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# Interval-valued picture fuzzy almost subsemigroups in semigroups

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**Abstract**

Our main results of this paper is to introduce an interval-valued picture fuzzy almost subsemigroup (IVPFAS) in a semigroup. Some properties of an IVPFAS in a semigroup will be studied. Moreover, we will investigate some relationships between an almost subsemigroup and an IVPFAS in a semigroup.

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***Keywords:*** *Almost subsemigroups, Interval-valued picture fuzzy set (IVPFS), Interval-valued picture fuzzy almost subsemigroup (IVPFAS).*

## 1. Introduction and Preliminaries

An almost subsemigroup was introduced by Iampan, Chinram, and Petchkaew [5] in 2021. An almost subsemigroup $A$ of a semigroup $S$ is a nonempty subset of $S$ that satisfies the condition $A^2 \cap A \neq \emptyset$. A fuzzy subset of a set $X$ is the map from each element in $X$ into a membership value between 0 and 1. The idea of fuzzy set was first presented by Zadeh [15] in 1965. After that, the theory of fuzzy subsets have been studied extensively. The concept of a fuzzy subset was applied to the groupoids and groups by Rosenfeld [10], to that rings by Liu [8], to that semigroups by Kuroki [6, 7], and to that po-ternary semihypergroups [11]. Again in 2021, Iampan, Chinram, and Petchkaew [5]

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defined a fuzzy almost subsemigroup in a semigroup. Moreover, they studied some properties of a fuzzy almost subsemigroup in a semigroup and investigated some relationship between an almost subsemigroup and a fuzzy almost subsemigroup in a semigroup. For a decade later, Zadeh [16] presented the basic concepts of interval-valued membership functions.

Let $CI[0,1]$ be the set of all simple closed subintervals of $[0,1]$, this is,

$$CI[0,1] = \{[s,s'] \mid 0 \leq s \leq s' \leq 1\}.$$

Let $[s,s'], [t,t'] \in CI[0,1]$.

(1) $[s,s'] \succeq [t,t'] \Leftrightarrow s \geq t$ and $s' \geq t'$.
(2) $[s,s'] \preceq [t,t'] \Leftrightarrow s \leq t$ and $s' \leq t'$.
(3) $[s,s'] = [t,t'] \Leftrightarrow s = t$ and $s' = t'$.
(4) $[s,s'] \succ [t,t'] \Leftrightarrow [s,s'] \succeq [t,t']$ and $[s,s'] \neq [t,t']$.
(5) $[s,s'] \prec [t,t'] \Leftrightarrow [s,s'] \preceq [t,t']$ and $[s,s'] \neq [t,t']$.

The refined minimum of $[s,s']$ and $[t,t']$, denoted by $\text{rmin}\{[s,s'],[t,t']\}$, is defined by

$$\text{rmin}\{[s,s'],[t,t']\} = [\min\{s,t\}, \min\{s',t'\}].$$

The refined maximum of $[s,s']$ and $[t,t']$, denoted by $\text{rmax}\{[s,s'], [t,t']\}$, is defined by

$$\text{rmax}\{[s,s'],[t,t']\} = [\max\{s,t\}, \max\{s',t'\}].$$

Let $[s,s']$ and $[t,t']$ be elements of $CI[0,1]$. Then

(1) If $[s,s'] \preceq [t,t']$ and $[s,s'] \neq [0,0]$, then $[t,t'] \neq [0,0]$.
(2) If $[s,s'] \preceq [t,t']$ and $[t,t'] \neq [1,1]$, then $[s,s'] \neq [1,1]$.
(3) If $[s,s'] \succeq [t,t']$ and $[t,t'] \neq [0,0]$, then $[s,s'] \neq [0,0]$.
(4) If $[s,s'] \succeq [t,t']$ and $[s,s'] \neq [1,1]$, then $[t,t'] \neq [1,1]$.

Next, let $\{[s_i, s'_i] | i \in \Lambda\}$ be the collection of close subintervals of $[0,1]$. We define

$$\inf_{i\in\Lambda} [s_i, s_i'] = \inf_{i\in\Lambda} s_i, \text{ and } \sup_{i\in\Lambda} [s_i, s_i'] = \sup_{i\in\Lambda} s_i'$$

$$\text{rinf}_{i\in\Lambda} [s_i, s_i'] = [\inf_{i\in\Lambda} s_i, \inf_{i\in\Lambda} s_i'], \text{ and } \text{rsup}_{i\in\Lambda} [s_i, s_i'] = [\sup_{i\in\Lambda} s_i, \sup_{i\in\Lambda} s_i'].$$

For the rest of this paper, let $X$ be a nonempty set and $S$ a semigroup.

An interval-valued fuzzy subset (shortly, IVFS) of $X$ is a function from $X$ into $CI[0,1]$.

Let $\overline{\tau}$ and $\overline{\pi}$ be IVFSs of $X$.

(1) $\overline{\tau} \subseteq \overline{\pi} \Leftrightarrow \overline{\tau}(x) \preceq \overline{\pi}(x)\ \forall x \in X$.
(2) $\overline{\tau} = \overline{\pi} \Leftrightarrow \overline{\tau}(x) = \overline{\pi}(x)\ \forall x \in X$.
(3) $(\overline{\tau} \cup \overline{\pi})(x) = \text{rmax}\{\overline{\tau}(x), \overline{\pi}(x)\}\ \forall x \in X$.
(4) $(\overline{\tau} \cap \overline{\pi})(x) = \text{rmin}\{\overline{\tau}(x), \overline{\pi}(x)\}\ \forall x \in X$.

**Proposition 1:** [16] Let $\overline{\tau}$, $\overline{\pi}$ and $\overline{\psi}$ be IVFSs of $X$.

(1) $\overline{\tau} \subseteq \overline{\tau} \cup \overline{\pi}$ and $\overline{\pi} \subseteq \overline{\tau} \cup \overline{\pi}$.

