

Secure 2-domination in graphs

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Abstract

Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a dominating set of G if every vertex in $V \setminus D$ is adjacent to at least one vertex in D . D is a 2-dominating set if every vertex in $V \setminus D$ has

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at least two neighbors in D . A secure dominating set S of a graph G is a dominating set with the property that every vertex u in $V \setminus S$ is adjacent to a vertex $v \in S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. In this paper, we define and study a new invariant of domination in graphs which is the secure 2-dominating set. A 2-dominating set S is secure 2-dominating if for every vertex u in $V \setminus S$, $\exists v \in (S \cap N(u))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a 2-dominating set. The secure 2-domination number $\gamma_s^2(G)$ is equal to the minimum cardinality of a secure 2-dominating set. First, we establish lower and upper bounds on $\gamma_s^2(G)$ of a graph and a triangle-free graph. Then we determine the secure 2-domination number $\gamma_s^2(G)$ of paths and cycles. After, we study the complexity of secure 2-domination problem by proving that the determining of the parameters value is NP-complete, also for split and bipartite graphs. Finally, we provide an upper bound of parameter in terms of clique number and independent number for perfect graphs.

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1. Introduction

In graph theory, the concept of domination was first introduced as the "coefficient of external stability" by Berge [2] in 1958. Oré in 1962 introduced it as "the domination number". An application of domination was given by Liu in his book [14] in 1968, the author discussed the concept of domination in communication networks. After, the domination has been intensively studied and several related parameters have been defined, among these parameters, Fink and Jacobson [11] introduced the concept of k -domination, of which 2-domination is a special case. Fair domination in the class of paths was studied by Alikhani and Safazadeh in [1]. The concept of secure domination was introduced by Cockayne et al. [8] and investigated in a series of papers [5, 6, 7, 9, 10, 12, 13, 16-18]. In this paper, motivated by these works, we have coupled the two concepts of secure domination and 2-domination to form the basis of this article, we define and study a new parameter of domination in graphs, which is the secure 2-dominating set. We begin, in Section 2, by introducing basic concepts and earlier results. In Section 3, first, we establish lower and upper bounds for the parameter for general graphs and also for triangle-free graphs, then we provide exact values of the parameter for paths and cycles. In Section 4, we compute computational complexity of secure 2-domination problem, while providing upper bounds of the secure 2-dominating number in terms of clique number and independent number for perfect graphs.

2. Definitions and earlier results

Let $G = (V, E)$ be a simple graph. For a vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is $N[S] =$

$N(S) \cup S$. $\delta(G)$ and $\Delta(G)$ are, respectively, the minimum and the maximum degree of the graph G . A set $S \subseteq V$ is a dominating set if $N[S] = V$. The domination number $\gamma(G)$ and the upper domination number $\Gamma(G)$ are, respectively, the minimum and maximum cardinalities of a minimal dominating set. S is a 2-dominating set if every vertex in $V \setminus S$ has at least two neighbors in S . The 2-domination number $\gamma_2(G)$ is equal to the minimum cardinality of a 2-dominating set. A secure dominating set S of a graph G is a dominating set with the property that every vertex u in $V \setminus S$ is adjacent to a vertex $v \in S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The secure domination number $\gamma_s(G)$ is equal to the minimum cardinality of a secure dominating set. A set S is an independent set if no two vertices in S are adjacent. The independent domination number $i(G)$ and the vertex independence number $\beta_0(G)$ are, respectively, the minimum and maximum cardinalities of a maximal independent set in G . The chromatic number $\chi(G)$ of a graph G is the minimal number of colors needed to color the vertices in such a way that no two adjacent vertices have the same color.

For a set of vertices $S \subseteq V$, the subgraph of G induced by S is the graph denoted by $G[S]$ having S for set of vertices. The edges of $G[S]$ are those of E whose both endpoints are in S . The complement of graph G is a graph $\overline{G}$ with the same vertex set but whose edge set consists of the edges not present in G , so only the edges are complemented.

In this work, we define a new domination invariant of graphs that we call secure 2-domination in graphs. A 2-dominating set S is secure 2-dominating if for every vertex u in $V \setminus S$, $\exists v \in (S \cap N(u))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a 2-dominating set. The secure 2-domination number $\gamma_2^s(G)$ is equal to the minimum cardinality of a secure 2-dominating set. We consider simple non-trivial graphs with order $n \geq 3$.

