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PBIB-designs with constant block sums arising from different graph parameters

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Abstract

Khattree introduced the concept of constant block sum partially balanced incomplete block designs (CBS-PBIBDs) in 2019. In this paper, we have tried to extend this concept to designs arising from different graph parameters. We have studied when some graph theoretical parameters such as diametral paths, maximum independent sets and geodetic sets taken as blocks yield CBS-PBIBD for some classes of graphs. We have also given the distance based association schemes for each design and found robustness of these designs using *A*-, *D*- and *E*-optimality criteria. Also we have disproved existence of a (v, β_0, μ) design arising from Shrikhande graph proved by Walikar, Acharya and Shirkol by establishing a flaw in their proof and also attempted one of the questions posed by Khattree on CBS-PBIBD.

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1. Introduction

Combinatorial design theory is a significant branch of combinatorial mathematics focused on existence, construction and characteristics of finite set systems arranged according to generalized principles of balance and symmetry. Main fields in it include balanced incomplete block designs (BIBDs) and partially balanced incomplete block designs (PBIBDs). In general, it involves studying collections of subsets with specific intersection properties. While BIBDs possess many optimal features, their high replication numbers often make them unsuitable

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for most experimental scenarios. Thus PBIBD was introduced in [1] through the concept of association scheme.

Recently, a new trend has begun in getting blocks of a particular design to have constant sum. Constant block sum designs (CBSDs) are a class of interesting designs that measure repeated experimentation where treatment levels are quantitative and all units have received the same constant total dosage. The CBS-PBIBDs were introduced by Khattree in [2]. He constructed PBIBDs using Parshvanath Yanthram, an ancient 4×4 magic square which sum to 34, inscribed on the entrance wall of an ancient temple in Khajuraho, India. In a series of papers [2-5], Khattree showed the existence and non-existence of CBS-PBIBDs. He has also given lucid examples of these to find applications in biomedical experimentation in terms of equi-spaced quantitative treatments. The goal was to reuse the same group of animals in experiments, despite the potential carry-over effects from multiple doses. While these effects cannot be entirely nullified, future studies can minimize their impact by ensuring all animals experience similar levels of carry-over effects. This can be accomplished by using a design in the initial study where the cumulative doses administered are kept constant or nearly constant. Consequently, CBS designs are valuable for their practical applications in biomedical clinical trials.

In [4], Khattree constructed more CBS-PBIBDs using methods such as pair-sums, radial and circular arrangements. He also investigated the connection between CBS-PBIBDs and mixture designs, along with optimization challenges that arise when constant block sums are not feasible for certain design classes with specified parameters. In [3], Khattree showed the non-existence of CBS-BIBDs. Recently, Bansal and Garg [6] proposed three series of CBS-PBIBDs based on regular graphs, demonstrating the existence of CBS-PBIBDs with their parameters for a specified association scheme using concentric circle reticles. Although there exist numerous CBS-PBIBDs and methods of construction, the problem of choices for v, b, r, k for the existence of fixed sum PBIBD keeps researchers working in this area [7, 8], etc. A highly sought-after optimization algorithm for identifying nearly CBS designs, including balanced incomplete blocks and other classical designs, is a key motivation for this study.

Inspired by all the above, here we approach CBS-PBIBDs through parameters of graphs, as graph theory is another major subfield of combinatorics. A subset of a graph of particular nature can be considered as a block to construct PBIBDs. Usually, a graph parameter or invariant like maximal independence number, domination number, connected domination number, biconnected domination number, vertex cover, edge cover, etc. can be used to get sets of these parameter size and build PBIBDs from them. Current research in the interplaying area of graph theory and design theory are due to Bose and Shrikhande [9], Huilgol and Vidya in [10-14], Ibrahim, Ramli and Hassan in [15], Innamorati and Faro in [16], Ionin and Shrikhande [17], Jen, Hsia and Hasan in [18], Walikar, Acharya and Shirkol in [19], to name a few.

In [3] Khattree had posed two questions. First one was to develop methods and algorithms to arrive at various PBIBDs which satisfy the property of CBS. Addressing this question, here we construct two types of constant block sum designs namely, the usual arithmetic sum and boolean sum (double boolean sum) PBIBDs that arise from important vertex subsets corresponding to fundamental graph parameters such as diameter $diam(G)$, maximum independence number β_0, geodetic number g, etc. Here we have used distance based association scheme for each of these designs. A distance based association scheme utilizes the distance between two vertices in a graph as its basis for association, that is, for a vertex, another vertex becomes an i^{th} associate if the distance from the considered vertex is i. We have also considered robustness of these designs using A-, D- and E-optimality criteria for further applications.

2. Preliminaries

We first list some important definitions pertaining to graph theory and design theory. Any undefined graph theoretical term here is used in the sense of [20, 21].

Definition 2.1: [20] The distance $d(x,y)$ between two vertices x and y in G is the shortest length of a path connecting them if any; otherwise $d(x,y) = \infty$. The shortest path between x and y is referred to as an $x - y$ geodesic.

Let G be a connected graph and x a vertex in G. The eccentricity $e(x)$ of x is the distance to farthest vertex from x. Thus $e(x) = max\{d(x,y): x \in V\}$.

Radius $rad(G)$ represents the minimum eccentricity and Diameter $diam(G)$ denotes the maximum eccentricity.

A graph G is self-centered if $rad(G) = diam(G)$.

Definition 2.2: [20] A k-regular graph on v vertices is strongly regular if there exist positive integers k, λ and μ such that every adjacent pair of vertices share λ common neighbours and every non-adjacent pair of vertices share μ common neighbours. Strongly regular graphs (SRGs) are determined by the four parameters (v, k, λ, μ).

These parametrs satisfy the relation $(v - k - 1)\mu = k(k - \lambda - 1)$. Connected strongly regular graphs have diameter 2. The complement of an SRG is an SRG with parameters $(v, v - k - 1, v - 2 - 2k + \mu, v - 2k + \lambda)$.

Definition 2.3: [14] A path joining eccentric vertices that are at distance $diam(G)$ is called a diametral path.

Definition 2.4: [20] A circulant graph $C_v(n_1, n_2, \dots, n_k)$ where $0 < n_1 < n_2 < \cdots < n_k < (v + 1)/2$ is a graph on v vertices $x_1, x_2, \dots, x_v$ with vertex x_i adjacent to each vertex $x_{i \pm n_j (mod v)}$. The values n_i are called jump sizes.

Definition 2.5: [22] A set of vertices in a graph G is called an independent set if no two vertices in the set are adjacent to each other. Such a set with maximum size is called a maximum independent set whose cardinality is known as independence number of a graph denoted as β_0.

Definition 2.6: [23] A balanced incomplete block design (BIBD) consists of v elements organized into b blocks, each containing k elements, such that each element appears in exactly r blocks and every pair of unordered elements in λ blocks. This combinatorial configuration is referred to as (v, b, r, k, λ)-design. A BIBD satisfies the following conditions.

$i)\quad vr = bk \quad ii)\quad \lambda(v-1) = r(k-1) \quad iii)\quad b \geq v.$

Definition 2.7: [1] Consider a set $V = \{1,2, \dots, v\}$ and an association scheme with m classes on V. A partially balanced incomplete block design (PBIBD) denoted as $(v, b, r, k, \lambda_1, \dots, \lambda_m)$, consists of b subsets of V called blocks, each containing k elements $(k < v)$ with the property that every element appears in r blocks and any two elements α and β that are i^{th} associates occur together in λ_i blocks, where the value of λ_i is independent of the specific pair chosen.

