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Norms (T^{nor} and S^{con}) over intuitionistic fuzzy B-subalgebras

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Abstract

This study attempts to introduce intuitionistic fuzzy and intuitionistic fuzzy normal of B-algebras with illustrations with respect to norms (T^{nor} and S^{con}). Cartesian product and intersection of them are defined and their properties of them are established. Next we explore the relation between them and an upper level and lower level cuts of B-algebras. Finally the image and pre-image of them under homomorphisms of B-algebras are analysed.

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1. Introduction

Zadeh [37] introduced fuzzy set theory and the fuzzy concept. Neggers and Kim [10, 11] defined and investigated the notions of d-algebras and BCK-algebras. The concept of fuzzy B-algebras was defined and analysed by Jun et. al. [8]. The idea of intuitionistic fuzzy sets is one of the extensions of fuzzy sets and suggested by Atanassov [3]. Many researchers [4, 5, 6] applied intuitionistic fuzzy sets appear in various fields. Hohle [7], Alsina et al. [2] applied norms as intersection and union of fuzzy sets. Sripeang et al. [36] investigated algebraic structures of Q-fuzzy UP-subalgebras and Senapati et al. [35] obtained some results about intuitionistic fuzzy BG-subalgebras of BG-algebras. As using norms, the author obtained many results of fuzzy algebraic structures [12-34]. In the

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present work, with respect norms (T^{nor} and S^{con}), we define the concepts of intuitionistic fuzzy and intuitionistic fuzzy normal of B-algebras and study some of their properties. We prove that every intuitionistic fuzzy and intuitionistic fuzzy normal of B-algebras can be realized as a B-subalgebra. Finally, some basic results which have been mentioned in the abstract will be obtained.

2. Preliminaries

We recall some basic definitions and preliminary results [1, 9, 18, 19].

Definition 2.1: We say that $X \neq \phi$ with a constant 0 and a binary operation is a B-algebra if:

(a) $x \bullet x = 0$,

(b) $x \bullet 0 = x$,

(c) $(x \bullet y) \bullet z = x \bullet (z \bullet (0 \bullet y))$

that $x, y, z \in X$. We define a partial ordering $\leq$ on X as $x \leq y$ if and only if $x \bullet y = 0$. Assume $\varnothing \neq N$ be a subset of B-algebra X. If for any $m, n \in N$ we get that $m \bullet n \in N$ then we say N is B-subalgebra of X. Also we define N as normal if $(m \bullet p) \bullet (n \bullet q) \in N$ which $m \bullet n \in N$ and $p \bullet q \in N$. Note that any normal subset N of a B-algebra X is a B-subalgebra of X, but the converse need not be true. N is called a normal B-subalgebra of X if it is both a B-subalgebra and normal.

Example 2.2: Let $X = \mathbb{R}^+$ define a binary operation $\bullet$ on X by

$$x \bullet y = \tfrac{n(x-y)}{n+y}.$$

Then $(X; \bullet, 0)$ will be a B-algebra.

Example 2.3: Let $(\mathbb{Z}, +)$ be the group which $\alpha \notin \mathbb{Z}$. Let $Y = \mathbb{Z} \cup x$ such that $x + 0 = x$, $x + n = n - 1$ where $n \neq 0$ in $\mathbb{Z}$ and $\mathrm{ix} + \mathrm{x} \in Y$. Suppose $\varphi : Y \to Y$ be a map with $\varphi(x) = 1$, $\varphi(n) = n$ where $n \in \mathbb{Z}$. Suppose $\bullet$ on Y by $m \bullet p = m + \varphi(p)$, then $(Y; \bullet, 0)$ is a non-group derived B-algebra.

Lemma 2.4: *As S is a B-algebra, so $s = 0 \bullet (0 \bullet s)$ for every $s \in S$.*

Definition 2.5: The map $g : (M; \bullet, 0) \to (N; \bullet', 0')$ of B-algebras is called B-homomorphism which $g(m \bullet n) = g(m) \bullet' g(n)$, for every $m, n \in M$.

Definition 2.6: Assume $x_1, y_1, z_1 \in \mathcal{I} = [0,1]$. Then

(i) A triangular norm is a map $T^{nor} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, by $T^{nor}(x_1, 1) = x_1$, $T^{nor}(x_1, y_1) \le T^{nor}(x_1, z_1)$ if $y_1 \le z_1$, $T^{nor}(x_1, y_1) = T^{nor}(y_1, x_1)$ and $T^{nor}(x, T^{nor}(y_1, z_1)) = T^{nor}(T^{nor}(x_1, y_1), z_1)$.

(ii) A triangular conorm is a function $S^{con} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, by $S^{con}(x_1, 0) = x_1$, $S^{con}(x_1, y_1) \le S^{con}(x_1, z_1)$ if $y_1 \le z_1$, $S^{con}(x_1, y_1) = S^{con}(y_1, x_1)$ and $S^{con}(x_1, S^{nor}(y_1, z_1)) = S^{con}(S^{nor}(x_1, y_1), z_1)$.