Before presenting our results, we present some results that have been obtained for the two parameters $\gamma_2(G)$ and $\gamma_s(G)$.

2.1 2-domination in graphs

Proposition 2.1: [4] If G is a graph with $\delta(G) \geq 2$, then $\gamma_2(G) \leq \beta_0(G)$.

Theorem 2.2: [4] For every nontrivial tree T , $\gamma_2(T) \geq i(T)$.

Theorem 2.3: [3] For every nontrivial tree T , $\gamma_2(T) \geq \beta_0(T)$, with equality if and only if T has a unique $\gamma_2(T)$ -set that also is a $\beta_0(T)$ -set.

2.2 Secure domination in graphs

Theorem 2.4: [8] For the path P_n and the cycle C_n ,

$$\gamma_s(P_n) = \gamma_s(C_n) = \left\lceil \frac{3n}{7} \right\rceil.$$

Theorem 2.5: [16] For every graph G , $\gamma_s(G) \leq \gamma(G) + \beta_0(G) - 1$.

Theorem 2.6: [16] For every connected triangle-free graph G , $\gamma_s(G) \leq \frac{3}{2}\beta_0(G)$.

Corollary 2.7: [16] For every tree T , $\gamma_s(T) \leq \beta_0(T)$.

3. Bounds and exact values for secure 2-domination in graphs

In this section, after we provide the formal definition of the parameter, we establish some bounds and its exact values.

Definition 3.1: A 2-dominating set S is secure 2-dominating (S2DS, for short) if for every vertex u in $V \setminus S$, $\exists v \in (S \cap N(u))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a 2-dominating set. The secure 2-domination number $\gamma_2^s(G)$ is equal to the minimum cardinality of a secure 2-dominating set.

We establish some properties of S2DS. For $w \in V \setminus X$, $v \in X$ is an X -internal private neighbour of w (X -ipn of w , for short) if $N(v) \cap V \setminus X = \{w\}$. Let $P(w, X)$ be the set of all X -ipn of w .

The following theorem provide the lower bound of $\gamma_2^s(G)$ in term of n , the order of the graph G , for triangle-free graph.

Theorem 3.2: If G is a triangle-free graph of order n , then

$$\gamma_2^s(G) \geq \frac{n+2}{2}.$$

Proof: Let X be a secure 2-dominating set. Let X_I be the set of vertices of $\bigcup_{w \in V \setminus X} P(w, X)$, $X_m = X \setminus X_I$, and $C = V \setminus X$. Let $|X_I| = x_I$, $|X_m| = x_m$ and $|C| = c$. Then

$$V = X_I \cup X_m \cup C \quad (\text{disjoint union}).$$

Therefore

$$\begin{aligned} n &= x_I + x_m + c, \\ c &= n - x_I - x_m. \end{aligned} \tag{1}$$

By counting $e(C, X)$, the number of edges from C to X , we obtain

$$2c \leq e(C, X) \leq x_I + \Delta x_m.$$

We use (1) to eliminate c from this inequality and get

$$\begin{aligned} x_I + \Delta x_m &\geq 2n - 2x_I - 2x_m, \\ 3x_I + (\Delta + 2)x_m &\geq 2n. \end{aligned}$$

Since G is a triangle-free graph then for each vertex $w \in V \setminus X$, $|P(w, X)| \geq 1$ and $|X_m| \geq 2$.

$$\begin{aligned} x &\geq c + 2, \\ 3x_I + (\Delta + 2)x_m &\geq 2n, \end{aligned} \tag{2}$$

$$x_I + x_m \geq c + 2. \tag{3}$$

The minimum value of $x_I + x_m$ subject to the constraints (2) and (3) and $x_I, x_m \geq 0$ is $\frac{(n+2)}{2}$. Then

$$\gamma_2^s(G) \geq \frac{n+2}{2}.$$

In Figure 1, we give an example for which $\gamma_2^s(G) = \frac{n+2}{2}$.