(2) $\overline{\tau} \cap \overline{\pi} \subseteq \overline{\tau}$ and $\overline{\tau} \cap \overline{\pi} \subseteq \overline{\pi}$.

(3) If $\overline{\tau} \subseteq \overline{\pi}$ and $\overline{\pi} \subseteq \overline{\psi}$, then $\overline{\tau} \subseteq \overline{\psi}$.

(4) If $\overline{\tau} \subseteq \overline{\pi}$, then $\overline{\tau} \cup \overline{\psi} \subseteq \overline{\pi} \cup \overline{\psi}$ and $\overline{\psi} \cup \overline{\tau} \subseteq \overline{\psi} \cup \overline{\pi}$.

(5) If $\overline{\tau} \subseteq \overline{\pi}$, then $\overline{\tau} \cap \overline{\psi} \subseteq \overline{\pi} \cap \overline{\psi}$ and $\overline{\psi} \cap \overline{\tau} \subseteq \overline{\psi} \cap \overline{\pi}$.

For a nonempty subset $A$ of $X$, we define IVFSs $\overline{\Pi}_A$ and $\overline{\Pi}'_A$ of $X$ by

$$\overline{\Pi}_A(x) = \begin{cases} [1,1] & \text{if } x \in A, \\ [0,0] & \text{if } x \notin A, \end{cases}$$

$$\overline{\Pi}'_A(x) = \begin{cases} [0,0] & \text{if } x \in A, \\ [1,1] & \text{if } x \notin A. \end{cases}$$

**Proposition 2:** [16] Let $A, B \in P(X)$.

(1) $A \subseteq B \Leftrightarrow \overline{\Pi}_A \subseteq \overline{\Pi}_B$.

(2) $\overline{\Pi}_A \cup \overline{\Pi}_B = \overline{\Pi}_{A\cup B}$.

(3) $\overline{\Pi}_A \cap \overline{\Pi}_B = \overline{\Pi}_{A\cap B}$.

**Proposition 3:** Let $A, B \in P(X)$.

(1) $A \subseteq B \Leftrightarrow \overline{\Pi}'_B \subseteq \overline{\Pi}'_A$.

(2) $\overline{\Pi}'_{A\cup B} \subseteq \overline{\Pi}'_A \cup \overline{\Pi}'_B$.

(3) $\overline{\Pi}'_A \cap \overline{\Pi}'_B \subseteq \overline{\Pi}'_{A\cap B}$.

(4) $\overline{\Pi}'_A \cup \overline{\Pi}'_B = \overline{\Pi}'_{A\cap B}$.

(5) $\overline{\Pi}'_A \cap \overline{\Pi}'_B = \overline{\Pi}'_{A\cup B}$.

***Proof:***

(1) Let $A \subseteq B$ and $x \in X$. If $x \in A$, then $x \in B$, so $\overline{\Pi}'_B(x) = [0,0] = \overline{\Pi}'_A(x)$. If $x \notin A$, then $\overline{\Pi}'_B(x) \preceq [1,1] = \overline{\Pi}'_A(x)$. Therefore, $\overline{\Pi}'_B \subseteq \overline{\Pi}'_A$.

Conversely, assume that $\overline{\Pi}'_B \subseteq \overline{\Pi}'_A$. Let $x \in A$. Then $\overline{\Pi}'_B(x) \preceq \overline{\Pi}'_A(x) = [0,0]$, which implies that $\overline{\Pi}'_B(x) = [0,0]$. Hence $x \in B$. Therefore, $A \subseteq B$

(2) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (1) we have $\overline{\Pi}'_{A\cup B} \subseteq \overline{\Pi}'_A$ and $\overline{\Pi}'_{A\cup B} \subseteq \overline{\Pi}'_B$.

Hence $\overline{\Pi}'_{A\cup B} \subseteq \overline{\Pi}'_A \cup \overline{\Pi}'_B$.

(3) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (1) we have $\overline{\Pi}'_A \subseteq \overline{\Pi}'_{A\cap B}$ and $\overline{\Pi}'_A \subseteq \overline{\Pi}'_{A\cap B}$.

Hence $\overline{\Pi}'_A \cap \overline{\Pi}'_B \subseteq \overline{\Pi}'_{A\cap B}$.

(4) Let $x \in X$. If $x \in A \cap B$, then $(\overline{\Pi}_A' \cup \overline{\Pi}_B')(x) = \text{rmax}\{\overline{\Pi}_A'(x), \overline{\Pi}_B'(x)\} = [0,0] = \overline{\Pi}_{A\cap B}'(x)$. If $x \notin A \cap B$, then $(\overline{\Pi}_A' \cup \overline{\Pi}_B')(x) = \text{rmax}\{\overline{\Pi}_A'(x), \overline{\Pi}_B'(x)\} = [1,1] = \overline{\Pi}_{A\cap B}'(x)$. Therefore, $\overline{\Pi}'_A \cup \overline{\Pi}'_B = \overline{\Pi}'_{A\cap B}$.

(5) Let $x \in X$. If $x \in A \cup B$, then $(\overline{\Pi}_A' \cap \overline{\Pi}_B')(x) = \text{rmin}\{\overline{\Pi}_A'(x), \overline{\Pi}_B'(x)\} = [0,0] = \overline{\Pi}_{A\cup B}'(x)$. If $x \notin A \cup B$, then $(\overline{\Pi}_A' \cap \overline{\Pi}_B')(x) = \text{rmin}\{\overline{\Pi}_A'(x), \overline{\Pi}_B'(x)\} = [1,1] = \overline{\Pi}_{A\cup B}'(x)$. Consequently, $\overline{\Pi}'_A \cap \overline{\Pi}'_B = \overline{\Pi}'_{A\cup B}$.