Definition 2.8: [2] A relationship that meets the following three criteria is referred to as a partially balanced association scheme with *m*-associate class. For a set $\{1, 2, 3, \cdots, v\}$ containing v elements, a relation that fulfills these conditions is termed an association scheme with *m* classes.

- Any two elements α and β are i^{th} associates for some i with $1 \leq i \leq m$ and this relation of being i^{th} associates is symmetric.
- The number of i^{th} associates of each element is n_i.
- If α and β are two elements which are i^{th} associates, then the number of elements which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β.

Definition 2.9: [24] If N is the incidence matrix of the design, then the concurrence matrix is NN^T, the information matrix is given by $L = rI - \frac{NN^T}{k}$ where I is the $v \times v$ identity matrix and the remaining notations have the usual meaning. The optimality of the design is calculated using the eigenvalues $e_1, e_2, \dots, e_{v-1}$ of the matrix $L^* = I - \frac{NN^T}{rk}$ using the following formulae.

A-optimality criterion is given by $A = (v-1)[\sum_{i=1}^{v-1} e_i^{-1}]^{-1}$

D-optimality criterion is given by $D = [\prod_{i=1}^{v-1} e_i]^{\frac{1}{v-1}}$

E-optimality criterion is given by $E = min\{e_1, e_2, e_3, \dots, e_{v-1}\}$

Remark 1: From the above definition, we see that A-, D-, and E- optimality conditions go to zero, whenever eigenvalue `0' is not a simple eigenvalue (that is, multiplicity is more than one).

Next we listout some theorems required to prove our results in the following sections.

Theorem 2.1: [19] *For Shrikhande graph,* (v, β_0, μ)*-design has parameters* $v = 16,\ \beta_0 = 4$ *and* $\mu = 1$.

Proposition 2.1: [19] Let G be any graph and $\{x_1, x_2, \dots, x_k\}$ be a set of independent vertices in G and let H be the graph obtained by removal of vertices in $\bigcup_{i=1}^{k} N[x_i]$ from G. If S is a maximal independent set in H, then $S \cup \{x_1, x_2, \dots, x_k\}$ is a maximal independent set in G.

In the next section we establish results on constant block arithmetic sum PBIBDs arising from different vertex subsets in a few classes of graphs.

3. Constant block arithmetic sum PBIBDs

3.1 *Petersen Graph*

Petersen graph is a distance regular, distance transitive and an SRG with diameter 2 having parameter $srg(10,3,0,1)$. It is a simple graph on ten vertices with regularity three.

Labeling 1: The vertices of Petersen graph are two-element subsets of a five-element set $X = \{1, 2, 3, 4, 5\}$ and its edges connect pairs of disjoint two-element subsets. We represent the two-element subsets $\{a, b\}$ as simply ab. For clarity in labeling, readers can refer to [22].

Lemma 3.1: *A CBS-PBIBD exists with* $v = 10$, $b = 5$, $r = 2$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$ *where the blocks represent* β_0 *sets of Petersen graph.*

Proof: The existence of a (v, β_0, μ)-design arising from β_0 sets in Petersen graph with parameters given in the statement is proved by Walikar, Acharya and Shirkol [19]. Hence in Petersen graph, each β_0 set is of size 4 and such five sets exist. With standard construction of Petersen graph as in [22], vertices get labels {12, 34, 15, 23, 45, 35, 25, 24, 14, 13}. We can easily observe that each block has a constant sum congruent to zero under modulo 3. For clarity, we consider an example of a β_0 set, {12, 15, 14, 13} which gives the modular sum 54 $\equiv$ 0(mod 3).

Clearly, this design has a 2-class association scheme. The parameters of second kind refer to the dds for each vertex, excluding the first entry, that is, $n_1 = 3$, $n_2 = 6$ and distance degree sequence (dds) as in [20],

$$P_1 = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

The spectrum of matrix L^* of the above design is $(1(5), 0.625(4), 0(1))$. Therefore we get A-, D- and E-optimality as 0.7895, 0.8115 and 0.625, respectively.

Corollary 3.1: *A CBS-PBIBD exists with* $v = 10$, $b = 5$, $r = 2$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$ *where the blocks are geodetic sets of the Petersen graph.*

Proof: It is established that in Petersen graph, every β_0 set also serves as a geodetic set, and hence we get a PBIBD with same parameter set as in Lemma 3.1, along with A-, D- and E-optimality.

Note: A constant block arithmetic sum PBIBD does not exist with blocks as diametral paths with vertices following Labeling 1 because, as each diametral path consists of three vertices, there are two diametral paths only differing in terminal vertex. If these two sets of three vertices each were to give a constant arithmetic sum, the third vertex has to be the same, which is absurd. Hence, such a constant block arithmetic sum PBIBD does not exist for Petersen graph with labels following Labeling 1.

3.2 *Square Lattice Graph* $L_2(n)$

Square lattice graph $L_2(n)$ is SRG with diameter 2. It can be obtained from complete bipartite graph $K_{n,n}$ by taking its line graph.

Labeling 2: A square lattice graph $L_2(n)$ for $n \geq 2$ is the graph which has $V = X \times X$ as its vertex set, where $X = \{1,2,\dots,n\}$. Two vertices (i,j) and (r,s) are connected by an edge in $L_2(n)$ if and only if $i = r, j \neq s$ or $i \neq r, j = s$. Thus when $X = \{1,2,\dots,n\}$ there are $n \times n = n^2$ ordered pairs which form the vertices of $L_2(n)$. Placing all vertices in an $n \times n$ array with all the vertices with identical first coordinate in the same row and vertices with identical second coordinate in the same column along with their adjacencies as mentioned above results in an $L_2(n)$. Hence there are n rows and n columns containing n^2 vertices. We label the coordinates (a,b) as simply ab.

For example, $L_2(4)$ is the product graph obtained from usual cartesian product of two complete graphs K_4 and hence the vertices follow similar labelings. Therefore, the labels of vertices in $L_2(4)$ are {11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44}.

Theorem 3.1: *A CBS-PBIBD exists with* $v = n^2$, $b = n!$, $r = (n-1)!$, $k = n$, $\lambda_1 = 0$, $\lambda_2 = (n-2)!$ *where blocks represent* β_0 *sets of an* $L_2(n)$.

Proof: A result in [19] proves the existence of (v,β_0,μ)-design arising from β_0 sets in $L_2(n)$. It is evident from the structure of $L_2(n)$ that vertices in each row and each column form a clique. Thus, each β_0 set comprises of a vertex from each row and each column, hence both coordinates of vertices will have all entries from 1 to n, thus giving constant block sum $(\frac{n(n+1)}{2},\frac{n(n+1)}{2})$.

$L_2(n)$ exhibits a 2-class distance based association scheme with parameters of second kind $n_1 = 2(n-1)$, $n_2 = (n-1)^2$, and

$$P_1 = \begin{bmatrix} n-2 & n-1 \\ n-1 & (n-1)(n-2) \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 2 & 2(n-2) \\ 2(n-2) & (n-2)^2 \end{bmatrix}.$$

Illustration: A β_0 set in $L_2(4)$ is $\{(1,2),(2,1),(3,4),(4,3)\}$ which gives the sum (10,10). A-, D- and E- optimality of the above CBS-PBIBD arising from $L_2(4)$ is 0.7693, 0.7841 and 0.6667, respectively, as calculated from Definition [2.9].

Corollary 3.2: *A CBS-PBIBD exists with* $v = n^2$, $b = n!$, $r = (n-1)!$, $k = n$, $\lambda_1 = 0$, $\lambda_2 = (n-2)!$ *where the blocks are geodetic sets of a square lattice graph* $L_2(n)$.