If for all $x_1 \in \mathcal{I}$ we get that $T^{nor}(x_1, x_1) = x_1$ and $S^{con}(x_1, x_1) = x_1$ then T^{nor} and S^{con} will be idempotent.

Definition 2.7: The functions T_n^{nor}, $S_n^{con} : \Pi_{i=1} \mathcal{I} \to \mathcal{I}$ are defined by

$$T_n^{nor}(x_1, x_2, \dots, x_n) = T^{nor}(x_i, T_{n-1}^{nor}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n))$$

and

$$S_n^{con}(x_1, x_2, \dots, x_n) = S^{con}(x_i, S_{n-1}^{con}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n))$$

for all $1 \le i \le n$, where $n \ge 2$ such that $T_2^{nor} = T^{nor}$, $S_2^{con} = S^{con}$ and T_1^{nor}, $S_1^{con} = id$ (identity).

Lemma 2.8: As $e_i, f_i \in \mathcal{I}$ whith $1 \le i \le n$ and $n \ge 2$ so

$$T_n^{nor}(T^{nor}(e_1, f_1), T^{nor}(e_2, f_2), \dots, T^{nor}(e_n, f_n)) = T^{nor}(T_n^{nor}(e_1, e_2, \dots, e_n), T_n^{nor}(f_1, f_2, \dots, f_n))$$

and

$$S_n^{con}(S^{con}(e_1, f_1), S^{con}(e_2, f_2), \dots, S^{con}(e_n, f_n)) = S^{con}(S_n^{con}(e_1, e_2, \dots, e_n), S_n^{con}(f_1, f_2, \dots, f_n)).$$

Lemma 2.9: *Assume* $p_1, p_2, \dots, p_n \in \mathcal{I}$, *where* $n \ge 2$, *then*

$$\begin{aligned} T_n^{nor}(p_1, p_2, \dots, p_n) &= T^{nor}(\dots T^{nor}(T^{nor}(T^{nor}(p_1, p_2), p_3), p_4), p_n) \\ &= T^{nor}(p_1, T^{nor}(p_2, T^{nor}(p_3, \dots T^{nor}(p_{n-1}, p_n) \dots))) \end{aligned}$$

and

$$\begin{aligned} S_n^{con}(p_1, p_2, \dots, p_n) &= S^{con}(\dots S^{con}(S^{con}(S^{con}(p_1, p_2), p_3), p_4), p_n) \\ &= S^{con}(p_1, S^{con}(p_2, S^{con}(p_3, \dots S^{con}(p_{n-1}, p_n) \dots))) \end{aligned}$$

Lemma 2.10: *Let* $q,d,w,z \in \mathcal{I}$ *we have that*

$$T^{nor}(T^{nor}(q,d),T^{nor}(w,z)) = T^{nor}(T^{nor}(q,w),T^{nor}(d,z))$$

and

$$S^{con}(S^{con}(q,d),S^{con}(w,z)) = S^{con}(S^{con}(q,w),S^{con}(d,z)),$$

Definition 2.11: Define $\nu : X \to \mathcal{I}$ is a fuzzy subset of an arbitrary set X. Assume $s,t \in \mathcal{I}$ the sets $\nu_s = \{x \in X : \nu(x) \geq s\}$ and $\nu_t = \{z \in X : \nu(z) \leq t\}$ are called an upper level and lower level of ν, respectively.

Definition 2.12: Assume $Z \neq \varnothing$ be a set. We say that a complex mapping $C = (\mu_C, \nu_C) : Z \to \mathcal{I}^2$ is an intuitionistic fuzzy set (in short, IFS) in Z if $\mu_C + \nu_C \leq 1$ that the mappings $\mu_C : X \to \mathcal{I}$ and $\nu_C : X \to \mathcal{I}$ will be the degree of membership (namely $\mu_C(z)$) and the degree of non-membership (namely $\nu_C(z)$) for every $z \in Z$ to A, respectively. Also we define $\varnothing_X$ and U_X as the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in Z such that $\varnothing_Z(z) = (0,1) \sim 0$ and $U_Z(z) = (1,0) \sim 1$, respectively. The set of all $IFSs$ in Z will be denoted by $IFS(Z)$.

Definition 2.13: Suppose $\omega : X \to Y$ be a function of sets with $M = (\mu_M, \nu_M) \in IFS(X)$ and $Z = (\mu_Z, \nu_Z) \in IFS(Y)$. Define $\omega(M)(y) = (\omega(\mu_M)(y), \omega(\nu_M)(y))$ as

$$\omega(\mu_M)(y_2) = \sup\{\mu_M(x_2) \mid x_2 \in X, \omega(x_2) = y_2\}$$

and

$$\omega(\nu_M)(y_2) = \inf\{\nu_M(x_2) \mid x_2 \in X, \omega(x_2) = y_2\}.$$

Also $\omega^{-1}(Z)(x_2) = (\omega^{-1}(\mu_Z)(x_2), \omega^{-1}(\nu_Z)(x_2)) = (\mu_Z(\omega(x_2)), \nu_Z(\omega(x_2)))$ such that $x_2 \in X$, $y_2 \in Y$.