Let $\overline{\tau}$ and $\overline{\pi}$ be two IVFSs of $S$. Define IVFSs $\overline{\tau} \mathbin{\bar{\circ}} \overline{\pi}$ and $\overline{\tau} \mathbin{\bar{\bullet}} \overline{\pi}$ of $S$ by

$$(\overline{\tau} \mathbin{\bar{\circ}} \overline{\pi})(x) = \begin{cases} \underset{x=yz}{\text{rsup}}\ \text{rmin}\{\overline{\tau}(y), \overline{\pi}(z)\} & \text{if } x \in S^2, \\ [0,0] & \text{if } x \notin S^2, \end{cases}$$

and

$$(\overline{\tau} \mathbin{\bar{\bullet}} \overline{\pi})(x) = \begin{cases} \underset{x=yz}{\text{rinf}}\ \text{rmax}\{\overline{\tau}(y), \overline{\pi}(z)\} & \text{if } x \in S^2, \\ [1,1] & \text{if } x \notin S^2. \end{cases}$$

It is well-known that the operations $\bar{\circ}$ and $\bar{\bullet}$ are associative.
The following two propositions are properties of $\bar{\circ}$ and $\bar{\bullet}$.

**Proposition 4:** [9] Let $\overline{\tau}$, $\overline{\pi}$ and $\overline{\psi}$ be IVFSs of $S$.

(1) If $\overline{\pi} \subseteq \overline{\psi}$, then $\overline{\tau} \mathbin{\bar{\circ}} \overline{\pi} \subseteq \overline{\tau} \mathbin{\bar{\circ}} \overline{\psi}$ and $\overline{\pi} \mathbin{\bar{\circ}} \overline{\tau} \subseteq \overline{\psi} \mathbin{\bar{\circ}} \overline{\tau}$.

(2) $\overline{\tau} \mathbin{\bar{\circ}} (\overline{\pi} \cup \overline{\psi}) = (\overline{\tau} \mathbin{\bar{\circ}} \overline{\pi}) \cup (\overline{\tau} \mathbin{\bar{\circ}} \overline{\psi})$.

(3) $\overline{\tau} \mathbin{\bar{\circ}} (\overline{\pi} \cap \overline{\psi}) = (\overline{\tau} \mathbin{\bar{\circ}} \overline{\pi}) \cap (\overline{\tau} \mathbin{\bar{\circ}} \overline{\psi})$.

(4) $(\overline{\tau} \cup \overline{\pi}) \mathbin{\bar{\circ}} \overline{\psi} = (\overline{\tau} \mathbin{\bar{\circ}} \overline{\psi}) \cup (\overline{\pi} \mathbin{\bar{\circ}} \overline{\psi})$.

(5) $(\overline{\tau} \cap \overline{\pi}) \mathbin{\bar{\circ}} \overline{\psi} = (\overline{\tau} \mathbin{\bar{\circ}} \overline{\psi}) \cap (\overline{\pi} \mathbin{\bar{\circ}} \overline{\psi})$.

Similar to Proposition 4, we have the next proposition.

**Proposition 5:** Let $\overline{\tau}, \overline{\pi}$ and $\overline{\psi}$ be IVFSs of $S$.

(1) If $\overline{\pi} \subseteq \overline{\psi}$, then $\overline{\tau} \mathbin{\bar{\bullet}} \overline{\pi} \subseteq \overline{\tau} \mathbin{\bar{\bullet}} \overline{\psi}$ and $\overline{\pi} \mathbin{\bar{\bullet}} \overline{\tau} \subseteq \overline{\psi} \mathbin{\bar{\bullet}} \overline{\tau}$.

(2) $\overline{\tau} \mathbin{\bar{\bullet}} (\overline{\pi} \cup \overline{\psi}) = (\overline{\tau} \mathbin{\bar{\bullet}} \overline{\pi}) \cup (\overline{\tau} \mathbin{\bar{\bullet}} \overline{\psi})$.

(3) $\overline{\tau} \mathbin{\bar{\bullet}} (\overline{\pi} \cap \overline{\psi}) = (\overline{\tau} \mathbin{\bar{\bullet}} \overline{\pi}) \cap (\overline{\tau} \mathbin{\bar{\bullet}} \overline{\psi})$.

(4) $(\overline{\tau} \cup \overline{\pi}) \mathbin{\bar{\bullet}} \overline{\psi} = (\overline{\tau} \mathbin{\bar{\bullet}} \overline{\psi}) \cup (\overline{\pi} \mathbin{\bar{\bullet}} \overline{\psi})$.

(5) $(\overline{\tau} \cap \overline{\pi}) \mathbin{\bar{\bullet}} \overline{\psi} = (\overline{\tau} \mathbin{\bar{\bullet}} \overline{\psi}) \cap (\overline{\pi} \mathbin{\bar{\bullet}} \overline{\psi})$.

**Proposition 6:** [9] Let $A, B \in P(S)$. Then $\overline{\Pi}_A \mathbin{\bar{\circ}} \overline{\Pi}_B = \overline{\Pi}_{AB}$.

Similar to Proposition 6, we have the next proposition.

**Proposition 7:** [9] Let $A, B \in P(S)$. Thus $\overline{\Pi}'_A \mathbin{\bar{\bullet}} \overline{\Pi}'_B = \overline{\Pi}'_{AB}$.

Narayanan and Manikantan [9] applied theory of IVFSs in semigroups in 2006. In 1983, Atanassov [1] extended a fuzzy subset to an intuitionistic fuzzy set (shortly, IFS). In IFSs, an element is expressed by a degree of membership and a degree of non-membership, where the sum of these two degrees of membership is always less than or equal to one. Later, Atanassov and Gargov [2] proposed an interval-valued IFS based on IFS and IVFSs. Many years later, in 2013, a picture fuzzy set (shortly, PFS) was first introduced by Cuong and Kreinovich [3, 4], which are extensions of the fuzzy subsets and IFSs. In general, PFSs are useful in situations where humans make four decisions: yes, abstain, no, and refuse. Additionally, a definition and properties of an interval-valued picture fuzzy set was simultaneously created. After a while, PFSs have been redefined by Yang et. al. [12] without mentioned the Cuong defining of PFSs as follows:

A PFS of $X$ is the set

$$\{(x, \upsilon(x), \rho(x), \varepsilon(x)) \mid x \in X\},$$

where $\upsilon, \rho, \varepsilon$ are fuzzy subsets of $X$ which satisfy the following condition:

$$0 \leq \upsilon(x) + \varepsilon(x) \leq 1 \text{ and } 0 \leq \upsilon(x) + \rho(x) + \varepsilon(x) \leq 2 \text{ for all } x \in X.$$

A PFS of $X$ is briefly denoted by $(\upsilon, \rho, \varepsilon)$.

