theta))+(2(\pi+1)+2) \\
& +(2(\pi+2)+\pi\delta_T(\theta))\ +(2(\pi+2)+6)+(2+6) \\
& -(2+6)-(6+8)-(8+8)\ -(2(\pi+1)+8) \\
& -((\pi-1)(\delta_T(\theta)-2)+2(\pi+1)) \\
>\ & \pi\delta_T(\theta)+\delta_T(\theta)+6\pi-24 \geq 12\pi-18>0.
\end{aligned}$$

If $l=2$ and $s>3$, then

$$\begin{aligned}
\mathcal{ND}_6(T)-\mathcal{ND}_6(T') \geq\ & \beta+[\deg_T(\theta)\delta_T(\theta)+\deg_T(\eta_1)\delta_T(\eta_1)] \\
& +[\deg_T(\eta_1)\delta_T(\eta_1)+\deg_T(\eta_2)\delta_T(\eta_2)] \\
& +[\deg_T(z_s)\delta_T(z_s)+\deg_T(z_{s-1})\delta_T(z_{s-1})] \\
& +[\deg_T(z_{s-1})\delta_T(z_{s-1})+\deg_T(z_{s-2})\delta_T(z_{s-2})] \\
& -[\deg_{T'}(\eta_2)\delta_{T'}(\eta_2)+\deg_{T'}(\eta_1)\delta_{T'}(\eta_1)] \\
& -[\deg_{T'}(\eta_1)\delta_{T'}(\eta_1)+\deg_{T'}(z_s)\delta_{T'}(z_s)] \\
& -[\deg_{T'}(z_s)\delta_{T'}(z_s)+\deg_{T'}(z_{s-1})\delta_{T'}(z_{s-1})] \\
& -[\deg_{T'}(z_{s-1})\delta_{T'}(z_{s-1})+\deg_{T'}(z_{s-2})\delta_{T'}(z_{s-2})] \\
=\ & \beta+(2(\pi+1)+\pi\delta_T(\theta))+(2(\pi+1)+2) \\
& +(2+6)+(6+8)-(2+6)-(6+8)-2(8+8) \\
>\ & \pi\delta_T(\theta)+4\pi-28 \geq 10\pi-28>0.
\end{aligned}$$

If $l=s=3$, then

$$\begin{aligned}
\mathcal{ND}_6(T)-\mathcal{ND}_6(T') \geq\ & \beta+[\deg_T(\theta)\delta_T(\theta)+\deg_T(\eta_1)\delta_T(\eta_1)] \\
& +[\deg_T(\eta_1)\delta_T(\eta_1)+\deg_T(\eta_2)\delta_T(\eta_2)] \\
& +[\deg_T(\eta_2)\delta_T(\eta_2)+\deg_T(\eta_3)\delta_T(\eta_3)] \\
& +[\deg_T(\theta)\delta_T(\theta)+\deg_T(z_1)\delta_T(z_1)] \\
& +[\deg_T(z_1)\delta_T(z_1)+\deg_T(z_2)\delta_T(z_2)] \\
& +[\deg_T(z_2)\delta_T(z_2)+\deg_T(z_3)\delta_T(z_3)] \\
& -[\deg_{T'}(\eta_3)\delta_{T'}(\eta_3)+\deg_{T'}(\eta_2)\delta_{T'}(\eta_2)] \\
& -[\deg_{T'}(\eta_2)\delta_{T'}(\eta_2)+\deg_{T'}(\eta_1)\delta_{T'}(\eta_1)]
\end{aligned}$$