Definition 2.14: Assume $H = (\mu_H, \nu_H)$ and $Q = (\mu_Q, \nu_Q)$ such that $H, Q \in IFS(X)$. We define the intersection of H and Q as $H \cap Q = (\mu_{H\cap Q}, \nu_{H\cap Q}) : X \to \mathcal{I}$ which $\mu_{H\cap Q}(x) = T^{nor}(\mu_H(x), \mu_Q(x))$ and $\nu_{H\cap Q}(x) = S^{con}(\nu_H(x), \nu_Q(x))$ for each $x \in X$.

Definition 2.15: The cartesian product of $M = (\mu_M, \nu_M) \in IFS(X)$ and $N = (\mu_N, \nu_N) \in IFS(Y)$ is defined with $M \times N : X \times Y \to \mathcal{I}$ which

$$\begin{aligned}(M\times N)(a_2,b_2) &= ((\mu_M,\nu_M)\times(\mu_N,\nu_N))(a_2,b_2)\\ &= (\mu_{M\times N},\nu_{M\times N})(a_2,b_2)\\ &= (\mu_{M\times N}(a_2,b_2),\nu_{M\times N}(a_2,b_2))\\ &= (T^{nor}(\mu_M(a_2),\mu_N(b_2)),S^{con}(\nu_M(a_2),\nu_N(b_2)))\end{aligned}$$

for every $(a_2,b_2)\in X\times Y$.

3. Main Results

Definition 3.1: Assume $Z=(\mu_Z,\nu_Z)\in IFS(E)$ that E is a B-algebra. We say Z as an intuitionistic fuzzy B-algebra of E with respect to norms if it:

(1) $\mu_Z(a_3\bullet b_3)\geq T^{nor}(\mu_Z(a_3),\mu_Z(b_3))$,
(2) $\nu_Z(a_3\bullet b_3)\leq S^{con}(\nu_Z(a_3),\nu_Z(b_3))$,
for each $a_3,b_3\in E$.

The set of all intuitionistic fuzzy of B-algebra E with respect norms will be denoted with $(T^{nor},S^{con})IFB(E)$,

Example 3.2: Assume $X=\{0,-1,-2\}$ be a set as:

$\bullet$	0	–1	–2
0	0	–2	–1
–1	–1	0	–2
–2	–2	–1	0

so $(X,\bullet,0)$ will be a B-algebra. Suppose $\mu_A: X\to\mathcal{I}$ which

$$\mu_A(x)=\begin{cases}0.35 & \text{such } x=0,\\ 0.25 & \text{such } x=-1,\\ 0.15 & \text{such } x=-2,\end{cases}$$

and $\nu_A: X\to\mathcal{I}$ as

$$\nu_A(x)=\begin{cases}0.1 & \text{such } x=0,\\ 0.55 & \text{such } x=-1,\\ 0.75 & \text{such } x=-2.\end{cases}$$

If $T^{nor}(e_4,\ f_4)=T_p^{nor}(e_4,\ f_4)=e_4f_4$ and $S^{con}(e_4,\ f_4)=S_p^{con}(e_4,\ f_4)=e_4+f_4-e_4f_4$ which $e_4,f_4\in\mathcal{I}$ then $A=(\mu_A,\ \nu_A)\in(T^{nor},\ S^{con})\ IFB(X)$.

The following result is easy to prove.

Proposition 3.3: Suppose $R=(\mu_R,\nu_R)\in(T^{nor},S^{con})IFB(X)$ and T^{nor},S^{con} be idempotent. Then

(1) $R(0)\supseteq R(x_5)$,

(2) $R(0\bullet x_5)=R(x_5)$,

(3) $R(x_5\bullet(0\bullet y_5))\supseteq(T^{nor}(\mu_R(x_5),\mu_R(y_5)),S^{con}(\nu_R(x_5),\nu_R(y_5)))$,

for every $x_5,y_5\in X$.

Proposition 3.4: If $A=(\mu_A,\nu_A)\in IFS(X)$ satisfies (2) and (3) in Proposition 3.3, so $A=(\mu_A,\nu_A)\in(T^{nor},S^{con})IFB(X)$.