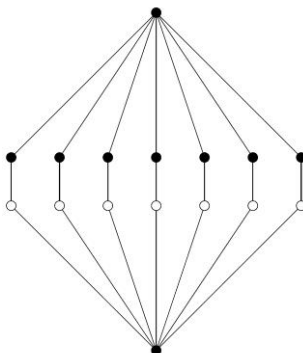


Figure 1
Minimum secure 2-dominating set in a triangle-free graph.

The second theorem provide the lower bound of $\gamma_2^s(G)$ in term of n , the order of the graph G , for any graphs.

Theorem 3.3: *If G is a graph of order n , then*

$$\gamma_2^s(G) \geq \frac{n+1}{2}.$$

Proof: Let x be a secure 2-dominating set. Let X_I be the set of vertices of $\bigcup_{w \in V \setminus X} P(w, X)$, $X_m = X \setminus X_I$, and $C = V \setminus X$. Let $|X_I| = x_I$, $|X_m| = x_m$ and $|C| = c$. Then

$$V = X_I \cup X_m \cup C \quad (\text{disjoint union}).$$

Therefore,

$$\begin{aligned} n &= x_I + x_m + c, \\ c &= n - x_I - x_m. \end{aligned} \tag{4}$$

By counting $e(C, X)$, the number of edges from C to X , we obtain

$$\begin{aligned} 2c &\leq e(C, X) \leq x_I + \Delta x_m, \\ x_I + \Delta x_m &\geq 2c. \end{aligned}$$

We use (4) to eliminate c from this inequality and get

$$\begin{aligned} x_I + \Delta x_m &\geq 2n - 2x_I - 2x_m, \\ 3x_I + (\Delta + 2)x_m &\geq 2n. \end{aligned}$$

Since G is a graph which can have a triangle, then for each vertex $w \in V \setminus X$, w can be in a triangle, then

$$\begin{aligned} x &\geq c + 1, \\ 3x_I + (\Delta + 2)x_m &\geq 2n, \end{aligned} \tag{5}$$

$$x_I + x_m \geq c + 1. \tag{6}$$

The minimum value of $x_l + x_m$ subject to the constraints (5) and (6) and $x_l, x_m \geq 0$ is $\frac{(n+1)}{2}$. Then

$$\gamma_2^s(G) \geq \frac{n+1}{2}.$$

In Figure 2, we give an example for which $\gamma_2^s(G) = \frac{n+1}{2}$.

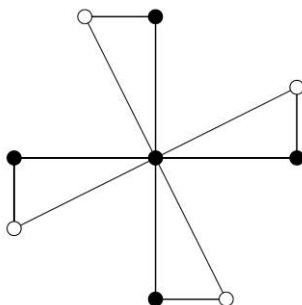


Figure 2
Minimum secure 2-dominating set.

Now, we provide exact values of $\gamma_2^s(G)$ for paths and cycles.

Proposition 3.4: For the path P_n and the cycle C_n ,

$$\gamma_2^s(P_n) = \begin{cases} 3 \left\lfloor \frac{n}{5} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod{5} \\ n - 2 \left\lfloor \frac{n}{5} \right\rfloor, & \text{if } n \equiv 1, 2, 3 \pmod{5}, \\ n - 2 \left\lfloor \frac{n}{5} \right\rfloor - 1, & n \equiv 4 \pmod{5}. \end{cases}$$

And

$$\gamma_2^s(C_n) = \begin{cases} \gamma_2^s(P_n) - 1 & \text{if } n \equiv 0, 3 \pmod{5}, \\ \gamma_2^s(P_n) & \text{if } n \equiv 1, 2, 4 \pmod{5}. \end{cases}$$

$$= \begin{cases} 3 \left\lfloor \frac{n}{5} \right\rfloor & \text{if } n \equiv 0 \pmod{5}, \\ n - 2 \left\lfloor \frac{n}{5} \right\rfloor & \text{if } n \equiv 1, 2 \pmod{5}, \\ n - 2 \left\lfloor \frac{n}{5} \right\rfloor - 1 & \text{if } n \equiv 3 \pmod{5}, \\ n - 2 \left\lfloor \frac{n}{5} \right\rfloor - 1 & n \equiv 4 \pmod{5}. \end{cases}$$