$$\begin{aligned}
&\quad -[\deg_{T'}(\eta_1)\delta_{T'}(\eta_1) + \deg_{T'}(z_3)\delta_{T'}(z_3)]\\
&\quad -[\deg_{T'}(z_3)\delta_{T'}(z_3) + \deg_{T'}(z_2)\delta_{T'}(z_2)]\\
&\quad -[\deg_{T'}(z_2)\delta_{T'}(z_2) + \deg_{T'}(z_1)\delta_{T'}(z_1)]\\
&\quad -[\deg_{T'}(z_1)\delta_{T'}(z_1) + \deg_{T'}(\theta)\delta_{T'}(\theta)]\\
&= \beta + 2(2(\pi+2) + \pi\delta_T(\theta)) + 2(2(\pi+1) + 6)\\
&\quad +2(2+6) - (2+6) - (6+8) - 2(8+8)\\
&\quad -(2(\pi+1) + 8) - ((\pi-1)(\delta_T(\theta) - 2)\\
&\quad +2(\pi+1))\\
&> \pi\delta_T(\theta) + \delta_T(\theta) + 2\pi - 28 \geq 8\pi - 18 > 0.
\end{aligned}$$

If $l = 3$ and $s > 3$, then

$$\begin{aligned}
\mathcal{ND}_6(T) - \mathcal{ND}_6(T') &\geq \beta + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\eta_1)\delta_T(\eta_1)]\\
&\quad +[\deg_T(\eta_1)\delta_T(\eta_1) + \deg_T(\eta_2)\delta_T(\eta_2)]\\
&\quad +[\deg_T(\eta_2)\delta_T(\eta_2) + \deg_T(\eta_3)\delta_T(\eta_3)]\\
&\quad +[\deg_T(z_s)\delta_T(z_s) + \deg_T(z_{s-1})\delta_T(z_{s-1})]\\
&\quad +[\deg_T(z_{s-1})\delta_T(z_{s-1}) + \deg_T(z_{s-2})\delta_T(z_{s-2})]\\
&\quad -[\deg_{T'}(\eta_3)\delta_{T'}(\eta_3) + \deg_{T'}(\eta_2)\delta_{T'}(\eta_2)]\\
&\quad -[\deg_{T'}(\eta_2)\delta_{T'}(\eta_2) + \deg_{T'}(\eta_1)\delta_{T'}(\eta_1)]\\
&\quad -[\deg_{T'}(\eta_1)\delta_{T'}(\eta_1) + \deg_{T'}(z_s)\delta_{T'}(z_s)]\\
&\quad -[\deg_{T'}(z_s)\delta_{T'}(z_s) + \deg_{T'}(z_{s-1})\delta_{T'}(z_{s-1})]\\
&\quad -[\deg_{T'}(z_{s-1})\delta_{T'}(z_{s-1}) + \deg_{T'}(z_{s-2})\delta_{T'}(z_{s-2})]\\
&= \beta + (2(\pi+2) + \pi\delta_T(\theta))\\
&\quad +(2(\pi+2) + 6) + 2(2+6) + (6+8)\\
&\quad -(2+6) - (6+8) - 3(8+8)\\
&> \pi\delta_T(\theta) + 4\pi - 26 \geq 10\pi - 26 > 0.
\end{aligned}$$

If $l, s > 3$, then

$$\begin{aligned}
\mathcal{ND}_6(T) - \mathcal{ND}_6(T') &\geq \beta + [\deg_T(\theta)\delta_T(\theta) + \deg_T(\eta_1)\delta_T(\eta_1)]\\
&\quad +[\deg_T(\eta_1)\delta_T(\eta_1) + \deg_T(\eta_2)\delta_T(\eta_2)]\\
&\quad +[\deg_T(z_s)\delta_T(z_s) + \deg_T(z_{s-1})\delta_T(z_{s-1})]\\
&\quad +[\deg_T(z_{s-1})\delta_T(z_{s-1}) + \deg_T(z_{s-2})\delta_T(z_{s-2})]\\
&\quad -[\deg_{T'}(\eta_2)\delta_{T'}(\eta_2) + \deg_{T'}(\eta_1)\delta_{T'}(\eta_1)]\\
&\quad -[\deg_{T'}(\eta_1)\delta_{T'}(\eta_1) + \deg_{T'}(z_s)\delta_{T'}(z_s)]\\
&\quad -[\deg_{T'}(z_s)\delta_{T'}(z_s) + \deg_{T'}(z_{s-1})\delta_{T'}(z_{s-1})]\\
&\quad -[\deg_{T'}(z_{s-1})\delta_{T'}(z_{s-1}) + \deg_{T'}(z_{s-2})\delta_{T'}(z_{s-2})]\\
&= \beta + (2(\pi+2) + \pi\delta_T(\theta)) + (2(\pi+2) + 8)\\
&\quad +(2+6) + (6+8) - 4(8+8)\\
&> \pi\delta_T(\theta) + 4\pi - 26 \geq 10\pi - 26 > 0.
\end{aligned}$$