Recently, Yiarayong [13, 14] applied the concept of PFSs to semigroups.

An interval-valued picture fuzzy set (shortly, IVPFS)) of $X$ is the set

$$\{(x, \overline{\upsilon}(x), \overline{\rho}(x), \overline{\varepsilon}(x)) \mid x \in X\},$$

where $\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon}$ are IVFSs of $X$ which satisfy the following condition

$$0 \leq \sup \overline{\upsilon}(x) + \sup \overline{\varepsilon}(x) \leq 1 \text{ and } 0 \leq \sup \overline{\upsilon}(x) + \sup \overline{\rho}(x) + \sup \overline{\varepsilon}(x) \leq 2 \text{ for all } x \in X.$$

The IVPFS of $X$ is briefly denoted by $(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon})$.

Let $(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1)$ and $(\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ be two IVPFSs of $X$. Then

(1) $(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1) \subseteq (\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ if and only if

$\overline{\upsilon}_1(x) \preceq \overline{\upsilon}_2(x)$, $\overline{\rho}_1(x) \succeq \overline{\rho}_2(x)$, and $\overline{\varepsilon}_1(x) \succeq \overline{\varepsilon}_2(x)$ for all $x \in X$.

(2) $(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1) = (\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ if and only if

$\overline{\upsilon}_1(x) = \overline{\upsilon}_2(x), \overline{\rho}_1(x) = \overline{\rho}_2(x)$, and $\overline{\varepsilon}_1(x) = \overline{\varepsilon}_2(x)$ for all $x \in X$.

(3) The union of $(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1)$ and $(\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ is defined by

$$(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1) \cup (\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2) = \{(x, (\overline{\upsilon}_1 \cup \overline{\upsilon}_2)(x), (\overline{\rho}_1 \cap \overline{\rho}_2)(x), (\overline{\varepsilon}_1 \cap \overline{\varepsilon}_2)(x)) \mid x \in X\}.$$

(4) The intersection of $(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1)$ and $(\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ is defined by

$$(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1) \cap (\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2) = \{(x, (\overline{\upsilon}_1 \cap \overline{\upsilon}_2)(x), (\overline{\rho}_1 \cup \overline{\rho}_2)(x), (\overline{\varepsilon}_1 \cup \overline{\varepsilon}_2)(x)) \mid x \in X\}.$$

The next results are following from Proposition 1.

**Proposition 8:** Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ be *IVPFSs* of $S$. Then

(1) $\mathcal{A}_1 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_2 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$.

(2) $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq \mathcal{A}_1$ and $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq \mathcal{A}_2$.

(3) If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_2 \subseteq \mathcal{A}_3$, then $\mathcal{A}_1 \subseteq \mathcal{A}_3$.

(4) If $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $\mathcal{A}_1 \cup \mathcal{A}_3 \subseteq \mathcal{A}_2 \cup \mathcal{A}_3$ and $\mathcal{A}_3 \cup \mathcal{A}_1 \subseteq \mathcal{A}_3 \cup \mathcal{A}_2$.

(5) If $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $\mathcal{A}_1 \cap \mathcal{A}_3 \subseteq \mathcal{A}_2 \cap \mathcal{A}_3$ and $\mathcal{A}_3 \cap \mathcal{A}_1 \subseteq \mathcal{A}_3 \cap \mathcal{A}_2$.

The characteristic IVPFS of a nonempty subset $A$ in $X$ is defined by

$$\left\{(x, \overline{\Pi}_A(x), \overline{\Pi}'_A(x), \overline{\Pi}'_A(x)) \mid x \in X\right\}.$$

We denote the characteristic IVPFS of $A$ in $X$ by $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A)$.

**Remark 9:** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be *IVPFSs* of $S$.

(1) If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_1 \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$, then $\mathcal{A}_2 \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$.

(2) If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_2 \neq (\overline{\Pi}_S, \overline{\Pi}'_S, \overline{\Pi}'_S)$, then $\mathcal{A}_1 \neq (\overline{\Pi}_S, \overline{\Pi}'_S, \overline{\Pi}'_S)$.

(3) If $\mathcal{A}_2 \subseteq \mathcal{A}_1$ and $\mathcal{A}_2 \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$, then $\mathcal{A}_1 \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$.

(4) If $\mathcal{A}_2 \subseteq \mathcal{A}_1$ and $\mathcal{A}_1 \neq (\overline{\Pi}_S, \overline{\Pi}'_S, \overline{\Pi}'_S)$, then $\mathcal{A}_2 \neq (\overline{\Pi}_S, \overline{\Pi}'_S, \overline{\Pi}'_S)$.

The next proposition follows from Proposition 2 and 3.

**Proposition 10:** Let $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A)$ and $(\overline{\Pi}_B, \overline{\Pi}'_B, \overline{\Pi}'_B)$ be two characteristic IVPFSs of a set $A$ and $B$ in $X$.

(1) $A \subseteq B$ if and only if $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A) \subseteq (\overline{\Pi}_B, \overline{\Pi}'_B, \overline{\Pi}'_B)$.

(2) $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A) \cup (\overline{\Pi}_B, \overline{\Pi}'_B, \overline{\Pi}'_B) = (\overline{\Pi}_{A\cup B}, \overline{\Pi}'_{A\cup B}, \overline{\Pi}'_{A\cup B})$.