Proof: Let P_n be a path of order n and let the vertex sequence of P_n be $(u_1, u_2, \dots, u_n)$. Let $n = 5k + r$, where $k \geq 0$ and $r \in \{1, 3, 4\}$. Define $Y = \bigcup_{k=0}^{\left\lfloor \frac{n}{5} \right\rfloor - 1} \{u_{5k+1}, u_{5k+3}, u_{5k+4}\}$ and

$$Z = \begin{cases} \{u_n\}, & n \equiv 0 \pmod{5}, \\ \{u_n\}, & n \equiv 1 \pmod{5}, \\ \{u_{n-1}, u_n\}, & n \equiv 2 \pmod{5}, \\ \{u_{n-2}, u_{n-1}, u_n\}, & n \equiv 3 \pmod{5}, \\ \{u_{n-3}, u_{n-1}, u_n\}, & n \equiv 4 \pmod{5}, \end{cases}$$

Then $X = Y \cup Z$ is a secure 2-dominating set of size $3 \left\lfloor \frac{n}{5} \right\rfloor + 1$ if $n \equiv 0 \pmod{5}$, $n - 2 \left\lfloor \frac{n}{5} \right\rfloor$ if $n \equiv 1, 2, 3 \pmod{5}$, and $n - 2 \left\lfloor \frac{n}{5} \right\rfloor - 1$ if $n \equiv 4 \pmod{5}$.

Figure 3 illustrates paths according to the previous cases, depending on $n \equiv k \pmod{5}$.

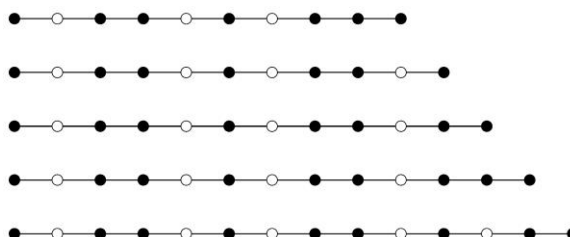


Figure 3
Minimum secure 2-dominating set on paths P_n .

Now, let S be a secure 2-dominating set. For each vertex u not in S , u must have at least two neighbors v and w in S , one of the two vertices v , w have at least another neighbor in S or v and w is adjacent in S , so u , v and w forms a triangle in S , but the path P_n is a graph with no triangle so at least one of the two vertices v , w have at least another neighbor in S . Therefore, S is the minimum secure 2-dominating set.

For the cycle C_n , we first consider the path P_n then we put an edge between the vertex u_1 and the vertex u_n . We distinguish two cases according to the value of n modulo 5.

Case 1: If $n \equiv 0, 3 \pmod{5}$.

In this case the addition of an edge on P_n generates an induced path $(u_{n-2}, u_{n-1}, u_n, u_1)$ such that the four vertices $u_{n-2}, u_{n-1}, u_n, u_1$ are in S , therefore removing u_{n-1} from S does not affect the secure 2-domination of S on G . In this case $\gamma_2^s(C_n) = \gamma_2^s(P_n) - 1$.

Case 2: If $n \equiv 1, 2, 4 \pmod{5}$.

In this case the addition of an edge on P_n generates an induced path (u_{n-1}, u_n, u_1) such that the three vertices u_{n-1}, u_n, u_1 are in S . Removing a vertex of the induced path loses the secure 2-domination of S on G so $\gamma_2^s(C_n) = \gamma_2^s(P_n)$.

After providing some bounds and exact values for γ_2^s , in the next section we will highlight the related problem complexity.

4. Computational complexity of secure 2-domination problem

In this section, we prove that secure 2-domination problem (S2DOM, for short) is an NP-complete problem for general graphs, making reduction to secure domination problem (SDOM, for short) which is known to be an NP-complete one [18].

For that we need to provide the decision version of the following related problems:

- 1) The decision problem for the secure domination problem (SDOM) of a graph is known to be NP-complete [18]:

Secure Domination Problem (SDOM)

Instance: An undirected and connected graph G and a positive integer k .

Question: Is $\gamma_s(G) \leq k$?

- 2) Formulated as a decision problem, we define the secure 2-domination problem, denoted by S2DOM, as follows:

Secure 2 –Domination Problem (S2DOM)

Instance: An undirected and connected graph G^* and a positive integer k^* .