Proof: Suppose $x_6,y_6\in X$. Thus

$$\begin{aligned}\mu_A(x_6\bullet y_6)=\mu_A(x_6\bullet(0\bullet(0\bullet y_6)))&\ge T^{nor}(\mu_A(x_6),\mu_A(0\bullet(0\bullet y_6)))\\&=T^{nor}(\mu_A(x_6),\mu_A(0\bullet y_6))\\&=T^{nor}(\mu_A(x_6),\mu_A(y_6))\end{aligned}$$

so $\mu_A(x_6\bullet y_6)\ge T^{nor}(\mu_A(x_6),\mu_A(y_6))$. Also

$$\begin{aligned}\nu_A(x_6\bullet y_6)=\nu_A(x_6\bullet(0\bullet(0\bullet y_6)))&\le S^{con}(\nu_A(x_6),\nu_A(0\bullet(0\bullet y_6)))\\&=S^{con}(\nu_A(x_6),\nu_A(0\bullet y_6))=S^{con}(\nu_A(x_6),\nu_A(y_6))\end{aligned}$$

then $\nu_A(x_6\bullet y_6)\le S^{con}(\nu_A(x_6),\ \nu_A(y_6))$. Therefore $A=(\mu_A,\ \nu_A)\in(T^{nor},\ S^{con})\ IFB(X)$.

Proposition 3.5: Assume $W=(\mu_W,\nu_W)\in IFS(X)$ which T^{nor},S^{con} be idempotent. If $W=(\mu_W,\nu_W)\in(T^{nor},S^{con})\ IFB(X)$, then

(1) $A_{p,q}=\{x\in X:W(x)\supseteq(p,q)\}\ne\varnothing$ will be a subalgebra of B-algebra X for each $p,q\in[0,1]$.

(2) $K=\{x\in X:A(x)=A(0)\}$ will be a subalgebra of X.

Example 3.6: Let $Y=\{0,10,20,30,40,50\}$ be a set which:

$\bullet$	0	10	20	30	40	50
0	0	20	10	30	40	50
10	10	0	20	40	50	30
20	20	10	0	50	30	40
30	30	40	50	0	20	10
40	40	50	30	10	0	20
50	50	30	40	20	10	0

Then $(Y, \bullet, 0)$ is a B-algebra. Define

$$\mu_A : Y \to \mathcal{I}$$

as

$$\mu_A(x) = \begin{cases} 0.45 & \text{by } y = 0, \\ 0.4 & \text{by } y = 10, \\ 0.35 & \text{by } y = 20, \\ 0.45 & \text{by } y = 30, \\ 0.25 & \text{by } y = 40, \\ 0.2 & \text{by } y = 50, \end{cases}$$

and

$$\nu_A : Y \to \mathcal{I}$$

as

$$\nu_A(x) = \begin{cases} 0.15 & \text{as } y = 0, \\ 0.2 & \text{as } y = 10, \\ 0.25 & \text{as } y = 20, \\ 0.45 & \text{as } y = 30, \\ 0.35 & \text{as } y = 40, \\ 0.4 & \text{as } y = 50. \end{cases}$$

Let $T^{nor}(g_6, h_6) = T_p^{nor}(g_6, h_6) = g_6 h_6$ and $S^{con}(g_6, h_6) = S_p^{con}(g_6, h_6) = g_6 + h_6 - g_6 h_6$ with $g_6, h_6 \in \mathcal{I}$ then $A = (\mu_A, \nu_A) \in (T^{nor}, S^{con})$ $IFB(Y)$.

Let $s, t = 0.4$ then

$$A_{0.4,0.4} = \{y \in Y : A(y) \supseteq (0.4, 0.4)\} = \{0, 10, 20\}$$

as

$\bullet$	0	10	20
0	0	20	10
10	10	0	20
20	20	10	0

thus $A_{0.4,0.4}$ is subalgebra Y.

There is no need to prove the following proposition.

Proposition 3.7:

(1) Let $M=(\mu_M,\nu_M)\in(T^{nor}, S^{con})$ $IFB(F)$ and $N=(\mu_N,\nu_N)\in(T^{nor}, S^{con})$ $IFB(F)$. Then $M\cap N=(\mu_{M\cap N},\nu_{M\cap N})\in(T^{nor}, S^{con})$ $IFB(F)$.

(2) Let $M=(\mu_M,\nu_M)\in(T^{nor}, S^{con})$ $IFB(F)$ and $N=(\mu_N,\nu_N)\in(T^{nor}, S^{con})$ $IFB(E)$. Then $M\times N\in(T^{nor}, S^{con})$ $IFB(F\times E)$.