(3) $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A) \cap (\overline{\Pi}_B, \overline{\Pi}'_B, \overline{\Pi}'_B) = (\overline{\Pi}_{A\cap B}, \overline{\Pi}'_{A\cap B}, \overline{\Pi}'_{A\cap B})$.

The product of IVPFSs $\mathcal{A}_1 = (\overline{v}_1, \overline{\rho}_1, \overline{\varepsilon}_1)$ and $\mathcal{A}_2 = (\overline{v}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ of $S$, written $\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_2$, is defined by

$$\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_2 = \{(x, (\overline{v}_1 \mathbin{\bar{\circ}} \overline{v}_2)(x), (\overline{\rho}_1 \mathbin{\bar{\bullet}} \overline{\rho}_2)(x), (\overline{\varepsilon}_1 \mathbin{\bar{\bullet}} \overline{\varepsilon}_2)(x)) \mid x \in X\}.$$

The Proposition 4 and 5 lead to the next proposition.

**Proposition 11:** Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ be *IVPFSs* of $S$.

(1) If $\mathcal{A}_2 \subseteq \mathcal{A}_3$, then $\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_2 \subseteq \mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_3$ and $\mathcal{A}_2 \mathbin{\bar{\circ}_p} \mathcal{A}_1 \subseteq \mathcal{A}_3 \mathbin{\bar{\circ}_p} \mathcal{A}_1$.

(2) $\mathcal{A}_1 \mathbin{\bar{\circ}_p} (\mathcal{A}_2 \cap \mathcal{A}_3) = (\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_2) \cap (\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_3)$.

(3) $(\mathcal{A}_1 \cap \mathcal{A}_2) \mathbin{\bar{\circ}_p} \mathcal{A}_3 = (\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_3) \cap (\mathcal{A}_2 \mathbin{\bar{\circ}_p} \mathcal{A}_3)$.

(4) $\mathcal{A}_1 \mathbin{\bar{\circ}_p} (\mathcal{A}_2 \cup \mathcal{A}_3) = (\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_2) \cup (\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_3)$.

(5) $(\mathcal{A}_1 \cup \mathcal{A}_2) \mathbin{\bar{\circ}_p} \mathcal{A}_3 = (\mathcal{A}_1 \mathbin{\bar{\circ}_p} \mathcal{A}_3) \cup (\mathcal{A}_2 \mathbin{\bar{\circ}_p} \mathcal{A}_3)$.

Proposition 6 and Proposition 7 yield the following result.

**Proposition 12:** For $A, B \in P(S)$, we get

$$(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A) \mathbin{\bar{\circ}_p} (\overline{\Pi}_B, \overline{\Pi}'_B, \overline{\Pi}'_B) = (\overline{\Pi}_{AB}, \overline{\Pi}'_{AB}, \overline{\Pi}'_{AB}).$$

In this paper, we use the concepts of an almost subsemigroup and an IVPFS in a semigroup to define an interval-valued picture fuzzy almost subsemigroup in a semigroup. Basic properties of an interval-valued picture fuzzy almost subsemigroup in a semigroup will be studied. Furthermore, we will study some relationships between an almost subsemigroup and its interval-valued picture fuzzification.

## 2. Main Results

We begin this section by introducing an interval-valued picture fuzzy almost subsemigroup in a semigroup.

**Definition 1:** Let $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ be an IVPFS of $S$ such that $(\overline{v}, \overline{\rho}, \overline{\varepsilon}) \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$. We call $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ an interval-valued picture fuzzy almost subsemigroup (shortly, IVPFAS) of $S$, if

$$((\overline{v}, \overline{\rho}, \overline{\varepsilon}) \mathbin{\bar{\circ}_p} (\overline{v}, \overline{\rho}, \overline{\varepsilon})) \cap (\overline{v}, \overline{\rho}, \overline{\varepsilon}) \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S).$$

**Theorem 13:** *Let $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ be an IVPFS of $S$ such that $(\overline{v}, \overline{\rho}, \overline{\varepsilon}) \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$. Then $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ is an IVPFAS of $S$ if and only if there is $x \in S$ that satisfies*

$$[(\overline{v} \mathbin{\bar{\circ}} \overline{v}) \cap \overline{v}](x) \neq [0,0], \textit{ or } [(\overline{\rho} \mathbin{\bar{\bullet}} \overline{\rho}) \cup \overline{\rho}](x) \neq [1,1], \textit{ or } [(\overline{\varepsilon} \mathbin{\bar{\bullet}} \overline{\varepsilon}) \cup \overline{\varepsilon}](x) \neq [1,1].$$

***Proof:*** We see that

$$\begin{aligned}((\overline{v}, \overline{\rho}, \overline{\varepsilon}) \mathbin{\bar{\circ}_p} (\overline{v}, \overline{\rho}, \overline{\varepsilon})) \cap (\overline{v}, \overline{\rho}, \overline{\varepsilon}) &= (\overline{v} \mathbin{\bar{\circ}} \overline{v}, \overline{\rho} \mathbin{\bar{\bullet}} \overline{\rho}, \overline{\varepsilon} \mathbin{\bar{\bullet}} \overline{\varepsilon}) \cap (\overline{v}, \overline{\rho}, \overline{\varepsilon}) \\ &= ((\overline{v} \mathbin{\bar{\circ}} \overline{v}) \cap \overline{v}, (\overline{\rho} \mathbin{\bar{\bullet}} \overline{\rho}) \cup \overline{\rho}, (\overline{\varepsilon} \mathbin{\bar{\bullet}} \overline{\varepsilon}) \cup \overline{\varepsilon}).\end{aligned}$$

Assume that $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ is an IVPFAS of $S$. Then

$$((\overline{v}, \overline{\rho}, \overline{\varepsilon}) \mathbin{\bar{\circ}_p} (\overline{v}, \overline{\rho}, \overline{\varepsilon})) \cap (\overline{v}, \overline{\rho}, \overline{\varepsilon}) \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S).$$

Then $((\overline{v} \mathbin{\bar{\circ}} \overline{v}) \cap \overline{v}, (\overline{\rho} \mathbin{\bar{\bullet}} \overline{\rho}) \cup \overline{\rho}, (\overline{\varepsilon} \mathbin{\bar{\bullet}} \overline{\varepsilon}) \cup \overline{\varepsilon}) \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$.