Proof: It is noted that β_0 sets and geodetic sets in square lattice graph $L_2(n)$ are same. Therefore, the proof is derived from the proof of the Theorem 3.1 along with A-, D- and E-optimalities.

Remark 2: There does not exist a constant block arithmetic sum PBIBD where blocks represent diametral paths of square lattice graph $L_2(n)$ following Labeling 2, as there are $n-1$ diametral paths in which initial two vertices are same and only terminal vertex differs. Hence arithmetic sum of vertices in each diametral path is not constant.

3.3 *Symmetric Generalized Petersen Graph GP(n,k)*

A generalized Petersen graph $GP(n,k)$ also denoted as $P(n,k)$ for $n \geq 3$ and $1 \leq k \leq (n-1)/2$ is a connected cubic graph made up of an inner star polygon $\{n,k\}$ and an outer cycle C_n with edges connecting corresponding vertices of the inner and outer polygons. Inner star polygons are circulants with jump size k. $GP(n,k)$ has vertex set $\{x_1, x_2, \dots, x_n\}$ for outer cycle and $\{y_1, y_2, \dots, y_n\}$ for inner polygon and edge set $\{x_i x_{i+1}, x_i y_i, \dots, y_i y_{i+k}\}$, for all i, $1 \leq i \leq n$.

There are seven symmetric generalized Petersen graphs which are, $GP(4,1) \cong Q_3$, $GP(5,2) \cong$ Petersen graph, $GP(8,3)$, $GP(10,2)$, $GP(10,3)$, $GP(12,5)$ and $GP(24,5)$.

We construct constant block arithmetic sum PBIBDs over symmetric generalized Petersen graphs $GP(n,k)$ using different subsets of its vertex set. But first we introduce a particular labeling on $GP(n,k)$ as follows:

Labeling 3: Label the outer polygon C_n of the generalized Petersen graph $GP(n,k)$ with even integers in clockwise direction and inner star polygon with odd integers in clockwise direction. Following this labeling to vertices of $GP(n,k)$, the edge set is $\{x_{2p}x_{2p\pm 2}, x_{2p}y_{2p-1}, y_{2p-1}y_{2p-1\pm 2k}\}$ for p, $1 \leq p \leq n$.

For example, in $GP(8,3)$, outer polygon C_8 vertices are labeled 2, 4, 6, 8, 10, 12, 14 and 16 in clockwise direction and vertices of inner star polygon are labeled with 1, 3, 5, 7, 9, 11, 13 and 15 along with adjacencies as described in Labeling 3.

Theorem 3.2: *A CBS-PBIBD exists with $v = 2n$, $b = 2$, $r = 1$, $k = n$, $\lambda_i = 0$ when i is odd, $\lambda_j = 1$ when j is even for all i, j, $1 \leq i, j \leq$ diam(G) where the blocks are β_0 sets of symmetric Generalized Petersen graphs $GP(4,1)$, $GP(8,3)$, $GP(10,3)$, $GP(12,5)$ and $GP(24,5)$.*

Proof: In a symmetric generalized Petersen graph $GP(n,k)$, n being even and k being odd, every alternate vertex from outer cycle and inner cycle contribute to β_0 set. Thus a β_0 set has $n/2$ number of vertices each from outer cycle and inner cycle, hence it has size n. All the remaining n vertices (remaining $n/2$ vertices each from outer and inner cycle) form another β_0 set. Therefore, there are two β_0 sets in each of the five symmetric generalized Petersen graphs of size n with repetition number r as 1.

Following Labeling 3 for vertices, a β_0 set is obtained by choosing alternate vertices from outer and inner cycles. In terms of labels, we get the independent sets as

$$\{2,6,10,\ldots, \text{ upto } n/2 \text{ terms}, 3,7,11,\ldots, \text{ upto } n/2 \text{ terms}\} \text{ and}$$

$$\{4,8,12,\ldots, \text{ upto } n/2 \text{ terms}, 1,5,9,\ldots, \text{ upto } n/2 \text{ terms}\}.$$

Each set is made up of two arithmetic progressions with common difference 4 having $n/2$ terms each. Hence taking sum of these two arithmetic progressions gives $(n^2 + n/2)$ as the sum. Taking β_0 sets as blocks gives constant block sum $(n^2 + n/2)$ for the design.

The spectrum of L^* is $(1(v-2),0(2))$. Thus, we see that A-, D- and E-optimality of the above CBS-PBIBD for $GP(n,k)$ are zeros by Remark 1.

Theorem 3.3: *A CBS-PBIBD exist in $GP(5,2)$ and $GP(10,2)$ with $v = 10$, $b = 5$, $r = 2$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$ and $v = 20$, $b = 5$, $r = 2$, $k = 8$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 = 0$, $\lambda_5 = 2$, respectively, where maximum independent sets are taken as blocks.*

Proof: Consider $GP(5,2)$ and $GP(10,2)$ with Labeling 3. Here each β_0 set has $(n-1)/2$ vertices from outer cycle with even labels and $(n-1)/2$ vertices from inner cycle having odd labels. Thus taking sum of labels of vertices in each set gives a constant sum zero under modulo 2.

For example, β_0 sets {2, 8, 3, 5} and {6, 10, 16, 20, 1, 3, 11, 13} has modular sums 18 $\equiv$ 0(mod 2) and 80 $\equiv$ 0(mod 2) in $GP(5,2)$ and $GP(10,2)$, respectively. The spectrum of L^* of designs arising from $GP(5,2)$ and $GP(10,2)$ are (1(5), 0.625(4),0(1)) and (1(15), 0.625(4), 0(1)), respectively, along with A-, D- and E- optimalities 0.7895, 0.8115 and 0.625, respectively, for $GP(5,2)$ and 0.8878, 0.9058 and 0.625, respectively, for $GP(10,2)$.

Theorem 3.4: *A CBS-PBIBD exists where the blocks are geodetic sets of $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$ and $GP(10,3)$.*

Proof: The diameter of $GP(4,1) \cong Q_3$ is 3 and a pair of eccentric vertices constitutes a geodetic set. Following Labeling 3 for the vertices, a geodetic set will have one even labeled vertex and the other odd labeled vertex. Hence, sum will always be odd giving a constant sum 1 under modulo 2, thus giving a constant block arithmetic sum PBIBD with $v = 8$, $b = 4$, $r = 1$, $k = 2$, $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 1$. For clarity, consider a geodetic set {3,8} in $GP(4,1)$ whose modular sum is 11≡1(mod 2). The spectrum of this design is (1(4), 0(4)) along with optimality zeros by Remark 1.

CBS-PBIBD arising from geodetic sets in $GP(5,2) \cong$ Petersen graph is explained in detail along with their A-, D- and E-optimality criteria in Corollary 3.1.

The diameter of $GP(10,2)$ and $GP(10,3)$ are 4 and a pair of eccentric vertices form a geodetic set. A vertex in outer cycle will have its eccentric vertex in outer cycle and similarly a vertex in inner cycle has its eccentric vertex in inner cycle. Thus we get ten geodetic sets each having size 2 amongst which five of them have vertex pairs with odd labels and the remaining with even labels. Taking sum of labels of vertices in each set gives a constant sum 0 under modulo 2, being even. Thus we get a CBS-PBIBD(arithmetic) with $v = 20$, $b = 10$, $r = 1$, $k = 2$, $\lambda_i = 0$ where $i, 1 \leq i \leq 4$ and $\lambda_5 = 1$ where the blocks are geodetic sets of $GP(10, 2)$ and $GP(10, 3)$. For example, geodetic set $\{1, 11\}$ in $GP(10,2)$ and $GP(10, 3)$ has modular sum 12 $\equiv$ 0(mod 2). The spectrum of L^* of this design is (1(10), 0(10)) with A-, D- and E-optimality zeros by Remark 1.