Example 3.8: Suppose $Y=\{0,11,12,13\}$ with:

$\bullet$	0	11	12	13
0	0	0	0	0
11	11	0	0	0
12	12	0	0	11
13	13	0	0	0

Then $(Y,*,0)$ is a B-algebra. Define

$$\mu_A : X\to\mathcal{I}$$

as

$$\mu_A(y)=\begin{cases}0.55 & \text{as } y=0,\\ 0.45 & \text{as } y=11,\\ 0.35 & \text{as } y=12,\\ 0.25 & \text{as } y=13,\end{cases}$$

and

$$\nu_A : X\to\mathcal{I}$$

as

$$\nu_A(y)=\begin{cases}0.15 & \text{with } y=0,\\ 0.25 & \text{with } y=11,\\ 0.35 & \text{with } y=12,\\ 0.45 & \text{with } y=13,\end{cases}$$

Let $T^{nor}(a_7,b_7)=T_m^{nor}(a_7,b_7)=\min\{a_7,b_7\}$ and $S^{con}(a_7,b_7)=S_m^{con}(a_7,b_7)=\max\{a_7,b_7\}$ for all $a_7,b_7\in\mathcal{I}$. Then $A=(\mu_A,\nu_A)\in(T^{nor},S^{con})$ $IFB(Y)$. Also Define

$$\mu_B : Y\to\mathcal{I}$$

as

$$\mu_B(m) = \begin{cases} 0.65 & \text{which } m = 0, \\ 0.55 & \text{which } m = 11, \\ 0.45 & \text{which } m = 12, \\ 0.35 & \text{which } m = 13, \end{cases}$$

and

$$\nu_B : Y \to \mathcal{I}$$

as

$$\nu_B(m) = \begin{cases} 0.25 & \text{as } m = 0, \\ 0.35 & \text{as } m = 11, \\ 0.45 & \text{as } m = 12, \\ 0.55 & \text{as } m = 13. \end{cases}$$

Let $T^{nor}(a,b) = T_m^{nor}(a,b) = \min\{a,b\}$ and $S^{con}(a,b) = S_m^{con}(a,b) = \max\{a,b\}$ for all $a,b \in \mathcal{I}$. Then $B = (\mu_B, \nu_B) \in (T^{nor}, S^{con})\ IFB(Y)$. Now we define

$$\mu_{A\cap B} : Y \to \mathcal{I}$$

as

$$(\mu_{A\cap B})(d) = T^{nor}(\mu_A(d), \mu_B(d)) = T_m^{nor}(\mu_A(d), \mu_B(d))$$
$$= \min\{\mu_A(d), \mu_B(d)\} = \begin{cases} 0.55 & \text{by } d = 0, \\ 0.45 & \text{by } d = 11, \\ 0.35 & \text{by } d = 12, \\ 0.35 & \text{by } d = 13, \end{cases}$$

and

$$(\nu_{A\cap B})(d) = S^{con}(\nu_A(d), \nu_B(d)) = S_m^{con}(\nu_A(d), \nu_B(d))$$
$$= \max\{\nu_A(d), \nu_B(d)\} = \begin{cases} 0.15 & \text{as } d = 0, \\ 0.25 & \text{as } d = 11, \\ 0.35 & \text{as } d = 12, \\ 0.45 & \text{as } d = 13. \end{cases}$$

Thus $A \cap B \in (T^{nor}, S^{con}) IFB(Y)$.

Corollary 3.9: *Let* $A_i = (\mu_{A_i}, \nu_{A_i}) \subseteq (T^{nor}, S^{con})$ $IFB(X)$ *which* $i = 1, 2, 3, 4, \ldots, n$. *Therefore* $\cap_{i=1,2,3,\ldots,n} A_i \in (T^{nor}, S^{con})$ $IFB(X)$.

Definition 3.10: Suppose $\varnothing \neq M \subseteq X$ and $A = (\mu_A, \nu_A) \in IFS(X)$. Define the characteristic function of M with $A_M = (\mu_{A_M}, \nu_{A_M}) : X \to \mathcal{I}$ as

$$A_M(x) = (\mu_{A_M}(x), \nu_{A_M}(x)) = \begin{cases} (1,0), & \text{if } x \in M \\ (0,1), & \text{if } x \notin M. \end{cases}$$

Proposition 3.11: Assume T^{nor}, S^{con} be idempotent. Therefore M will be a B-subalgebra of X if and only if $A_M = (\mu_{A_M}, \nu_{A_M}) \in (T^{nor}, S^{con})$ $IFB(X)$.

Corollary 3.12: *Let* $A_i = (\mu_{A_i}, \nu_{A_i}) \in (T^{nor}, S^{con}) IFB(Z_i)$ *where* $1 \le i \le n$, *then*

$$A = (\mu_A, \nu_A) = \prod_{i=1}^{n} A_i \in (T^{nor}, S^{con}) IFB(\prod_{i=1}^{n} Z_i = Z)$$

such that

$$\mu_A(z) = (\prod_{i=1}^{n} \mu_{A_i}(z_1, z_2, \ldots, z_n) = T_n^{nor}(\mu_{A_1}(z_1, \mu_{A_2}(z_2, \ldots, \mu_{A_n}(z_n))$$

and

$$\nu_A(z) = (\prod_{i=1}^{n} \nu_{A_i}(z_1, z_2, \ldots, z_n) = S_n^{con}(\nu_{A_1}(z_1, \nu_{A_2}(z_2, \ldots, \nu_{A_n}(z_n))$$

for each $z = (z_1, z_2, \ldots, z_n) \in Z$.