Question: Is $\gamma_2^s(G) \leq k^*$?

The NP-Completeness of secure 2-domination problem is given by the following theorem.

Theorem 4.1: *The secure 2-domination problem is NP-complete for general graphs.*

Proof: To prove the theorem, we use a reduction to the secure domination problem (Figure 4), with the decision version given above, and known to be NP-complete [18].

Clearly, S2DOM is a member of NP, since we can check in polynomial time, not exceed $O(k^*n^*)$, whether or not a given set S^* , $|S^*| \leq k^*$ of vertices is a S2DS of G^* , with n^* is the order of G^* .

Now, let us show how a polynomial time algorithm for S2DOM could be used to solve SDOM in polynomial time.

Given a graph G of order n and a positive integer k , we construct a graph G^* by attaching to each node $u \in V(G)$ a vertex x_u (Figure 4). It is worth pointing out that $G^* = (V^*, E^*)$ such that $V^* = V \cup \{x_u, \forall u \in V\}$ and $E^* = E \cup \{ux_u, \forall u \in V\}$ can be constructed from G in polynomial time.

Next, we shall show that G has a secure dominating set SD with $|SD| \leq k$ if and only if G^* has a secure 2-dominating set $S2SD$ with $|S2SD| \leq k^* = k + n$.

Let SD be a secure dominating set of size at most k , let $S2SD = SD \cup \{x_u, \forall u \in V\}$. Accordingly, by definition of a secure dominating set, there exists,

for each vertex $v \in V(G) \setminus SD$, a vertex $u \in SD$ such that $(SD \setminus \{u\}) \cup \{v\}$ is also a dominating set for G . Thereafter $(S2SD \setminus \{u\}) \cup \{v\}$ is a 2-dominating set of G^* , as desired.

Let's consider now, a Secure 2-Dominating Set ($S2SD^*$) of G^* such that $|S2SD^*| \leq k^*$ then we have i) $\{\forall u \in V, x_u \in S2DS\}$ because for otherwise, x_u become dominated by at most one vertex which is u , contradiction since $S2SD^*$ is supposed a 2-dominating set for G^* . ii) $SSD^* = S2SD^* \cap V$ is a secure dominating set for G because if there exists a node v such that $\{\forall u \in SSD^*, (SSD^* \setminus \{u\}) \cup \{v\}\}$ is not a dominating set then $\{\forall w \in S2SD^*, (S2SD^* \setminus \{w\}) \cup \{v\}\}$ will not be 2-dominating set for G^* , contradiction, since $S2SD^*$ is an $S2SD$ for G^* .

Accordingly, $SSD^* = S2SD^* \cap V$ is a secure dominating set with:

$$SSD^* = |S2SD^*| - n \leq k^* - n = k.$$

Then a polynomial time algorithm used to solve secure domination problem on G can be used to solve secure 2-domination on G^* , and vice versa. Therefore, the secure 2-domination problem is NP-complete.

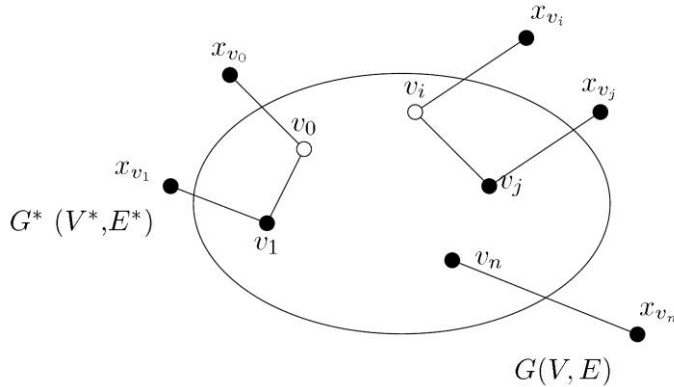


Figure 4

Reduction Secure Domination $\propto$ secure 2-domination on graphs(black vertices represent the $S2DS$ of G^*).

4.1 $S2DOM$ for bipartite graphs

The following proposition shows the NP-completeness of the secure 2-domination problem for bipartite graphs.

Proposition 4.2: The secure 2-domination problem is NP-complete for bipartite graphs.

