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Fibonacci range labeling on direct product and union of graphs

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Abstract

In this paper, we present a novel graph labeling parameter, referred to as Fibonacci range labeling or φ-labeling. We explore its applicability to $K_{1,n} \times K_{1,m}$, $P_n \times P_2$, $C_n \cup C_m$, $C_n \cup P_m$ and determine conditions with labeling φ.

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1. Introduction

Graph labeling plays a crucial role in combinatorial mathematics, with applications in network theory, coding, and cryptography [6]. Various labeling techniques, including harmonious and geometric labeling, have been explored to analyze graph structures and their properties [10], [11]. Fibonacci range labeling, or φ-labeling, is a novel approach where vertex labels are assigned distinct Fibonacci numbers, ensuring that edge labels conform to the golden ratio [1]. The concept of φ-labeling extends previous work on vertex valuations [4], harmonious graphs [8], and tree congruence properties [9]. Early research on graph labeling problems [5] and direct product structures [13] provides a foundation for analyzing the applicability of Fibonacci range labeling.

The discussion considers finite, undirected graphs without multiple edges. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two graphs. The union of these graphs, represented as $G_1 \cup G_2$, forms a new graph with a vertex set of $V_1 \cup V_2$ and the edge set of $E_1 \cup E_2$ when combining n identical copies of a graph G the resulting graph is indicated by nG.

Weichsel [12], establish that if for any two graphs (say G and H) then the graph G and H are finitely connected if and only if G and H are connected and neither is bipartite. In particular, Zhihui and Xianglan [7], gave an explicit result on the connectivity of the graphs and gave a variation on the connectivity of the edges. In general, in two product graphs G and H, two vertices (L, M) and (l, m) are adjacent in $G \times H$ iff $xx' \in E(H)$ and $yy' \in E(G)$.

In this context, we define Fibonacci labeling for graphs whose domain is given by $V(G)$, i.e. the vertex set and the elements are mapped to Fibonacci numbers. More precisely, if a graph G with $|E(G)| = q$ with vertices $x \in V$ with distinct labels $f(x) \rightarrow \{F_2, F_3, F_4, \dots, F_{q+1}\}$ is a label φ provided that $f^*(uv) = \left\lceil \frac{f(u)^2 + f(v)^2}{f(u)+f(v)} \right\rceil$ for q pairs of adjacent vertices u and v are distinct and the ratio between each edge and the subsequent edge conforms to the golden ratio. A φ-labeling is also known as Fibonacci range labeling. If f^* is an φ label of G, then there exists a positive integer n in $\mathbb{Z}^+$ so that for each edge $(u, v) \in E(G)$, we have $f^*(u) \leq n \leq f^*(v)$ $\forall$ $f^*(uv) \in \mathbb{N}^+$. Consequently, f^* is indeed a φ-labeling of G.

2. Product and Union of graphs

In this section we will make use of the following result to illustrate our proofs.

Definition 2.1: [12] The direct product of two graphs $\mathcal{G} \times \mathcal{H}$ is defined as a graph having vertex set $V(\mathcal{G}) \times V(\mathcal{H})$. In this product, two vertices (x, y) and (x', y') are adjacent iff the vertices x and x' are adjacent in $\mathcal{G}$ i.e., $xx' \in E(\mathcal{H})$ and y' and y' are adjacent in $\mathcal{H}$ i.e., $yy' \in E(\mathcal{G})$.

Lemma 2.1: [2] *For any two graphs* G *and* H *its direct product is given by*

$$G_{a,b} \times H_{c,d} \equiv G_{\alpha,\beta}$$

Corollary 2.1: [3] *Let (α,β) be the vertex label, then any path P_n, where $n \geq 3$ is a φ-labeling.*

Theorem 2.1: *The ladder graph $L_n = P_2 \times P_n$, for $n \geq 2$ admits a φ-labeling.*

Proof: Let L_n be the graph with v_n and u_m be the vertices of the two path joined by a single edge (see Figure 1). We define a function $f: V(G)$ defined by $f^*(uv)$ and we assign the vertex labels as $f(v_i) = F_{i+1}$ and $f(u_i) = \sum_{i=1}^{n} F_{n+(i+1)}$. Setting the edge set $f(v_i v_{i+1}) = \mathcal{M}_j^i$, $f(u_i u_{i+1}) = \mathcal{N}_j^i$, where $j = i + 1$ and $f(u_i v_j) = \mathcal{O}_j^i$.