Proposition 3.13: Assume $\rho : (D; \bullet, 0) \to (E; \bullet', 0')$ be a B-homomorphism of $\mathbb{B}$-algebras and $A = (\mu_A, \nu_A) \in (T^{nor}, S^{con})$ $IFB(D)$. Then $\rho(A) \in (T^{nor}, S^{con})$ $IFB(E)$.

Proof: Let $d_i \in D$ and $e_i \in E$ with $\rho(d_i) = e_i$ and $i = 1, 2$. Then

$$\begin{aligned} \rho(\mu_A)(e_1 \bullet' e_2) &= \sup\{\mu_A(d_1 \bullet d_2) \mid d_1 \bullet d_2 \in D, \rho(d_1 \bullet d_2) = e_1 \bullet' e_2\} \\ &\ge \sup\{T^{nor}(\mu_A(d_1), \mu_A(d_2)) \mid d_i \in D, \rho(d_i) = e_i\} \\ &= T^{nor}(\sup\{\mu_A(d_1) \mid d_1 \in D, \rho(d_1) = e_1\}, \sup\{\mu_A(d_2) \mid d_2 \in D, \rho(d_2) = e_2\}) \\ &= T^{nor}(\rho(\mu_A)(e_1), \rho(\mu_A)(e_2)) \end{aligned}$$

thus

$$\rho(\mu_A)(e_1 \bullet' e_2) \ge T^{nor}(\rho(\mu_A)(e_1), \rho(\mu_A)(e_2)).$$

Also

$$\begin{aligned}\rho(\nu_A)(e_1 \bullet' e_2) &= \inf\{\nu_A(d_1 \bullet d_2) \mid d_1 \bullet d_2 \in D, \rho(d_1 \bullet d_2) = e_1 \bullet' e_2\} \\ &\le \inf\{S^{con}(\nu_A(d_1), \nu_A(d_2)) \mid d_i \in D, \rho(d_i) = e_i\} \\ &= S^{con}(\inf\{\nu_A(d_1) \mid d_1 \in D, \rho(d_1) = e_1\}, \inf\{\nu_A(d_2) \mid d_2 \in D, \rho(d_2) = e_2\}) \\ &= S^{con}(\rho(\nu_A)(e_1), \rho(\nu_A)(e_2))\end{aligned}$$

so

$$\rho(\nu_A)(e_1 \bullet' e_2) \le S^{con}(\rho(\nu_A)(e_1), \rho(\nu_A)(e_2)).$$

Therefore $\rho(A) \in (T^{nor}, S^{con})IFB(E)$.

Proposition 3.14: If $L = (\mu_L, \nu_L) \in (T^{nor}, S^{con})$ $IFB(P)$ and $\varphi : (Q;\bullet,0) \to (P;\bullet',0')$ be a $\mathbb{B}$-homomorphism of $\mathbb{B}$-algebras, then $\varphi^{-1}(L) \in (T^{nor}, S^{con})$ $IFB(Q)$.

Proof: Let $q_1, q_2 \in Q$.

$$\begin{aligned}\varphi^{-1}(\mu_L)(q_1 \bullet q_2)) &= \mu_L(\varphi(q_1 \bullet q_2)) \\ &= \mu_L(\varphi(q_1) \bullet \varphi(q_2)) \\ &\ge T^{nor}(\mu_L(\varphi(q_1), \mu_L(\varphi(q_2)) \\ &= T^{nor}(\varphi^{-1}(\mu_L)(q_1)), \varphi^{-1}(\mu_L)(q_2))\end{aligned}$$

then

$$\varphi^{-1}(\mu_L)(q_1 \bullet q_2)) \ge T^{nor}(\varphi^{-1}(\mu_L)(q_1)), \varphi^{-1}(\mu_L)(q_2)).$$

As

$$\begin{aligned}\varphi^{-1}(\nu_L)(q_1 \bullet q_2)) &= \nu_L(\varphi(q_1 \bullet q_2)) \\ &= \nu_L(\varphi(q_1) \bullet \varphi(q_2)) \\ &\le S^{con}(\nu_L(\varphi(q_1), \nu_L(\varphi(q_2)) \\ &= S^{con}(\varphi^{-1}(\nu_L)(q_1)), \varphi^{-1}(\nu_L)(q_2))\end{aligned}$$

so

$$\varphi^{-1}(_L)(q_1 \bullet q_2)) \le S^{con}(\varphi^{-1}(\nu_L)(q_1)), \varphi^{-1}(\nu_L)(q_2)).$$

Then $\varphi^{-1}(L) \in (T^{nor}, S^{con})IFB(Q)$.