A geodetic set in $GP(8, 3)$ consists of two pairs of eccentric vertices. Thus the size of each such set is four and containing all four even labeled vertices or all four odd labeled vertices or two even and two odd labeled vertices. Being a unique eccentric vertex graph with diameter 4, $GP(8, 3)$ contains eight pairs of eccentric vertices. Each pair appears with the remaining seven pairs to form geodetic set of size four giving repetition number seven. Hence we get (8×7)/2 geodetic sets. Thus taking sum of labels of vertices in each of the above cases gives a CBS-PBIBD with $v = 16$, $b = 28$, $r = 7$, $k = 4$, $\lambda_i = 1$ where $i, 1 \leq i \leq 3$ and $\lambda_4 = 7$ where blocks are geodetic sets.

Example: The set {2, 10, 4, 12} is a geodetic set in $GP(8, 3)$ whose modular sum is 28 $\equiv$ 0(mod 2). The spectrum of L^* of this design is (1(8), 0.5714(7), 0(1)) with A-, D- and E- optimalities 0.7407, 0.7701 and 0.5714, respectively.

3.4 *Even Cycle* C_{2n}

Even cycle is a self-centered, unique eccentric vertex graph. It is also a distance degree regular (DDR) graph. Here, we define a labeling of even cycles that will help us in getting a CBS-PBIBD.

Labeling 4: Let $V(C_{2n}) = \{x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{2n}\}$. Then we relabel the vertices such that $x_1 = 1$, $x_2 = 3$, $x_3 = 5$, ..., $x_n = 2n - 1$, $x_{n+1} = 2n$, $x_{n+2} = 2n - 2$, ..., $x_{2n-1} = 4$, $x_{2n} = 2$. That is, the vertices on right (left) half of the cycle get even integers and the vertices on left (right) half get odd integers in ascending order.

Theorem 3.5: *A CBS-PBIBD exist with* $v = 2n$, $b = n$, $r = 1$, $k = 2$, $\lambda_i = 0$ *for all* $i, 1 \leq i \leq n - 1$, $\lambda_n = 1$ *where the blocks represent geodetic sets of an even cycle* C_{2n} *(following Labeling 4).*

Proof: Huilgol and Vidya [13] proved the existence of geodetic design arising from even cycles. A geodetic set in C_{2n} comprises of a pair of eccentric vertices. Following Labeling 4 for C_{2n}, vertices in each geodetic set is such that one vertex has odd label and the other has even label, that is, for vertex 1 its eccentric vertex has label $2n$, vertex 2 will have vertex labeled $2n-1$ as its eccentric vertex and so on. Hence taking sum of labels on vertices of each geodetic set gives a constant block sum $2n+1$, thus giving a constant block arithmetic sum PBIBD.

The spectrum of matrix L^* of the above design for even cycle C_{2n} is $(1(n), 0(n))$. Thus A-, D- and E-optimality go to zero by Remark 1.

Illustration: The geodetic sets in C_8 are $\{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$ which gives the constant sum 9.

This design exhibits a 4-class distance based association scheme for C_8 with parameters of second kind $n_1 = 2,\ n_2 = 2,\ n_3 = 2,\ n_4 = 1$ and

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ P_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$P_4 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 3: There does not exist a CBS-PBIBD having vertices of diametral paths as blocks in an even cycle C_{2n} with given Labeling 4 because when we consider a diametral path between two vertices, say labeled 1 and $2n$, there exist two diametral paths between them as $(1,2,4,\dots,2n)$ and $(1,3,5,\dots,2n-1,2n)$ whose sums of the labels are n^2+n+1 and n^2+2n, respectively.

Lemma 3.2: *There does not exist a CBS-PBIBD where the blocks are maximum independent sets in an even cycle C_{2n} with the given Labeling 4.*

Proof: We know that C_{2n} has two β_0 sets each of size n which forms its bipartition. Hence depending on the parity of n, the following two cases arise.

Case 1: Let *n* be even

We can see that each set comprises of $n/2$ vertices with even labels and $n/2$ vertices with odd labels. Each of them forms an arithmetic progression with common difference 4. One of the set has arithmetic progression of even labels starting with 2 and arithmetic progression of odd labels starting with 3. The other set has arithmetic progression of even labels starting with 4 and arithmetic progression of odd labels starting with 1. Therefore each of the set is a combination of two arithmetic progressions. The β_0 sets are

{2, 6, 10, ..., upto $n/2$ terms, 3, 7, 11, ..., upto $n/2$ terms} and

{4, 8, 12, ..., upto $n/2$ terms, 1, 5, 9, ..., upto $n/2$ terms}.

Taking sum of arithmetic progressions in each set gives sum as $(n^2 + n/2)$.

Case 2: Let *n* be odd

One of the β_0 set has $\lceil n/2 \rceil$ even labeled vertices and $\lfloor n/2 \rfloor$ odd labeled vertices and vice versa in the other β_0 set. The first set has $(n+1)/2$ even labeled vertices in arithmetic progression starting with 2 and $(n-1)/2$ odd labeled vertices starting with 3 in arithmetic progression with common difference 4. Similarly, the other set has $(n-1)/2$ even labeled vertices starting with 4 and $(n+1)/2$ odd labeled vertices starting with 1 which forms two different arithmetic progressions with common difference 4. Therefore, β_0 sets are

{2, 6, 10, ..., upto $(n+1)/2$ terms, 3, 7, 11, ..., upto $(n-1)/2$ terms} and
{4, 8, 12, ..., upto $(n-1)/2$ terms, 1, 5, 9, ..., upto $(n+1)/2$ terms}.

Taking sum of arithmetic progressions in each set gives sum as $\frac{2n^2+n+1}{2}$ for the first set and $\frac{2n^2+n-1}{2}$ for the second set. Thus sums are not constant. Hence, there does not exist a CBS-PBIBD in an even cycle C_{2n} for all n where the blocks are β_0 sets with Labeling 4.

Lemma 3.3: *There does not exist a CBS-PBIBD where the blocks are diametral paths of an odd cycle C_{2n+1} following Labeling 4.*

Proof: We know that in C_{2n+1}, each vertex has two eccentric vertices at distance n. Hence there are two distinct diametral paths with initial vertex same but terminal vertices different. Suppose we consider a vertex with label 1, then its eccentric vertices have labels $2n+1$ and $2n$. Thus the two distinct diametral paths arising from vertex labeled 1 are $(1,2,4,\ldots,2n)$ and $(1,3,5,\ldots,2n+1)$ whose sums of the labels are n^2+n+1 and n^2+2n+1 respectively. Thus, we get atleast one pair of blocks whose sums are not constant. Hence, there does not exist a CBS-PBIBD where the blocks are diametral paths of odd cycle C_{2n+1} where vertices follow Labeling 4.

Remark 4: There does not exist a CBS-PBIBD where the blocks are maximum independent sets in an odd cycle C_{2n+1} (following Labeling 4).

Proof: There are $2n+1$ β_0 sets in C_{2n+1} each of size n. Amongst these, two of them are such that $n-1$ elements are same and only one element different. Hence taking their sum gives different values.