Proposition 4: If $T \in T_{\nu,\Delta}$ is a starlike with $\Delta \geq 3$ such that T has two legs of length greater than 1, then there is a starlike $T' \in T_{\nu,\Delta}$ with $\mathcal{ND}_6(T') < \mathcal{ND}_6(T)$.

Proof: Assume that ϑ be the center of T and $\vartheta\theta_1\theta_2 \dots \theta_l$, $\vartheta\gamma_1\gamma_2 \dots \gamma_s$ be two legs with $l, s \geq 2$. Let $T' = (T - \{\theta_1\theta_2\}) \cup \{\theta_2\gamma_s\}$. Then

$$\begin{aligned}
\beta =& \textstyle\sum_{i=3}^{\Delta} [\deg_T(\vartheta)\delta_T(\vartheta) + \deg_T(\vartheta_i)\delta_T(\vartheta_i)] \\
&- \textstyle\sum_{i=3}^{\Delta} [\deg_{T'}(\vartheta)\delta_{T'}(\vartheta) + \deg_{T'}(\vartheta_i)\delta_{T'}(\vartheta_i)] \\
=& \textstyle\sum_{i=3}^{\Delta} [\Delta\delta_T(\vartheta) + \deg_T(\vartheta_i)\delta_T(\vartheta_i) - \Delta(\delta_T(\vartheta) - 1) - \deg_T(\vartheta_i)\delta_T(\vartheta_i)] \\
=& \textstyle\sum_{i=3}^{\Delta} \Delta = \Delta(\Delta - 2) > 0.
\end{aligned}$$

If $l = s = 2$, then

$$\begin{aligned}
\mathcal{ND}_6(T) - \mathcal{ND}_6(T') \geq\ & \beta + [\deg_T(\vartheta)\delta_T(\vartheta) + \deg_T(\theta_1)\delta_T(\theta_1)] \\
&+[\deg_T(\theta_1)\delta_T(\theta_1) + \deg_T(\theta_2)\delta_T(\theta_2)] \\
&+[\deg_T(\vartheta)\delta_T(\vartheta) + \deg_T(\gamma_1)\delta_T(\gamma_1)] \\
&+[\deg_T(\gamma_1)\delta_T(\gamma_1) + \deg_T(\gamma_2)\delta_T(\gamma_2)] \\
&-[\deg_{T'}(\vartheta)\delta_{T'}(\vartheta) + \deg_{T'}(\theta_1)\delta_{T'}(\theta_1)] \\
&-[\deg_{T'}(\theta_2)\delta_{T'}(\theta_2) + \deg_{T'}(\gamma_2)\delta_{T'}(\gamma_2)] \\
&-[\deg_{T'}(\gamma_2)\delta_{T'}(\gamma_2) + \deg_{T'}(\gamma_1)\delta_{T'}(\gamma_1)] \\
&-[\deg_{T'}(\gamma_1)\delta_{T'}(\gamma_1) + \deg_{T'}(\vartheta)\delta_{T'}(\vartheta)] \\
=\ & \beta + 2(2(\Delta + 1) + \Delta\delta_T(\vartheta)) + 2(2(\Delta + 1) + 2) \\
&-(2 + 6) - (2(\Delta + 2) + 6) - (\Delta(\delta_T(\theta) - 1) + \Delta) \\
=\ & \beta + 5\Delta - 10 > 0.
\end{aligned}$$