Definition 3.15: Let $K = (\mu_K, \nu_K) \in IFS(F)$ and F be a B-algebra. Define K as an intuitionistic fuzzy normal B-algebra of F with respect norms if

$$\mu_K((h \bullet a) \bullet (y \bullet b)) \ge T^{nor}(\mu_K(h \bullet y), \mu_K(a \bullet b))$$

and

$$\nu_K((h\bullet a)\bullet(y\bullet b)) \le S^{con}(\nu_K(h\bullet y), \nu_K(a\bullet b))$$

for all $h, y, a, b \in F$.

Denote by $(T^{nor}, S^{con})IFNB(F)$, the set of all intuitionistic fuzzy normal B-algebra of F under norms (t-norm T^{nor} and s-norm S^{con}).

$\bullet$	0	100	200	300
0	0	300	200	100
100	100	0	300	200
200	200	100	0	300
300	300	200	100	0

Example 3.16: Let $W = \{0, 100, 200, 300\}$ be a set which:

Then $(W, \bullet, 0)$ is a B-algebra. Define $\mu_Z : W \to \mathcal{I}$ as

$$\mu_Z(w) = \begin{cases} 0.25 & \text{by } w = 0, 100, \\ 0.45 & \text{by } w = 200, 300. \end{cases}$$

Also define $\nu_Z : W \to \mathcal{I}$ as

$$\nu_Z(w) = \begin{cases} 0.15 & \text{by } w = 0, \\ 0.2 & \text{by } w = 100, \\ 0.3 & \text{by } w = 200, \\ 0.4 & \text{by } w = 300. \end{cases}$$

Assume

$$T^{nor}(a_{10}, b_{10}) = T_b^{nor}(a_{10}, b_{10}) = \max\{0, a_{10} + b_{10} - 1\}$$

and

$$S^{con}(a_{10}, b_{10}) = S_b^{con}(a_{10}, b_{10}) = \min\{1, a_{10} + b_{10}\}$$

for each $a_{10}, b_{10} \in \mathcal{I}$ so

$$Z = (\mu_Z, \nu_Z) \in (T^{nor}, S^{con})IFNB(W).$$

There is no need to prove the following proposition that proves that every (T^{nor}, S^{con}) $IFNB(W)$ will be fuzzy (T^{nor}, S^{con}) $IFB(W)$.

Proposition 3.17: As $P = (\mu_P, \nu_P) \in (T^{nor}, S^{con})$ $IFNB(Q)$ so $P = (\mu_P, \nu_P) \in (T^{nor}, S^{con})$ $IFB(Q)$.

Remark 3.18: We prove that the converse of Proposition 3.17 is not true. Assume $L=\{0,11,22,33,44,55\}$ be a set such that:

$\bullet$	0	11	22	33	44	55
0	0	22	11	33	44	55
11	11	0	22	44	55	33
22	22	11	0	55	33	44
33	33	44	55	0	22	11
44	44	55	33	11	0	22
55	55	33	44	22	11	0

Then $(L,\bullet,0)$ is a B-algebra. Define $\mu_A : L \to \mathcal{I}$ as

$$\mu_A(l)=\begin{cases}0.7 & \text{if } l=0,33\\ 0.1 & \text{if } l=11,22,44,55\end{cases}$$

and $\nu_A : L \to \mathcal{I}$ as

$$\nu_A(l)=\begin{cases}0.1 & \text{if } l=0,33\\ 0.7 & \text{if } l=11,22,44,55.\end{cases}$$

Suppose $T^{nor}\,(a_{11},b_{11}) = T_m^{nor}\,(a_{11},b_{11}) = \min\,\{a_{11},b_{11}\}$ and $S^{con}\,(a_{11},b_{11}) = S_m^{con}\,(a_{11},b_{11}) = \max\,\{a_{11},b_{11}\}$ for every $a_{11},\ b_{11}\in\mathcal{I}$ then $A = (\mu_A,\nu_A) \in (T^{nor},\ S^{con})$ $IFB(L)$. But since $\mu_A((22 \bullet 55) \bullet (44 \bullet 11)) = \mu_A(22) = 0.1 \ngeq T^{nor}(\mu_A\ (22\bullet 44),\ \mu_A\ (55\bullet 11)) = T^{nor}\ (\mu_A(33),\ \mu_A(33)) = T^{nor}(0.7,\ 0.7) = 0.7$ and $\nu_A((22 \bullet 55) \bullet (44 \bullet 11)) = \nu_A(22) = 0.7 \nleq S^{con}(\nu_A\ (22\bullet 44),\ \nu_A\ (55\bullet 11)) = S^{con}\ (\nu_A(33),\ \nu_A(33)) = S^{con}(0.1,\ 0.1) = 0.1$ thus $A=(\mu_A,\ \nu_A) \notin (T^{nor},\ S^{con})$ $IFNB(L)$.

Proposition 3.19: Let $R=(\mu_R,\ \nu_R) \in (T^{nor},\ S^{con})$ $IFNB(X)$ and $T^{nor},\ S^{con}$ be idempotent. Hence

(1) $K=\{x_{11}\in X : R(x_{11})=A(0)\}$ will be a normal subalgebra of X.