$$\mathcal{M}_j^i = \begin{cases} \mathcal{M}_2^1 : 1 & i = 1 \\ \mathcal{M}_j^i : f(v_i) + F_{i-1} & i = 2,3,\ldots,n,m \end{cases} \tag{1}$$

Additionally, the edges between the paths are labeled

$$\mathcal{N}_j^i, \mathcal{O}_j^i = \begin{cases} \mathcal{N}_j^i : f(u_i) + F_{n+(i-1)} & i = 1,2,3,\ldots,n,m \\ \mathcal{O}_j^i : f(u_i) - f(v_i) & i = 1,2,3,4 \\ \mathcal{O}_j^i : f(u_i) - \{f(v_i) - 1\} & i = 5,6,7,\ldots,n,m \end{cases} \tag{2}$$

By observing these label, we see that the edge values are distinct, and the ratios between consecutive edge label follow the golden ratio.

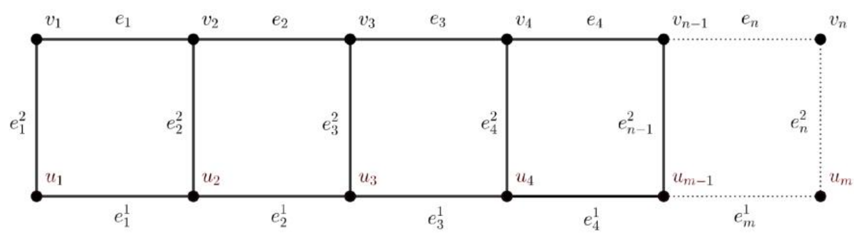

Figure 1
Generalized graph of $P_2 \times P_n$

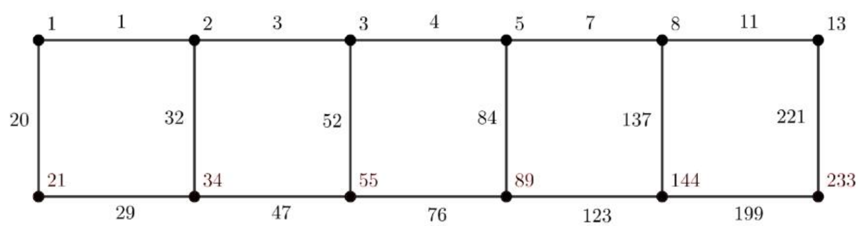

Figure 2
Fibonacci range graph of $P_2 \times P_6$

The calculated ratio of the induced edge values for $P_2 \times P_6$ (see Figure 2)

$R_t(e_i e_{i+1})$ = 3, 1.33, 1.750, 1.571,1.636, ...,

$R_t(e_i^1 e_{i+1}^1)$ = 1.617, 1.618, 1.617, 1.618, 1.618, ...,

$R_t(e_i^2 e_{i+1}^2)$ = 1.606, 1.622, 1.616, 1.625, 1.615, 1.616, ...,

For larger i, $R_t(e_i e_{i+1})$, $R_t(e_i^1 e_{i+1}^1)$, $R_t(e_i^2 e_{i+1}^2) \to \varphi$, where φ = 1.618. Hence, we conclude that the graph $L_n = P_n \times P_2$ admits a φ-labeling.

Corollary 2.2: *The product graph $P_n \times K_3$, for $n \geq 3$ is a φ-labeling.*

Corollary 2.3*:* *The product graph $P_n \times K_m$, for $n \geq 3$ and $m \geq 2$ is a φ-labeling.*

Theorem 2.2: *Let (α,β) be the vertex label, then graph $K_{1,n} \times K_{1,m}$, where $n \geq 2$ and $m \geq 2$ has a φ-labeling.*

Proof: We consider G be the graph of $K_{1,n} \times K_{1,m}$ (see Figure 3) with p vertices and q edges and let $\{u_1, u_2, u_3 \ldots, u_n, u_{n+1}, \ldots u_m\}$ be the vertices with central vertex v_1, v_2. Here v_1 are adjacent to u_i's and v_2 are adjacent to $u_{n+1} \ldots u_m$. Define the labeling function as in Theorem 2.1. Assign the vertex label $f(v_1)$ = 1, $f(v_2)$ = 2 and for 1≤ i ≤ m, label $f(u_i) = F_{i+3}$.