Lemma 3.4: There does not exist a CBS-PBIBD where the blocks are geodetic sets in an odd cycle C_{2n+1} (following Labeling 4).

Proof: The non-existence of geodetic design in odd cycles have been shown in [13]. Thus the proof follows.

4. Constant block boolean sum PBIBDs

In this section, we construct constant block boolean and double boolean sum PBIBDs arising from various vertex subsets in graphs.

Labeling vertices of a graph with integers do not always give us a CBS-PBIBD though a PBIBD exists. So in this section we label vertices with n-tuple boolean vectors to define boolean sum and to obtain constant block boolean sum PBIBD.

Boolean sum, denoted by $\vee$ is same as boolean join wherein bit-wise sum is taken, that is, boolean sum of binary digits or bits is given by $0 \vee 0 = 0$, $0 \vee 1 = 1$, $1 \vee 0 = 1$, $1 \vee 1 = 1$.

For example, a boolean sum of triples 001 and 111 is 001 $\vee$ 111 = 111.

Boolean sum of labels of vertices in a block is obtained by taking boolean join of corresponding coordinates of n-tuple binary digits of different vertices in a block.

Double boolean sum is obtained by taking boolean join of the coordinates of n-tuple obtained after taking boolean sum of the vertex labels of a block as explained above.

Weight of an n-tuple boolean vector is the number of 1's in it.

4.1 *Hypercube* Q_n

First we give some definitions and labelings that help us in deriving our results.

The hypercube Q_n of dimension n is an n-regular graph on 2^n vertices and $2^{n-1}n$ edges. Its vertex set consists of all n-dimensional boolean vectors, with edges connecting vectors that differ by exactly one coordinate. For clarity, refer [22] for labeling. The distance between two vertices is the number of coordinates that differ in their labels. Note that hypercube of dimension n is a self-centered, DDR graph of diameter n with unique eccentric vertex [25]. Hence the following remark is quite evident.

Remark 5: [19] There are $n!$ distinct shortest paths between any pair of eccentric vertices in Q_n.

Next, we consider different sets in hypercube and check for their constant block boolean sum PBIBDs. First of all, we consider vertices belonging to diametral paths in a hypercube as blocks. Using the structure of Q_n we get the following result.

Lemma 4.1: *There are* $2^{n-1}n!$ *distinct diametral paths in* Q_n.

Proof: Since Q_n is a unique eccentric vertex graph and if x and y are two vertices in Q_n such that $d(x,y) = diam(Q_n)$, then there exist $n!$ distinct shortest paths between x and y and by Menger's theorem, [20], it is an n-connected graph. Hence there are $2^{n-1}n!$ diametral paths.

Theorem 4.1: *A constant block boolean sum PBIBD exists with* $v = 2^n$, $b = 2^{n-1}n!$, $r = (n+1)!/2$, $k = n+1$, $\lambda_{k'} = k'!\,(n-k'+1)!$ *for all* $k', 1 \leq k' \leq n$ *where blocks represent vertices of diametral paths of hypercube* Q_n, $n \geq 2$.

Proof: There are $2^{n-1}n!$ diametral paths in Q_n. Note that for any vertex x with binary n-tuple labeling has a vertex y as its eccentric vertex if and only if it differs in all n-entries of its label, so that its distance is n. Hence if i^{th} coordinate has 1 in any vertex, then its eccentric vertex will have 0 in the i^{th} coordinate and vice-versa for all $1 \leq i \leq n$. The vertices on a diametral path alternates between the vertices of odd parity and even parity. As transition from odd or even parity vertex to even or odd parity vertex happens, one of the coordinate changes, and when the eccentric vertex is reached, all the n coordinates differ from the initial vertex. Hence taking boolean sum of labels of vertices on each diametral path, an n-tuple containing all 1's is obtained. Thus a constant block boolean sum is obtained for all $b = 2^{n-1}n!$ diametral paths. Hence we get a CBS-PBIBD(boolean) with parameters as in the statement. Existence of this design is proved in [11].

A-, D- and E- optimality of constant block boolean sum PBIBD arising from diametral paths of hypercube Q_3 is 0.8537, 0.8553 and 0.8553, respectively. Optimalities for other values of n can be calculated from formula given in Definition 2.9.

Illustration: For a diametral path in Q_3 100-000-010-011 has the boolean sum 111.

Above design for Q_3 exhibits distance based association scheme with parameters of second kind $n_1 = 3$, $n_2 = 3$, $n_3 = 1$ with $P_1 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Remark 6: [26] The number of maximal independent sets in Q_n is asymptotically equal to $2n2^{N/4}$ where $N = 2^n$.

Theorem 4.2: *A CBS-PBIBD(boolean) exists with design* $v = 2^n$, $b = 2$, $r = 1$, $k = 2^{n-1}$, $\lambda_i = 0$, $\lambda_j = 1$ *where* i *is odd and* j *is even for all* i, j, $1 \leq i, j \leq n$, *where blocks represent maximum independent sets of size* 2^{n-1} *of* Q_n.

Proof: From the structure of Q_n, we see that all vertices of odd parity forms an independent set and all vertices of even parity including the vertex with 0 in all coordinates forms another independent set. These are the maximum sized sets possible for independent sets as Q_n is bipartite. Hence, Q_n has two β_0 sets of size 2^{n-1} each.

We now check whether β_0 sets have constant boolean sum or not.

Case 1: Let n be odd.

The odd parity β_0 set has vertex labeled with all 1's. Hence, taking boolean sum of labels of all odd parity vertices gives an n-tuple of all 1's. The even parity

β_0 set has $\binom{n}{2}$ vertices having two 1's in their labels, $\binom{n}{4}$ number of vertices with four 1's and so on. The $\binom{n}{2}$ vertices with two 1's in their labels together span all n coordinates giving an n-tuple of all 1's when their boolean sum is taken. Similarly, $\binom{n}{4}$ vertices containing four 1's in their labels together span all n coordinates giving n-tuple of all 1's as their boolean sum and so on. Taking all these even parity vertices in a set and taking boolean sum of their labels again gives n-tuple of all 1's.

Case 2: Let n be even.

The even parity β_0 set has vertex labeled with all 1's. Hence, taking boolean sum of labels of all even parity vertices gives an n-tuple containing all 1's. The odd parity β_0 set has $\binom{n}{1}$ number of vertices with a single 1 in their labels, $\binom{n}{3}$ vertices with three 1's in their labels and so on. Similar to the above case, boolean sum gives n-tuple containing all 1's.

Since Q_n is bipartite, every vertex at even distance from a vertex occurs in the same set giving $\lambda_j = 1$ for even j and hence $\lambda_i = 0$, for odd i. Hence, we get a constant block boolean sum PBIBD with parameters $(2^n,\ 2, 1,\ 2^{n-1}, 0, 1, 0, 1, \ldots)$.

The spectrum of matrix L^* of the design is $(1(v-2), 0(2))$. Therefore, A-, D- and E- optimality of CBS-PBIBDs of hypercubes Q_n are all zeros by Remark 1.

Now, we consider designs arising from geodetic sets in hypercube taken as blocks.

Theorem 4.3: *A CBS-PBIBD(boolean) exist with $v = 2^n$, $b = 2^{n-1}$, $r = 1$, $k = 2$, $\lambda_i = 0$ for all i, $1 \le i \le n-1$, $\lambda_n = 1$ where the blocks are geodetic sets of an n-dimensional hypercube Q_n.*

Proof: From Remark 5 it follows that there are $n!$ distinct shortest paths between every pair of eccentric vertices. From the structure of hypercube, it is evident that these $n!$ shortest paths together span the entire vertex set of Q_n. Thus two element set containing eccentric vertices forms a geodetic set. Hence there are 2^{n-1} geodetic sets. Clearly, from the labels of eccentric vertices of Q_n, boolean sum of labels of vertices in each geodetic set gives a constant block sum n-tuple containing all 1's.