Proof: To prove the proposition from a given bipartite graph $G = (X, Y, E)$ with independent sets X and Y , we do the same construction to general graphs by attaching a vertex x_u to each node in $u \in X \cup Y$. It is clear that the result is also bipartite graph $G^* = (X^*, Y^*, E^*)$ with partite sets $X^* = X \cup \{x_u : \forall u \in Y\}$, $Y^* = Y \cup \{x_u : \forall u \in X\}$.

Thus, we can deduce that S2DOM is also an NP-complete problem even for bipartite graph, at the same manner of previous section, since SDOM is also NP-complete for bipartite graph class [16].

The following theorem states that secure 2-domination remains an NP-complete problem even for split graphs.

Theorem 4.3: *The secure 2-domination problem is NP-complete for split graphs.*

Proof: For split graphs, we use also a reduction to the secure domination problem which is known to be NP-complete even for split graphs [16].

Consider a split graph $G = (V, E)$ with clique Q and stable I , in which we construct a split graph $G^* = (V^*, E^*)$, by attaching an additional node x to each one of V . Therefore G^* has $Q^* = Q \cup \{x\}$ as clique and $I^* = I$ as stable and $E^* = E \cup (\{x_u : \forall u \in V\})$. It is clear that G^* can be constructed from G in polynomial time.

Let's consider that G has a secure dominating set such as $|SD| \leq k$, and the aim is to show whether the graph G^* has a secure 2-dominating set of cardinality at most $k^* = k + 1$. The set $S2SD = SD \cup \{x\}$ is a S2DS for G^* as follows, for each $v \in (V \setminus SD)$, $\exists u \in SD : SD^* = (SD - v) \cup \{u\}$ is a dominating set, and therefore $SD^* \cup \{x\}$ is 2-dominating set of G^* . Thus $S2SD = SD \cup \{x\}$ is a secure 2-dominating set for G^* with $|S2SD| \leq k + 1$.

Let's consider now a secure 2-dominating set SSD^* for G^* with $|SSD^*| < k^*$, and the aim is to show whether the graph G has a secure dominating set SD such as $|SD| \leq k^* - 1 = k$. Thus $SD = SSD^* - x$ is a secure dominating set for G with $|SD| \leq k^* - 1 = k$. Because if there exists a node $v \in V(G)$ such as $\forall u \in SD, (SD - u) \cup \{v\}$, is not a dominating set, then $\forall w \in SSD^*, (SSD^* - w) \cup \{v\}$ will not be a 2-domination set, contradiction.

Therefore the secure 2-domination problem is NP-complete even for split graphs.

After, we showed that secure 2-domination problem is NP-Complete even for split and bipartite graph classes. In what follows, we provide an upper bound of secure 2-domination number γ_2^s , in terms of independence number β_0 , for *perfect graph class*, the super-class that include both split and bipartite graph classes. For that, we need the following further definitions.

Definition 4.4: A perfect graph is a graph G such that for every induced subgraph of G , the clique number equals the chromatic number, i.e., $\omega(G) = \chi(G)$.

Definition 4.5: The clique covering number $\theta(G)$ of a graph G is the minimum number of cliques in G needed to cover the vertex set of G , the clique covering number is also given by $\theta(G) = \chi(\overline{G})$, with $\overline{G}$ is the graph complement of G .

Lemma 4.6: *For each perfect graph G , we have $\theta(G) = \beta_0(G)$.*

Proof: By Definition 4.5, for each graph G we have

$$\theta(G) = \chi(\overline{G}).$$

And by Definition 4.4, for each perfect graph G we have

$$\chi(G) = \omega(G).$$

So, for each perfect graph we obtain

$$\theta(G) = \omega(\overline{G}).$$

Thus,

$$\theta(G) = \beta_0(G),$$

since $\beta_0(G) = \omega(\overline{G})$ for each graph G .

The following theorem states this upper-bound of γ_2^s , in terms of independence number β_0 , for perfect graphs.

Theorem 4.7: For each perfect graph G , we have $\gamma_2^s(G) \leq 2\beta_0(G)$.