Thus there exists $x \in S$ such that $[(\overline{v} \mathbin{\bar{\circ}} \overline{v}) \cap \overline{v}](x) \neq \overline{\Pi}'_S(x) = [0,0]$, or $[(\overline{\rho} \mathbin{\bar{\bullet}} \overline{\rho}) \cup \overline{\rho}](x) \neq \overline{\Pi}_S(x) = [1,1]$, or $[(\overline{\varepsilon} \mathbin{\bar{\bullet}} \overline{\varepsilon}) \cup \overline{\varepsilon}](x) \neq \overline{\Pi}_S(x) = [1,1]$.

Conversely, assume that there exists $x \in S$ such that

$$[(\overline{v} \mathbin{\bar{\circ}} \overline{v}) \cap \overline{v}](x) \neq [0,0], \text{ or } [(\overline{\rho} \mathbin{\bar{\bullet}} \overline{\rho}) \cup \overline{\rho}](x) \neq [1,1], \text{ or } [(\overline{\varepsilon} \mathbin{\bar{\bullet}} \overline{\varepsilon}) \cup \overline{\varepsilon}](x) \neq [1,1].$$

Then $[(\overline{v} \mathbin{\bar{\circ}} \overline{v}) \cap \overline{v}](x) \neq \overline{\Pi}'_S(x)$, or $[(\overline{\rho} \mathbin{\bar{\bullet}} \overline{\rho}) \cup \overline{\rho}](x) \neq \overline{\Pi}_S(x)$, or $[(\overline{\varepsilon} \mathbin{\bar{\bullet}} \overline{\varepsilon}) \cup \overline{\varepsilon}](x) \neq \overline{\Pi}_S(x)$. Thus

$$((\overline{v}, \overline{\rho}, \overline{\varepsilon}) \mathbin{\bar{\circ}_p} (\overline{v}, \overline{\rho}, \overline{\varepsilon})) \cap (\overline{v}, \overline{\rho}, \overline{\varepsilon}) \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S).$$

Hence, $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ is an IVPFAS of $S$.

**Theorem 14:** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be IVPFSs of $S$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. If $\mathcal{A}_1$ is an IVPFAS of $S$, then $\mathcal{A}_2$ is an IVPFAS of $S$.*

***Proof:*** Let $\mathcal{A}_1$ be an IVPFAS of $S$ and $\mathcal{A}_2$ be an IVPFS of $S$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Then $(\mathcal{A}_1 \,\bar{\circ}_p\, \mathcal{A}_1) \cap \mathcal{A}_1 \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$. Since $\mathcal{A}_1 \subseteq \mathcal{A}_2$, it follows from Proposition 11 and 8 that $(\mathcal{A}_1 \,\bar{\circ}_p\, \mathcal{A}_1) \cap \mathcal{A}_1 \subseteq (\mathcal{A}_2 \,\bar{\circ}_p\, \mathcal{A}_2) \cap \mathcal{A}_2$. We obtain that $(\mathcal{A}_2 \,\bar{\circ}_p\, \mathcal{A}_2) \cap \mathcal{A}_2 \neq (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$. Hence, $\mathcal{A}_2$ is an IVPFAS of $S$.

**Corollary 15:** *A union of two IVPFASs of $S$ is also an IVPFAS of $S$.*

**Example 16:** Consider the semigroup $\mathbb{Z}_5$ under usual addition. IVPFSs $(\overline{v}_1, \overline{\rho}_1, \overline{\varepsilon}_1)$ and $(\overline{v}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ of $\mathbb{Z}_5$ are defined by

$$(\overline{v}_1, \overline{\rho}_1, \overline{\varepsilon}_1) = \{ (\overline{0}, [0.3,0.4], [0.2,0.5], [0.1,0.1]), (\overline{1}, [0.0,0.0], [1.0,1.0], [1.0,1.0]), (\overline{2}, [0.0,0.0], [1.0,1.0], [1.0,1.0]), (\overline{3}, [0.5,0.8], [0.2,0.2], [0.0,0.0]), (\overline{4}, [0.0,0.0], [1.0,1.0], [1.0,1.0])\},$$ and

$$(\overline{v}_2, \overline{\rho}_2, \overline{\varepsilon}_2) = \{ (\overline{0}, [0.0,0.0], [1.0,1.0], [1.0,1.0]), (\overline{1}, [0.0,0.3], [0.1,0.4], [0.1,0.1]), (\overline{2}, [0.1,0.1], [0.4,0.6], [0.1,0.1]), (\overline{3}, [0.0,0.0], [1.0,1.0], [1.0,1.0]), (\overline{4}, [0.2,0.2], [0.0,0.0], [0.0,0.0])\}.$$

We have that,

$[(\overline{v}_1 \,\bar{\circ}\, \overline{v}_1) \cap \overline{v}_1](\overline{0}) = [0.3,0.4], [(\overline{\rho}_1 \,\bar{\bullet}\, \overline{\rho}_1) \cup \overline{\rho}_1](\overline{0}) = [0.2,0.5], [(\overline{\varepsilon}_1 \,\bar{\bullet}\, \overline{\varepsilon}_1) \cup \overline{\varepsilon}_1](\overline{0}) = [0.1,0.1]$,

$[(\overline{v}_1 \,\bar{\circ}\, \overline{v}_1) \cap \overline{v}_1](\overline{1}) = [0.0,0.0], [(\overline{\rho}_1 \,\bar{\bullet}\, \overline{\rho}_1) \cup \overline{\rho}_1](\overline{1}) = [1.0,1.0], [(\overline{\varepsilon}_1 \,\bar{\bullet}\, \overline{\varepsilon}_1) \cup \overline{\varepsilon}_1](\overline{1}) = [1.0,1.0]$,