We see that the spectrum of matrix L^* of this design is $(1(b), 0(b))$. Therefore, A-, D- and E- optimalities go to zero by Remark 1.

Illustration: The geodetic sets in Q_3 are $\{100,011\}$, $\{010,101\}$, $\{001,110\}$, $\{000,111\}$ each of whose boolean sum is 111.

4.2 *Petersen Graph*

To obtain a CBS-PBIBD where the blocks are diametral paths, we assign labels to the vertices of the Petersen graph using subsets of 4-tuple boolean vectors having odd parity vertices (that is, vertices with weights 1 and 3) along with vertices with all four coordinates 0 and 1 as shown in Figure 1.

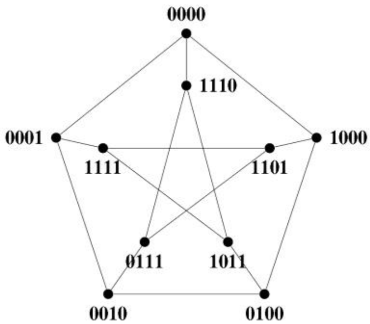

Figure 1
Petersen graph with boolean vector labels

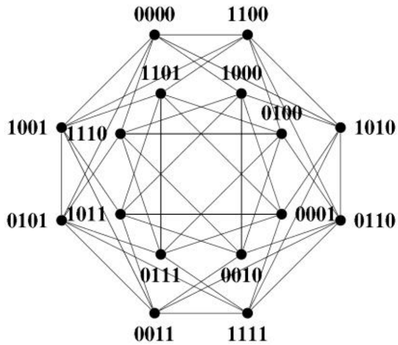

Figure 2
Shrikhande graph

Theorem 4.4: *A constant block double boolean sum PBIBD exist with* $v = 10$, $b = 30$, $r = 9$, $k = 3$, $\lambda_1 = 4$ *and* $\lambda_2 = 1$ *where the blocks represent diametral paths of Petersen graph.*

Proof: Petersen graph is an SRG with diameter 2. Huilgol and Vidya [14] have shown the existence of PBIBD where blocks are diametral paths with given parameters for the Petersen graph. Taking boolean sum of labels of vertices on each diametral path we get 1 in atleast one coordinate and hence gives double boolean sum 1.

The spectrum of matrix L^* of the above design is $(0.9259(4), 0.5925(5), 0(1))$. Therefore we get A-, D- and E-optimality as 0.7054, 0.7225 and 0.5925, respectively.

Illustration: A diametral path in the Petersen graph 0000-1000-0100 gives boolean sum 1100 and double boolean sum 1.

4.3 *Symmetric Generalized Petersen Graph GP(n,k)*

Here we label the vertices with boolean vectors to get a constant block double boolean sum PBIBD where the blocks are diametral paths of $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$ and $GP(10,3)$.

Theorem 4.5: *A constant block double boolean sum PBIBD exist where the blocks are diametral paths of* $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$ *and* $GP(10,3)$.

Proof: Constant block double boolean sum PBIBD arising from diametral paths of $GP(4,1) \cong Q_3$ is explained in detail along with their A-, D- and E-optimality criteria in Theorem 4.1 with labels as boolean vectors. Referring to Theorem 4.1 we get a $(8,24,12,4,6,4,6)$-design arising from diametral paths in Q_3 with binary triples as their labels. The 24 blocks that are diametral paths have constant block boolean sum of 1, as they comprise of triples with all 1's. We can add the labels of these sets again for double boolean sum to get 1 again. Hence we get constant double boolean sum maintained.

Constant block double boolean sum PBIBD arising from diametral paths of $GP(5,2) \cong$ Petersen graph is explained in detail along with their A-, D- and E-optimality criteria in Theorem 4.4 with labels as 4-tuples boolean vectors. Referring to Theorem 4.4 we get a $(10,30,9,3,4,1)$-design arising from vertices of diametral paths in Petersen graph with binary 4-tuple labelings.

The existence of PBIBD arising from diametral paths of $GP(8,3)$, $GP(10,2)$ and $GP(10,3)$ is shown in [11].

We label the vertices of $GP(8,3)$ with 4-tuple boolean vectors. The existence of PBIBD arising from diametral paths of $GP(8,3)$ is shown in [11] having design parameters as $v=16$, $b=48$, $r=15$, $k=5$, $\lambda_1=8$, $\lambda_2=3$, $\lambda_3=4$ and $\lambda_4=6$. Diameter of the graph is 4. Hence each diametral path is of length 4 having five vertices each. Since labels of vertices are boolean vectors, taking boolean sum of vertex labels of a diametral path will have atleast one 1 in the 4-tuple. Again taking boolean sum gives sum 1. Hence a constant block double boolean sum PBIBD is obtained with constant sum 1.

For example, diametral design arising from $GP(8,3)$ exhibits a 4-class distance based association scheme with parameters of second kind as $n_1=3$, $n_2=6$, $n_3=5$, $n_4=1$,

$$P_1=\begin{bmatrix}0&2&0&0\\2&0&4&0\\0&4&0&1\\0&0&1&0\end{bmatrix},\ P_2=\begin{bmatrix}1&0&2&0\\0&4&0&1\\2&0&3&0\\0&1&0&0\end{bmatrix},\ P_3=\begin{bmatrix}0&2&0&1\\2&0&4&0\\0&4&0&0\\1&0&0&0\end{bmatrix} \text{ and}$$

$$P_4=\begin{bmatrix}0&0&3&0\\0&6&0&0\\3&0&2&0\\0&0&0&0\end{bmatrix}.$$

The spectrum of matrix L^* of the above design is $(0.9724(4), 0.96(4), 0.7876(4), 0.64(3), 0(1))$ and corresponding A-, D- and E- optimalities are 0.8312, 0.8426 and 0.64, respectively.

We label the vertices of $GP(10,2)$ and $GP(10,3)$ with 5-tuple boolean vectors of weights 2 and 3. There are ten vertices each with weights 2 and 3. One of the labelings with 5-tuples for $GP(10,2)$ mark vertices of outer polygon with weight 2 labels and inner star polygon with weight 3 labels.

A result in [11] proves that collection of all diametral paths of $GP(10,2)$ constitutes a PBIBD characterized by parameters $v=20$, $b=60$, $r=18$, $k=6$, $\lambda_1=10$, $\lambda_2=4$, $\lambda_3=3$, $\lambda_4=4$ and $\lambda_5=6$. Each diametral path is of length 5 having six vertices each as diameter of $GP(10,2)$ is 5. Taking double boolean sum of labels of vertices of each diametral path gives sum 1. Hence a constant block double boolean sum PBIBD exist with the given parameters with sum 1. Clearly, this design gives a 5-class distance based association scheme with parameters of second kind as $n_1 = 3,\ n_2 = 6,\ n_3 = 6,\ n_4 = 3,\ n_5 = 1$,

$$P_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},\ P_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 0 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } P_5 = \begin{bmatrix} 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The spectrum of matrix L^* of the above design is (0.9946(3), 0.9722(4), 0.9167(4), 0.7778(5), 0.7461(3), 0(1)) and the corresponding A-, D- and E-optimalities are 0.8659, 0.8716 and 0.7461, respectively.