Proof: Let's consider a perfect graph G , for each clique we need two vertices, as a secure 2-dominating set for the clique, therefore we can obtain a 2-dominating set does not exceed $2\theta(G)$. Therefore, we have $\gamma_2^s(G) \leq 2\theta(G)$. This implies that $\gamma_2^s(G) \leq 2\beta_0(G)$, since the graph is perfect.

Remark 1: Note that the bound is reached for complete graphs, for which we have, $\gamma_2^s(G) = 2\beta_0(G) = 2$.

In what follows, we reduce again the previous bounds for *proper interval graphs*. Given that, a proper interval graph is a perfect (C_4 -free, claw-free) graph.

Theorem 4.8: For each proper interval graph G , we have $\gamma_2^s(G) \leq (3\beta_0(G) + 1)/2$.

Proof: A graph $G = (V, E)$ is a proper interval graph if and only if G has a consecutive ordering [15]. Given that, $(v_1, v_2, \dots, v_n)$ is a consecutive ordering of G if and only if $i < j < k$ and $v_i v_k \in E(G)$ imply $v_i v_j \in E(G)$ and $v_j v_k \in E(G)$.

Assume that the consecutive order of the vertices of a proper interval graph G is given as follows: $(v_0, v_2, \dots, v_{n-1})$, so we construct a secure 2-dominating set $S2SD$ as follows:

It is clear that $G[v_0, \dots, v_i]$, with v_i is the maximum order among the vertices adjacent to v_0 , is a maximum clique. Also we have $G[v_i, \dots, v_j]$, V_j is the maximum order vertex among the vertices adjacent to v_i , is maximum one. The set $\{v_1, v_i, v_j\}$ is a secure 2-dominating set for $G[v_1, \dots, v_j]$. We repeat the process for the remaining graph after adding the tree nodes to $S2SD$ and deleting

¹ A graph G is an interval graph if the vertex set V can be put into one-to-one correspondence with a set of intervals I on the real line, with no interval in I properly contains another, such that two vertices are adjacent in G if and only if their corresponding intervals have nonempty intersection.

subgraph induced by the two maxim cliques in question, until v_{n-1} is reached. However, if the number of cliques is odd, we have to add two further vertices for the last one. Thus we obtain $S2SD \leq \frac{3}{2}\theta(G) + 2$ and therefore $\gamma_2^s(G) \leq \frac{3}{2}(\theta(G) + 1) = \frac{3}{2}(\beta_0(G) + 1)$, as desired.

Remark 2: The bound is not assured for all perfect graphs (see Figure 5), whenever the consecutive ordering does not exist. In Figure 5, we have $\beta_0(G) = 4$ and $\gamma_2^s(G) = 7 \geq (3\beta_0 + 1)/2$, as discussed.

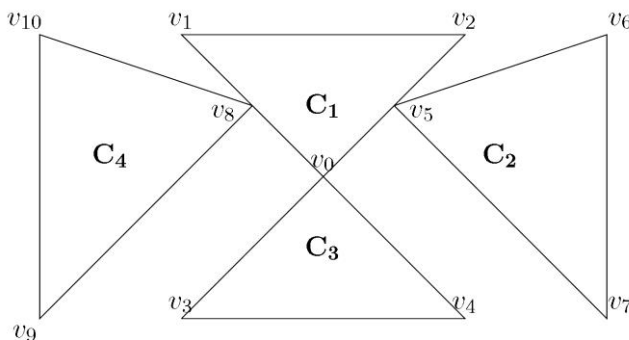


Figure 5

A perfect graph for which the bound $\gamma_2^s(G) \leq (3\beta_0(G) + 1)/2$ is not checked.

5. Conclusion

In this paper, we introduced and studied the secure 2-domination problem through the parameter's definition, the determination of exact values, bounds and computation of the related problem complexity, for the latter, we proved that corresponding problem is NP-complete for general case and even for split and bipartite graphs.

Future works in this topic may lead to generalize the parameter by introducing secure k -dominating problem. It is also interesting to characterize graphs and/or trees with equal 2-dominating and secure 2-dominating numbers in one hand and secure dominating with secure 2-dominating in the other one. Finally, it can be useful to compare the secure 2-dominating parameter by 2-secure dominating one, a newly introduced parameter [10]. This leads to carrying out a relationship that allows computing one of the parameter in function of the other, at least for certain graph classes.

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