If $l = 2$ and $s = 3$, then

$$\begin{aligned}
\mathcal{ND}_6(T) - \mathcal{ND}_6(T') \geq\ & \beta + [\deg_T(\vartheta)\delta_T(\vartheta) + \deg_T(\theta_1)\delta_T(\theta_1)] \\
&+[\deg_T(\theta_1)\delta_T(\theta_1) + \deg_T(\theta_2)\delta_T(\theta_2)] \\
&+[\deg_T(\vartheta)\delta_T(\vartheta) + \deg_T(\gamma_1)\delta_T(\gamma_1)] \\
&+[\deg_T(\gamma_1)\delta_T(\gamma_1) + \deg_T(\gamma_2)\delta_T(\gamma_2)] \\
&+[\deg_T(\gamma_2)\delta_T(\gamma_2) + \deg_T(\gamma_3)\delta_T(\gamma_3)] \\
&-[\deg_{T'}(\vartheta)\delta_{T'}(\vartheta) + \deg_{T'}(\theta_1)\delta_{T'}(\theta_1)] \\
&-[\deg_{T'}(\theta_2)\delta_{T'}(\theta_2) + \deg_{T'}(\gamma_3)\delta_{T'}(\gamma_3)] \\
&-[\deg_{T'}(\gamma_3)\delta_{T'}(\gamma_3) + \deg_{T'}(\gamma_2)\delta_{T'}(\gamma_2)] \\
&-[\deg_{T'}(\gamma_2)\delta_{T'}(\gamma_2) + \deg_{T'}(\gamma_1)\delta_{T'}(\gamma_1)] \\
&-[\deg_{T'}(\gamma_1)\delta_{T'}(\gamma_1) + \deg_{T'}(\vartheta)\delta_{T'}(\vartheta)] \\
=\ & \beta + (2(\Delta + 1) + \Delta\delta_T(\vartheta)) + (2(\Delta + 1) + 2) \\
&+(2(\Delta + 2) + \Delta\delta_T(\vartheta)) \\
&+(2(\Delta + 2) + 2) + (2 + 6) + (2(\Delta + 2) + 6)
\end{aligned}$$

$$
\begin{aligned}
&-(2+6)-(6+8)-(2(\Delta+2)+8)\\
&-(2(\Delta+2)+\Delta(\delta_T(\vartheta)-1))-(\Delta(\delta_T(\theta)-1)+\Delta)\\
=\ &\beta+5\Delta-10>0.
\end{aligned}
$$

If $l=2$ and $s>3$, then

$$
\begin{aligned}
\mathcal{ND}_6(T)-\mathcal{ND}_6(T') \geq\ & \beta+[\deg_T(\vartheta)\delta_T(\vartheta)+\deg_T(\theta_1)\delta_T(\theta_1)]\\
&+[\deg_T(\theta_1)\delta_T(\theta_1)+\deg_T(\theta_2)\delta_T(\theta_2)]\\
&+[\deg_T(\gamma_s)\delta_T(\gamma_s)+\deg_T(\gamma_{s-1})\delta_T(\gamma_{s-1})]\\
&+[\deg_T(\gamma_{s-1})\delta_T(\gamma_{s-1})+\deg_T(\gamma_{s-2})\delta_T(\gamma_{s-2})]\\
&-[\deg_{T'}(\vartheta)\delta_{T'}(\vartheta)+\deg_{T'}(\theta_1)\delta_{T'}(\theta_1)]\\
&-[\deg_{T'}(\theta_2)\delta_{T'}(\theta_2)+\deg_{T'}(\gamma_s)\delta_{T'}(\gamma_s)]\\
&-[\deg_{T'}(\gamma_s)\delta_{T'}(\gamma_s)+\deg_{T'}(\gamma_{s-1})\delta_{T'}(\gamma_{s-1})]\\
&-[\deg_{T'}(\gamma_{s-1})\delta_{T'}(\gamma_{s-1})+\deg_{T'}(\gamma_{s-2})\delta_{T'}(\gamma_{s-2})]\\
=\ &\beta+(2(\Delta+1)+\Delta\delta_T(\vartheta))\\
&+(2(\Delta+1)+2)+(2+6)+(6+8)\\
&-(\Delta(\delta_T(\theta)-1)+\Delta)-(2+6)-(6+8)\\
&-(8+8)\\
=\ &\beta+4\Delta-10>0.
\end{aligned}
$$