The vertices of $GP(10, 3)$ are labeled as mentioned above. The collection of all diametral paths of $GP(10, 3)$ constitutes a PBIBD characterized by $v = 20$, $b = 120$, $r = 36$, $k = 6$, $\lambda_1 = 20$, $\lambda_2 = 8$, $\lambda_3 = 6$, $\lambda_4 = 8$ and $\lambda_5 = 12$ [11]. Diameter of the graph is 5 with diametral paths having six vertices each. Taking double boolean sum of labels of vertices of each diametral path gives the sum 1. Hence a constant block double boolean sum PBIBD exists with the given parameters with sum 1. Clearly, this design gives a 5-class distance based association scheme with parameters of second kind as $n_1 = 3$, $n_2 = 6$, $n_3 = 6$, $n_4 = 3$, $n_5 = 1$,

$$P_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},\ P_3 = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } P_5 = \begin{bmatrix} 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The spectrum of matrix L^* of the above design is $(1(1), 0.9722(4), 0.9629(5), 0.7778(5), 0.7685(4), 0(1))$ and the corresponding A-, D- and E-optimality are 0.8660, 0.8716 and 0.7685, respectively.

Illustration: A diametral path in $GP(8, 3)$ 0000-1100-1010-1001-0110 gives boolean sum 1111 and double boolean sum 1.

A diametral path in $GP(10,2)$ 10100-11100-11001-10101-01010-01001 gives boolean sum 11111 and double boolean sum 1.

A diametral path in $GP(10,3)$ 11100-10110-01110-00111-11001-10011 gives the boolean sum 11111 and double boolean sum 1.

Note: The non-existence of PBIBD based on the diametral paths of $GP(12,5)$ and $GP(24,5)$ has been demonstrated by Huilgol and Vidya in [11], indicating that CBSDs cannot be formed.

4.4 *Clebsch Graph*

Clebsch graph $srg(16,5,0,2)$ is an SRG on 16 vertices and 40 edges. It can be obtained from tesseract (Q_4) by adding edges connecting the eccentric vertices.

Vertices in Clebsch graph can be labeled as binary 4-tuples where any two vertices are adjacent if and only if their labels differ in one coordinate or in all four coordinates.

Theorem 4.6: *A constant block double boolean sum PBIBD exists with* $v = 16$, $b = 160$, $r = 30$, $k = 3$, $\lambda_1 = 8$ *and* $\lambda_2 = 2$ *where blocks represent diametral paths of Clebsch graph.*

Proof: Each diametral path in Clebsch graph consists of three vertices. The labels on eccentric vertices differ either in two coordinates or in three coordinates. Thus, taking the boolean sum of labels of these vertices will have 1 in atleast one of the four coordinates. Hence we get double boolean sum as 1. A result in [14] proves that collection of all diametral paths constitute a PBIBD with given parameters as mentioned in the statement.

The spectrum of matrix L^* of diametral design of Clebsch graph is $(0.8889(5), 0.6222(10), 0(1))$. Thus we get A-, D- and E-optimality as 0.6913, 0.7008 and 0.6222, respectively.

Illustration: A diametral path 0000-0010-0110 gives boolean sum 0110 and double boolean sum 1.

Theorem 4.7: *A constant block boolean sum PBIBD exists with* $v = 16$, $b = 16$, $r = 5$, $k = 5$, $\lambda_1 = 0$, $\lambda_2 = 2$ *where blocks represent maximum independent sets of Clebsch graph.*

Proof: The existence of PBIBD with parameters as given in the statement arising from β_0 sets in Clebsch graph is proved in [19]. It follows that there are sixteen β_0 sets each of size five containing open neighbourhood of some vertex. Further, labels on adjacent vertices are such that four of them differ in exactly one coordinate and the fifth one in all four coordinates. So 1 will appear in all four coordinates when boolean sum is taken, hence giving a constant block boolean sum 4-tuple containing all $1's$.

For clarity, consider a β_0 set $\{0101, 0000, 1100, 0110, 1011\}$ whose boolean sum is 1111. Spectrum of L^* of this design is (0.96(10), 0.64(5), 0(5)) and A-, D- and E- optimalities are 0.8229, 0.8386 and 0.64, respectively.

Corollary 4.1: *A CBS-PBIBD(boolean) exists with* $v = 16$, $b = 16$, $r = 5$, $k = 5$, $\lambda_1 = 0$, $\lambda_2 = 2$ *where the blocks are geodetic sets of the Clebsch graph.*

Proof: Clebsch graph is an SRG with diameter two. As $\mu = 2$, between any pair of non-adjacent vertices there are two disjoint paths of length two. Thus shortest paths connecting a set of mutually non-adjacent vertices span the entire vertex set. Thus there exist sixteen geodetic sets each of size five giving a constant block boolean sum 4-tuple containing all $1's$. Thus, geodetic set in Clebsch graph coincides with β_0 set and hence proof follows from Theorem 4.7.

Next, let us consider complement of Clebsch graph wherein adjacencies and non-adjacencies of Clebsch graph are just reversed to get the graph.

4.5 *Complement of Clebsch graph*

Complement of Clebsch graph is an SRG on 16 vertices and 80 edges with parameters $srg(16, 10, 6, 6)$. The vertices in complement of Clebsch graph are labeled same as that of Clebsch graph with just adjacencies and non-adjacencies reversed. Thus, neighbours of a vertex are those vertices whose labels differ in 2 and 3 coordinates.

Theorem 4.8: *A constant block double boolean sum PBIBD exists with* $v = 16$, $b = 240$, $r = 45$, $k = 3$, $\lambda_1 = 6$ *and* $\lambda_2 = 6$ *where blocks represent diametral paths of complement of Clebsch graph.*

Proof: The existence of PBIBD arising from complement of Clebsch graph where blocks are diametral paths is shown in [14]. A vertex x in complement of Clebsch graph has five eccentric vertices, of which four are those whose label differ in exactly one coordinate and one whose label differ in all the four coordinates. Thus, taking boolean sum of labels of vertices on each diametral path will have 1 in atleast one of the four coordinates and hence gives double boolean sum 1. The spectrum of matrix L^* of diametral design of complement of Clebsch graph is $(0.7111(15), 0(1))$. Therefore, A-, D- and E-optimality are 0.7111.

Example: A diametral path 0000-1101-0001 has boolean sum 1101 and double boolean sum 1.

Theorem 4.9: *A constant block double boolean sum PBIBD exists with* $v = 16$, $b = 40$, $r = 5$, $k = 2$, $\lambda_1 = 0$ *and* $\lambda_2 = 1$ *where blocks constitute* β_0 *sets of complement of Clebsch graph.*

Proof: Each vertex in complement of Clebsch graph has five eccentric vertices. From the structure of complement of Clebsch graph, it is clear that eccentric vertices of a vertex form a clique. Hence they do not form part of any independent set. Thus every pair of eccentric vertices forms a β_0 set. Therefore, by counting we get 40 β_0 sets each of size 2 with $r = 5$, $\lambda_1 = 0$ and $\lambda_2 = 1$. Taking double boolean sum of vertices in each block gives a constant boolean sum 1.

The spectrum of matrix L^* of the above design for the complement of Clebsch graph is $(0.8(5), 0.4(10), 0(1))$. Therefore, A-, D- and E-optimality are 0.48, 0.504 and 0.4, respectively.

Illustration: For example, a β_0 set 0000, 1000 gives boolean sum 1000 and double boolean sum 1.