(2) $R(x_{11}\bullet y_{11})=R(y_{11}\bullet x_{11})$ for all $x_{11},y_{11}\in X$.

Proposition 3.20: Assume $T^{nor},\ S^{con}$ be idempotent and $M=(\mu_M,\ \nu_m) \in IFS(U)$ such that $M = (\mu_M,\nu_M) \in (T^{nor},\ S^{con})$ $IFNB(U)$. Then $M_{i,j} = \{x\in U\ :\ M(u) \supseteq (i,j)\}$ is either empty or a normal subalgebra of B-algebra X for every $i,j\in[0,1]$.

Proof: Assume $M=(\mu_M,\nu_M)\in(T^{nor},\,S^{con})\ IFNB(U)$ and $M_{i,j}\neq\varnothing$. If $u_{11}\bullet v_{11}\in M_{i,j}$, then $\mu_M(u_{11}\bullet v_{11})\geq i$ and $\nu_M(u_{11}\bullet v_{11})\leq j$, and if $a_{11}\bullet b_{11}\in M_{i,j}$, then $\mu_M(a_{11}\bullet b_{11})\geq i$ and $\nu_M(a\bullet b)\leq j$. Now

$$\mu_M((u\bullet a)\bullet(v\bullet b))\geq T^{nor}(\mu_M(u\bullet v),\mu_M(a_{11}\bullet b_{11}))\geq T^{nor}(i,i)=i$$

and

$$\nu_M((u_{11}\bullet a_{11})\bullet(v_{11}\bullet b_{11}))\leq S^{con}(\nu_M(u_{11}\bullet v_{11}),\nu_M(a_{11}\bullet b_{11}))\leq S^{con}(j,j)=j$$

thus

$$M((u_{11}\bullet a_{11})\bullet(v_{11}\bullet b_{11}))=(\mu_M((u_{11}\bullet a_{11})\bullet(v_{11}\bullet b_{11})),\nu_M((u_{11}\bullet a_{11})\bullet(v_{11}\bullet b_{11}))\supseteq(i,j)$$

and then $(u_{11}\bullet a_{11})\bullet(v_{11}\bullet b_{11})\in M_{i,j}$ and so the set $M_{i,j}$ will be normal subalgebra of B-algebra U for every $i,j\in[0,1]$.

Proposition 3.21:

(1) Let $J=(\mu_J,\,\nu_J)\in(T^{nor},\,S^{con})\ IFNB(X)$ and $H=(\mu_H,\,\nu_H)\in(T^{nor},\,S^{con})\ IFNB(X)$. Thus $J\cap H=(\mu_{J\cap H},\,\nu_{J\cap H})\in(T^{nor},\,S^{con})\ IFNB(X)$.

(2) Let $J=(\mu_J,\,\nu_J)\in(T^{nor},\,S^{con})\ IFNB(X)$ and $H=(\mu_H,\nu_H)\in(T^{nor},\,S^{con})\ IFNB(Y)$. Then $J\times H=(\mu_{J\times H},\,\nu_{J\times H})\in(T^{nor},\,S^{con})\ IFNB\ (X\times Y)$.

Corollary 3.22: *Let* $A_i=(\mu_{A_i},\nu_{A_i})\in(T^{nor},S^{con})IFNB(Q_i)$ *where* $1\leq i\leq n$, *then*

$$A=(\mu_A,\nu_A)=\prod_{i=1}^{n}A_i\in(T^{nor},C)IFNB(\prod_{i=1}^{n}Q_i=Q)$$

which

$$\mu_A(q)=(\prod_{i=1}^{n}\mu_{A_i}(q_1,q_2,...,q_n)=T_n^{nor}(\mu_{A_1}(q_1,\mu_{A_2}(q_2,...,\mu_{A_n}(q_n))$$

and

$$\nu_A(q)=(\prod_{i=1}^{n}\nu_{A_i}(q_1,q_2,...,q_n)=S_n^{con}(\nu_{A_1}(q_1,\nu_{A_2}(q_2,...,\nu_{A_n}(q_n))$$

for all $q=(q_1,q_2,...,q_n)\in Q$.

Proposition 3.23: If $U=(\mu_U,\nu_U)\in(T^{nor},\,S^{con})\ IFNB(E)$ that $\chi:(E;\bullet,0)\to(F;\bullet',0')$ be a B-homomorphism of B-algebras, then $\chi(U)\in(T^{nor},S^{con})$ $IFNB(F)$.

Proposition 3.24: As $Z = (\mu_Z, \nu_Z) \in (T^{nor}, S^{con})$ $IFNB(Q)$ and $\omega : (W; \bullet, 0) \to (Q; \bullet', 0')$ be a B-homomorphism of B-algebras, so $\omega^{-1}(Z) \in (T^{nor}, S^{con})$ $IFNB(W)$.

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