If $l=s=3$, then

$$
\begin{aligned}
\mathcal{ND}_6(T)-\mathcal{ND}_6(T') \geq\ & \beta+[\deg_T(\vartheta)\delta_T(\vartheta)+\deg_T(\theta_1)\delta_T(\theta_1)]\\
&+[\deg_T(\theta_1)\delta_T(\theta_1)+\deg_T(\theta_2)\delta_T(\theta_2)]\\
&+[\deg_T(\theta_2)\delta_T(\theta_2)+\deg_T(\theta_3)\delta_T(\theta_3)]\\
&+[\deg_T(\vartheta)\delta_T(\vartheta)+\deg_T(\gamma_1)\delta_T(\gamma_1)]\\
&+[\deg_T(\gamma_1)\delta_T(\gamma_1)+\deg_T(\gamma_2)\delta_T(\gamma_2)]\\
&+[\deg_T(\gamma_2)\delta_T(\gamma_2)+\deg_T(\gamma_3)\delta_T(\gamma_3)]\\
&-[\deg_{T'}(\vartheta)\delta_{T'}(\vartheta)+\deg_{T'}(\theta_1)\delta_{T'}(\theta_1)]\\
&-[\deg_{T'}(\theta_3)\delta_{T'}(\theta_3)+\deg_{T'}(\theta_2)\delta_{T'}(\theta_2)]\\
&-[\deg_{T'}(\theta_2)\delta_{T'}(\theta_2)+\deg_{T'}(\gamma_3)\delta_{T'}(\gamma_3)]\\
&-[\deg_{T'}(\gamma_3)\delta_{T'}(\gamma_3)+\deg_{T'}(\gamma_2)\delta_{T'}(\gamma_2)]\\
&-[\deg_{T'}(\gamma_2)\delta_{T'}(\gamma_2)+\deg_{T'}(\gamma_1)\delta_{T'}(\gamma_1)]\\
&-[\deg_{T'}(\gamma_1)\delta_{T'}(\gamma_1)+\deg_{T'}(\vartheta)\delta_{T'}(\vartheta)]\\
=\ &\beta+2(2(\Delta+2)+\Delta\delta_T(\vartheta))+2(2(\Delta+2)+6)\\
&+2(2+6)-(2+6)-(6+8)-(8+8)\\
&-(2(\Delta+2)+8)-(2(\Delta+2)+\Delta(\delta_T(\vartheta)-1))\\
&-(\Delta(\delta_T(\theta)-1)+\Delta)\\
=\ &\beta+5\Delta-10>0.
\end{aligned}
$$