Theorem 4.10: *A constant block boolean sum PBIBD exists with v=16, b=16, r=6, k=6, $\lambda_1 = 2$ and $\lambda_2 = 2$ where blocks are geodetic sets of complement of Clebsch graph.*

Proof: Each geodetic set in complement of Clebsch graph has six vertices comprising of a vertex along with its eccentric vertices, as each vertex has five eccentric vertices. Since, $\mu = 6$, there are six geodesics between any pair of non-adjacent vertices. The set of geodesics between a vertex and all its eccentric vertices span the entire vertex set of graph. Thus, there are sixteen geodetic sets each of size six with $r = 6$, $\lambda_1 = 2$ and $\lambda_2 = 2$. Taking boolean sum of labels of these six vertices gives a CBS 4-tuples containing all 1's.

The spectrum of matrix L^* of the above design is $(0.8889(15), 0(1))$. Therefore we get A-, D- and E-optimality as 0.8889.

Example: A geodetic set 0000, 1000, 0100, 0010, 0001, 1111 gives boolean sum 1111.

Note: $L_2(n)$ is a uniquely determined SRG with parameters $srg(n^2, 2(n-1), n-2,2)$. In case when $n = 4$, we have exactly one graph with same parameters as that of $L_2(4)$ but not isomorphic to $L_2(4)$. Shrikhande called this pseudo-L_2 scheme; which is now known as the Shrikhande graph [27]. We consider the CBS-PBIBDs arising from Shrikhande graph in the next section.

4.6 *Shrikhande graph*

Shrikhande graph is an SRG on 16 vertices and 48 edges. It is a self centered graph with parameter $srg(16,6,2,2)$ same as that of $L_2(4)$, but not isomorphic to $L_2(4)$.

Labeling 5: Label the vertices of Shrikhande graph with binary 4-tuples with outer eight vertices getting even parity including the vertex containing 0 in all four coordinates. The remaining inner eight vertices get odd parity labels.

Shrikhande graph with boolean vectors as labels following Labeling 5 is shown in Figure 2.

Theorem 4.11: A constant block double boolean sum PBIBD exists with $v = 16$, $b = 144$, $r = 27$, $k = 3$, $\lambda_1 = 6$ and $\lambda_2 = 2$ where blocks are diametral paths of Shrikhande graph.

Proof: Each vertex in the Shrikhande graph has six adjacent vertices and nine non-adjacent vertices. Since Shrikhande graph is an SRG with diameter two, each diametral path consists of three vertices. Taking boolean sum of labels of vertices in each block will have atleast one 1 in one of the four coordinates, thus giving double boolean sum as 1. Huilgol and Vidya [14] proved the existence of PBIBD arising from diametral paths in Shrikhande graph with parameters as given in the statement.

The spectrum of matrix L^* of the above design is $(0.7901(9), 0.5926(6), 0(1))$. Therefore, we get A-, D- and E-optimality as 0.6972, 0.7042 and 0.5926, respectively.

Example: A diametral path 0000-1110-0100 gives boolean sum 1110 and double boolean sum 1.

Remark 7: In Theorem 2.1 taken from [19], the authors had proved the existence of a (v, β_0, μ)-design obtained from Shrikhande graph. Following their proof, if we consider two non-adjacent vertices, say 1001 and 1111 in Shrikhande graph G, then we get their neighbourhoods as

$$N(1001) = \{0000, 1100, 0101, 0011, 1101, 1011\} \text{ and}$$
$$N(1111) = \{0110, 1010, 0011, 0101, 0001, 0111\}.$$

The subgraph G_1 is induced by $V(G) - N[1001] \cup N[1111]$. Therefore from Proposition 2.1 as given in [19], {1000, 0100, 1001, 1111} is the only β_0 set that include 1001 and 1111 in G as the set {1000, 0100} is the only β_0 set in G_1.

Further, consider two non-adjacent vertices 0000 and 1111 in G. We know that,

$$N(0000) = \{1100, 1010, 1000, 1110, 0101, 1001\} \text{ and}$$
$$N(1111) = \{0110, 1010, 0011, 0101, 0001, 0111\}.$$

The subgraph G_2 is induced by $V(G) - N[0000] \cup N[1111]$. From Proposition 2.1, {1101, 0010} and {1011, 0100} are the β_0 sets in G_2. Thus we get two β_0 sets {0000, 1111, 1101, 0010} and {0000, 1111, 1011, 0100} in G containing vertices 0000 and 1111. Therefore we see that every pair of non-adjacent vertices do not occur the same number of times. Hence, the argument presented in Proposition 2.1 in [19] is not valid. Therefore, the result stands incorrect. The major inconsistency in this proof is the consideration of union of neighbourhoods equalling the whole vertex set. We can rewrite a revised statement as follows to prove the non-existence of a (v, β_0, μ)-design. This clearly disproves Proposition 2.1 given in [19].

Proposition 4.1: Let G be any graph and $\{x_1, x_2, \dots, x_k\}$ be a set of independent vertices in G and let H be the graph obtained by removal of vertices in $\bigcup_{i=1}^{k} N[x_i]$ such that $\bigcup_{i=1}^{k} N[x_i] \neq V(G)$ from G. If S is a maximal independent set in H, then S $\cup \{x_1, x_2, \dots, x_k\}$ is a maximal independent set in G.

Proof: Let H be a graph as defined in the statement. Then

$V(H) = V(G) - \bigcup_{i=1}^{k} N[x_i]$ and let S be a maximal independent set in H. Then, every vertex in $V(H) - S$ is adjacent to atleast one vertex in S. Now, let $y \in V(G) - S \cup \{x_1, x_2, \dots, x_k\}$. We know that

$V(G) - S \cup \{x_1, x_2, \dots, x_k\} = (V(H) - S) \cup \left(\bigcup_{i=1}^{k} N[x_i]\right)$. In the case when $\bigcup_{i=1}^{k} N[x_i] = V(G)$ no maximal independent set exists in H, as vertices have their neighbourhood as $V(G)$. On the otherhand, if $\bigcup_{i=1}^{k} N[x_i] \neq V(G)$, implies that $v \in V(H) - S$ or $v \in \bigcup_{i=1}^{k} N[x_i]$.

If $y \in V(H) - S$, then y is adjacent to atleast one vertex of S as S is a maximal independent set in H.

If $y \in \bigcup_{i=1}^{k} N[x_i] = V(G)$, then naturally y is adjacent to some vertex x_j in $\{x_1, x_2, \dots, x_k\}$.

In either case, y is adjacent to atleast one vertex in $V(G) - S \cup \{x_1, x_2, \dots, x_k\}$. Hence, $S \cup \{x_1, x_2, \dots, x_k\}$ is a maximal independent set in G.

From the Proposition 4.1, we get the following result.

Theorem 4.12: *There does not exist a (v, β_0, μ) design for the Shrikhande graph.*

Proof: Proof follows from Remark 7 and Proposition 4.1.

Corollary 4.2: *There does not exist a PBIBD from Shrikhande graph where blocks represent geodetic sets.*

Proof: In Shrikhande graph, β_0 sets and geodetic sets are the same. Hence proof follows from the above Theorem 4.12.

5. Conclusion

In this paper, we have found out some PBIBDs giving constant block sum and their robustness using A-, D- and E-optimality criteria. Also we disprove the existence of a (v, β_0, μ)-design arising from the Shrikhande graph as proved by [19] by establishing a flaw in their proof. We have rectified this result and added a condition where such a design exists. Addressing one of the questions posed by [3], we have given some instances of constant block arithmetic and boolean (double boolean) sum PBIBD constructions arising from different graphs. This can be further extended to many such designs using different methods keeping applications of CBS-PBIBDs in focus.

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