$[(\overline{v}_1 \,\bar{\circ}\, \overline{v}_1) \cap \overline{v}_1](\overline{2}) = [0.0,0.0], [(\overline{\rho}_1 \,\bar{\bullet}\, \overline{\rho}_1) \cup \overline{\rho}_1](\overline{2}) = [1.0,1.0], [(\overline{\varepsilon}_1 \,\bar{\bullet}\, \overline{\varepsilon}_1) \cup \overline{\varepsilon}_1](\overline{2}) = [1.0,1.0]$,

$[(\overline{v}_1 \,\bar{\circ}\, \overline{v}_1) \cap \overline{v}_1](\overline{3}) = [0.3,0.4], [(\overline{\rho}_1 \,\bar{\bullet}\, \overline{\rho}_1) \cup \overline{\rho}_1](\overline{3}) = [0.2,0.5], [(\overline{\varepsilon}_1 \,\bar{\bullet}\, \overline{\varepsilon}_1) \cup \overline{\varepsilon}_1](\overline{3}) = [0.1,0.1]$,

$[(\overline{v}_1 \,\bar{\circ}\, \overline{v}_1) \cap \overline{v}_1](\overline{4}) = [0.0,0.0], [(\overline{\rho}_1 \,\bar{\bullet}\, \overline{\rho}_1) \cup \overline{\rho}_1](\overline{4}) = [1.0,1.0], [(\overline{\varepsilon}_1 \,\bar{\bullet}\, \overline{\varepsilon}_1) \cup \overline{\varepsilon}_1](\overline{4}) = [1.0,1.0]$.

Then $(\overline{v}_1, \overline{\rho}_1, \overline{\varepsilon}_1)$ is an IVPFAS of $\mathbb{Z}_5$. Similarly, $(\overline{v}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ is an IVPFAS of $\mathbb{Z}_5$.

We obtain that $(\overline{v}_1, \overline{\rho}_1, \overline{\varepsilon}_1) \cup (\overline{v}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ is an IVPFAS of $\mathbb{Z}_5$ but $(\overline{v}_1, \overline{\rho}_1, \overline{\varepsilon}_1) \cap (\overline{v}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$ is not an IVPFAS of $\mathbb{Z}_5$ because $(\overline{v}_1, \overline{\rho}_1, \overline{\varepsilon}_1) \cap (\overline{v}_2, \overline{\rho}_2, \overline{\varepsilon}_2) = (\overline{\Pi}'_S, \overline{\Pi}_S, \overline{\Pi}_S)$.

**Theorem 17:** *Let $A$ be a nonempty subset of $S$. Then $A$ is an almost subsemigroup of $S$ if and only if $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A)$ is an IVPFAS of $S$.*

***Proof:*** Let $A$ be an almost subsemigroup of $S$. Then $A^2 \cap A \neq \emptyset$. Thus $x \in A^2 \cap A$ for some $x \in S$. Then

$$[(\overline{\Pi}_A \circ \overline{\Pi}_A) \cap \overline{\Pi}_A](x) = [\overline{\Pi}_{A^2} \cap \overline{\Pi}_A](x) = \overline{\Pi}_{A^2 \cap A}(x) = [1,1] \neq [0,0].$$

By Theorem 13, we have that $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A)$ is an IVPFAS of $S$.

Conversely, let $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A)$ be an IVPFAS of $S$. Thus there exists $x \in S$ such that $\overline{\Pi}_{A^2 \cap A}(x) \neq [0,0]$ or $\overline{\Pi}'_{A^2 \cap A}(x) \neq [1,1]$. Hence, $x \in A^2 \cap A$, this is, $A^2 \cap A \neq \emptyset$. Therefore, $A$ is an almost subsemigroup of $S$.

**Definition 2:** Let $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ be an *IVPFS* of $X$. Define

$$supp(\overline{v}, \overline{\rho}, \overline{\varepsilon}) = \{x \in X | \overline{v}(x) \neq [0,0] \text{ or } \overline{\rho}(x) \neq [1,1] \text{ or } \overline{\varepsilon}(x) \neq [1,1]\}.$$

We called $supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ a *support* of $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$.

**Theorem 18:** *If $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ is an IVPFAS of $S$, then $supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ is an almost subsemigroup of $S$.*

***Proof:*** Let $(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ be an IVPFAS of $S$. Then there exists $x \in S$ such that

$$[(\overline{v} \circ \overline{v}) \cap \overline{v}](x) \neq [0,0], \text{ or } [(\overline{\rho} \bullet \overline{\rho}) \cup \overline{\rho}](x) \neq [1,1], \text{ or } [(\overline{\varepsilon} \bullet \overline{\varepsilon}) \cup \overline{\varepsilon}](x) \neq [1,1].$$

**Case 1:** $[(\overline{v} \circ \overline{v}) \cap \overline{v}](x) \neq [0,0]$. Then $(\overline{v} \circ \overline{v})(x) \neq [0,0]$ and $\overline{v}(x) \neq [0,0]$. Choose $a, b \in S$ such that $x = ab$, $\overline{v}(a) = \operatorname{rsup}_{x=uv} \overline{v}(u)$, and $\overline{v}(b) = \operatorname{rsup}_{x=uv} \overline{v}(v)$. Thus

$$\begin{aligned}\operatorname{rmin}\{\overline{v}(a), \overline{v}(b)\} &= \operatorname{rmin}\{\operatorname{rsup}_{x=uv} \overline{v}(u), \operatorname{rsup}_{x=uv} \overline{v}(v)\} = \operatorname{rsup}_{x=uv} \operatorname{rmin}\{\overline{v}(u), \overline{v}(v)\} \\ &= (\overline{v} \circ \overline{v})(x) \neq [0,0].\end{aligned}$$

Hence, $x, a, b \in supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$. So that $x = ab \in (supp(\overline{v}, \overline{\rho}, \overline{\varepsilon}))^2 \cap supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$.