If $l=3$ and $s>3$, then

$$
\mathcal{ND}_6(T)-\mathcal{ND}_6(T') \geq \beta+[\deg_T(\vartheta)\delta_T(\vartheta)+\deg_T(\theta_1)\delta_T(\theta_1)]
$$

$$
\begin{aligned}
& +[\deg_T(\theta_1)\delta_T(\theta_1)+\deg_T(\theta_2)\delta_T(\theta_2)] \\
& +[\deg_T(\theta_2)\delta_T(\theta_2)+\deg_T(\theta_3)\delta_T(\theta_3)] \\
& +[\deg_T(\gamma_s)\delta_T(\gamma_s)+\deg_T(\gamma_{s-1})\delta_T(\gamma_{s-1})] \\
& +[\deg_T(\gamma_{s-1})\delta_T(\gamma_{s-1})+\deg_T(\gamma_{s-2})\delta_T(\gamma_{s-2})] \\
& -[\deg_{T'}(\vartheta)\delta_{T'}(\vartheta)+\deg_{T'}(\theta_1)\delta_{T'}(\theta_1)] \\
& -[\deg_{T'}(\theta_3)\delta_{T'}(\theta_3)+\deg_{T'}(\theta_2)\delta_{T'}(\theta_2)] \\
& -[\deg_{T'}(\theta_2)\delta_{T'}(\theta_2)+\deg_{T'}(\gamma_s)\delta_{T'}(\gamma_s)] \\
& -[\deg_{T'}(\gamma_s)\delta_{T'}(\gamma_s)+\deg_{T'}(\gamma_{s-1})\delta_{T'}(\gamma_{s-1})] \\
& -[\deg_{T'}(\gamma_{s-1})\delta_{T'}(\gamma_{s-1})+\deg_{T'}(\gamma_{s-2})\delta_{T'}(\gamma_{s-2})] \\
= \; & \beta+(2(\Delta+2)+\Delta\delta_T(\vartheta)) \\
& +(2(\Delta+2)+6)+2(2+6)+(6+8) \\
& -(\Delta(\delta_T(\theta)-1)+\Delta)-(2+6)-(6+8) \\
& -2(8+8) \\
= \; & \beta+4\Delta-10>0.
\end{aligned}
$$

If $l, s > 3$, then

$$
\begin{aligned}
\mathcal{ND}_6(T)-\mathcal{ND}_6(T') \geq \; & \beta+[\deg_T(\vartheta)\delta_T(\vartheta)+\deg_T(\theta_1)\delta_T(\theta_1)] \\
& +[\deg_T(\theta_1)\delta_T(\theta_1)+\deg_T(\theta_2)\delta_T(\theta_2)] \\
& +[\deg_T(\gamma_s)\delta_T(\gamma_s)+\deg_T(\gamma_{s-1})\delta_T(\gamma_{s-1})] \\
& +[\deg_T(\gamma_{s-1})\delta_T(\gamma_{s-1})+\deg_T(\gamma_{s-2})\delta_T(\gamma_{s-2})] \\
& -[\deg_{T'}(\vartheta)\delta_{T'}(\vartheta)+\deg_{T'}(\theta_1)\delta_{T'}(\theta_1)] \\
& -[\deg_{T'}(\theta_2)\delta_{T'}(\theta_2)+\deg_{T'}(\gamma_s)\delta_{T'}(\gamma_s)] \\
& -[\deg_{T'}(\gamma_s)\delta_{T'}(\gamma_s)+\deg_{T'}(\gamma_{s-1})\delta_{T'}(\gamma_{s-1})] \\
& -[\deg_{T'}(\gamma_{s-1})\delta_{T'}(\gamma_{s-1})+\deg_{T'}(\gamma_{s-2})\delta_{T'}(\gamma_{s-2})] \\
= \; & \beta+(2(\Delta+2)+\Delta\delta_T(\vartheta)) \\
& +(2(\Delta+2)+8)+(2+6)+(6+8) \\
& -(\Delta(\delta_T(\theta)-1)+\Delta)-3(8+8) \\
= \; & \beta+4\Delta-10>0.
\end{aligned}
$$

This completes the proof.