That is $(supp(\overline{v}, \overline{\rho}, \overline{\varepsilon}))^2 \cap supp(\overline{v}, \overline{\rho}, \overline{\varepsilon}) \neq \emptyset$. Therefore, $supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ is an almost subsemigroup of $S$.

**Case 2:** $[(\overline{\rho} \bullet \overline{\rho}) \cup \overline{\rho}](x) \neq [1,1]$. Then $(\overline{\rho} \bullet \overline{\rho})(x) \neq [1,1]$ and $\overline{\rho}(x) \neq [1,1]$. Choose $a, b \in S$ such that $x = ab$, $\overline{\rho}(a) = \operatorname{rinf}_{x=uv} \overline{\rho}(u)$, and $\overline{\rho}(b) = \operatorname{rinf}_{x=uv} \overline{\rho}(v)$. Thus

$$\begin{aligned}\operatorname{rmax}\{\overline{\rho}(a), \overline{\rho}(b)\} &= \operatorname{rmax}\{\operatorname{rinf}_{x=uv} \overline{\rho}(u), \operatorname{rinf}_{x=uv} \overline{\rho}(v)\} = \operatorname{rinf}_{x=uv} \operatorname{rmax}\{\overline{\rho}(u), \overline{\rho}(w)\} \\ &= (\overline{\rho} \bullet \overline{\rho})(x) \neq [1,1].\end{aligned}$$

Hence, $x, a, b \in supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$. So that $x = ab \in (supp(\overline{v}, \overline{\rho}, \overline{\varepsilon}))^2 \cap supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$.

That is $(supp(\overline{v}, \overline{\rho}, \overline{\varepsilon}))^2 \cap supp(\overline{v}, \overline{\rho}, \overline{\varepsilon}) \neq \emptyset$. Therefore, $supp(\overline{v}, \overline{\rho}, \overline{\varepsilon})$ is an almost subsemigroup of $S$.

**Case 3:** $[(\overline{\varepsilon} \bullet \overline{\varepsilon}) \cup \overline{\varepsilon}](x) \neq [1,1]$. The proof of this case is similar to the proof of Case 2.

**Example 19:** Consider the semigroup $S = \{a, b, c, d\}$ with multiplication Table 1 as follows:

**Table 1**

**The multiplications of two elements of $S$**

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $b$ | $a$ |
| $d$ | $a$ | $a$ | $b$ | $b$ |

.

An IVPFS $(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon})$ of $S$ is defined by

$$(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon}) = \{ (a, [0,0], [1,1], [1,1]), (b, [0,0], [0,0], [1,1]), (c, [0,0], [1,1], [1,1]), (d, [0,0], [1,1], [0,0])\}.$$

Then $supp(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon}) = \{b, d\}$. Since $(supp(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon}))^2 \cap supp(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon}) = \{a, b\} \cap \{b, d\} = \{b\} \neq \emptyset$, we get $supp(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon}) = \{b, d\}$ is an almost subsemigroup of $S$. However,

$$[(\overline{\upsilon} \circ \overline{\upsilon}) \cap \overline{\upsilon}](a) = [0,0], [(\overline{\rho} \bullet \overline{\rho}) \cup \overline{\rho}](a) = [1,1], [(\overline{\varepsilon} \bullet \overline{\varepsilon}) \cup \overline{\varepsilon}](a) = [1,1],$$
$$[(\overline{\upsilon} \circ \overline{\upsilon}) \cap \overline{\upsilon}](b) = [0,0], [(\overline{\rho} \bullet \overline{\rho}) \cup \overline{\rho}](b) = [1,1], [(\overline{\varepsilon} \bullet \overline{\varepsilon}) \cup \overline{\varepsilon}](b) = [1,1],$$
$$[(\overline{\upsilon} \circ \overline{\upsilon}) \cap \overline{\upsilon}](c) = [0,0], [(\overline{\rho} \bullet \overline{\rho}) \cup \overline{\rho}](c) = [1,1], [(\overline{\varepsilon} \bullet \overline{\varepsilon}) \cup \overline{\varepsilon}](c) = [1,1],$$
$$[(\overline{\upsilon} \circ \overline{\upsilon}) \cap \overline{\upsilon}](d) = [0,0], [(\overline{\rho} \bullet \overline{\rho}) \cup \overline{\rho}](d) = [1,1], [(\overline{\varepsilon} \bullet \overline{\varepsilon}) \cup \overline{\varepsilon}](d) = [1,1].$$

Hence, $(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon})$ is not an IVPFAS of $S$.

## Conclusion

An IVPFAS in a semigroup was defined by using the concepts of an almost subsemigroup in a semigroup and an IVPFS. Its some properties were investigated. If $(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1)$ is an IVPFAS in a semigroup $S$ and $(\overline{\upsilon}_1, \overline{\rho}_1, \overline{\varepsilon}_1) \subseteq (\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$, then so is $(\overline{\upsilon}_2, \overline{\rho}_2, \overline{\varepsilon}_2)$. Hence, we obtain that a union of two IVPFASs of $S$ is an IVPFAS of $S$ but an intersection of two IVPFASs of $S$ need not be an IVPFAS of $S$. A nonempty subset $A$ in a semigroup $S$ is an almost subsemigroup of $S$ if and only if $(\overline{\Pi}_A, \overline{\Pi}'_A, \overline{\Pi}'_A)$ is an IVPFAS of $S$. Furthermore, if $(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon})$ is an IVPFAS of $S$, then the support of $(\overline{\upsilon}, \overline{\rho}, \overline{\varepsilon})$ is an almost subsemigroup of $S$. However, the converse is not true. These are some relationships between almost subsemigroups in a semigroup and their interval-valued picture fuzzifications.

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