Theorem 5: *If $T \in T_{\nu,\Delta}$, then*

$$
\mathcal{ND}_6(T) \geq \begin{cases}
\Delta^3+\Delta^2, & \text{if } \Delta=\nu-1 \\
\Delta^3+2\Delta^2+3\Delta+6, & \text{if } \Delta=\nu-2 \\
\Delta^3+2\Delta^2+3\Delta+22, & \text{if } \Delta=\nu-3 \\
16\nu+\Delta^3+2\Delta^2-13\Delta-26, & \text{if } \Delta<\nu-3.
\end{cases}
$$

The equality holds if and only if T is a starlike with at most one leg of length greater than 1.

Proof: Suppose that $T' \in T_{\nu,\Delta}$ such that $\mathcal{ND}_6(T') \leq ND_3(T)$ when $T \in T_{\nu,\Delta}$. Rooted T' at ϑ such that $\deg_{T'}(\vartheta) = \Delta$. First let $\Delta = 2$. Hence T' is a path and the result is immediate. Now let $\Delta \geq 3$. By the selection of T' and Lemmas 1, 2 and 3, we conclude that T' is a starlike centered at ϑ. By Proposition 4, we deduce that T' has at most one leg of length more than one. If T' is a star, then $\Delta = \nu - 1$ and $\mathcal{ND}_6(T') = \Delta(\Delta^2 + \Delta) = \Delta^3 + \Delta^2$. Now let $\Delta < \nu - 1$. If $\Delta = \nu - 2$, then

$$\begin{aligned}\mathcal{ND}_6(T') = &\ (\Delta - 1)(\Delta(\Delta + 1) + \Delta)\\ &+(\Delta(\Delta + 1) + 2(\Delta + 1)) + (2(\Delta + 1) + 2)\\ = &\ \Delta^3 + 2\Delta^2 + 3\Delta + 6,\end{aligned}$$

now if $\Delta = \nu - 3$, then

$$\begin{aligned}\mathcal{ND}_6(T') = &\ (\Delta - 1)(\Delta(\Delta + 1) + \Delta)\\ &+(\Delta(\Delta + 1) + 2(\Delta + 2)) + (2(\Delta + 2) + 2) + 8\\ = &\ \Delta^3 + 2\Delta^2 + 3\Delta + 22,\end{aligned}$$

and if $\Delta < \nu - 3$, then

$$\begin{aligned}\mathcal{ND}_6(T') = &\ (\Delta - 1)(\Delta(\Delta + 1) + \Delta)\\ &+(\Delta(\Delta + 1) + 2(\Delta + 2))\\ &+(2(\Delta + 2) + 8) + 22 + 16(\nu - \Delta - 4)\\ = &\ 16\nu + \Delta^3 + 2\Delta^2 - 13\Delta - 26.\end{aligned}$$

Then the proof is complete.

The following observation is immediately achieved from the definition of $\mathcal{ND}_6$ index.

Observation 6: If $\mathcal{S}$ be a graph and $\varepsilon \notin E(\mathcal{S})$, then $\mathcal{ND}_6(\mathcal{S} + \varepsilon) > \mathcal{ND}_6(\mathcal{S})$.

Application of Theorem 5 and Observation 6, we yield the next theorem.

Theorem 7: *If $\mathcal{S}$ be a graph with ν vertices and maximum degree Δ, then*

$$\mathcal{ND}_6(\mathcal{S}) \geq \begin{cases} \Delta^3 + \Delta^2, & \text{if } \Delta = \nu - 1\\ \Delta^3 + 2\Delta^2 + 3\Delta + 6, & \text{if } \Delta = \nu - 2\\ \Delta^3 + 2\Delta^2 + 3\Delta + 22, & \text{if } \Delta = \nu - 3\\ 16\nu + \Delta^3 + 2\Delta^2 - 13\Delta - 26, & \text{if } \Delta < \nu - 3.\end{cases}$$

The equality holds if and only if $\mathcal{S}$ is a starlike with at most one leg of length more than one.

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