Then by the labeling function the edge set are labeled as

$f(v_1 v_2) = 1$ $f(v_1 u_i) = \sum_{i=1}^{n} f(u_i) - 1$, and for $f(v_2 u_i) = \sum_{i=n+1}^{m} f(u_i) - 2$.

By the above operation the edge values are distinct and follows the same.

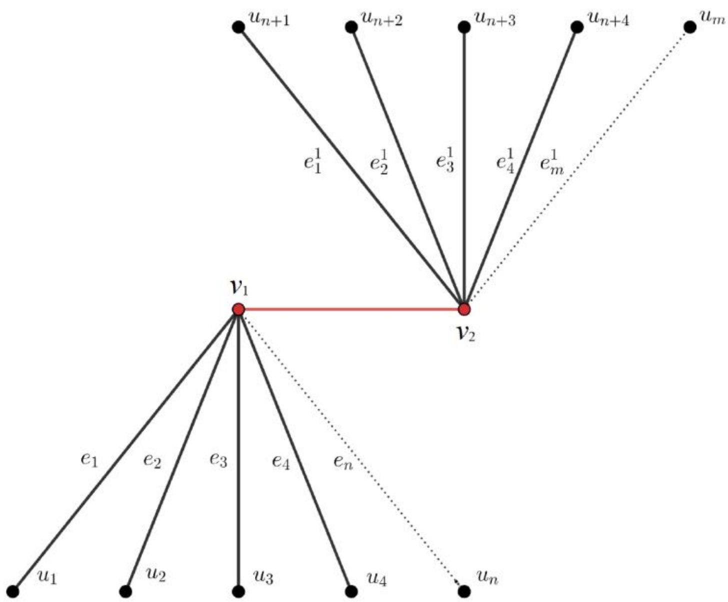

Figure 3
Generalized graph of $K_{1,n} \times K_{1,m}$

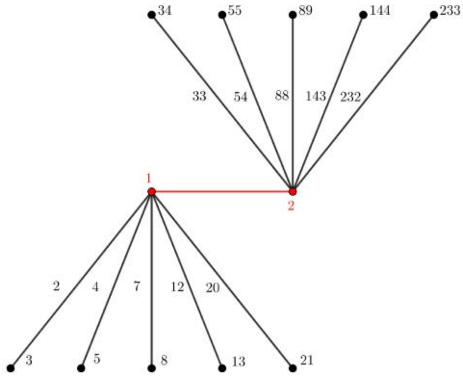

Figure 4
Fibonacci range graph of $K_{1,5} \times K_{1,5}$

The calculated value of the induced edge values for $K_{1,5} \times K_{1,5}$ (see Figure 4)

$R_t(e_i e_{i+1})$ = 2, 1.75, 1.714, 1.666, 1.650, 1.636, 1.619, 1.618, 1.618, ...,

$R_t(e_j^1 e_{j+1}^1)$ = 1.626, 1.623, 1.621, 1.620, 1.619, 1.618, 1.618, ...,

For higher order of (i, j), $R_t(e_i e_{i+1})$ or $R_t(e_j^1\ e_{j+1}^1) \to \varphi$. Hence, we conclude that the graph $K_{1,n} \times K_{1,m}$, where $n \geq 2$ and $m \geq 2$ is a φ-labeling.

Theorem 2.3: *For $m \geq 3$, $n \geq 2$, the graph $C_m \cup P_n$ admits a φ-labeling.*

Proof: Consider the cycle with $u_1, u_2, u_3, \ldots. u_m u_1$ and path $v_1, v_2, v_3 \ldots v_n$. The graph has $p = (m + n)$ vertices and $q = m + (n - 1)$ edges (Figure 5). Using the labeling function as in the case of Theorem 2.2. Label $f(u_i) = F_{i+1}$, $f(v_i) = \sum_{i=1}^{n,m} F_{m+i} + F_{m+(i-1)}$. Then by this operation we have $f(u_1u_2) = 1$, $f(u_i u_{i+1}) = \sum_{i=2}^{m-1} f(u_i) + F_{i-1}$ and it follows for $f(u_m u_1) = f(u_m) - 1$ and for $f(v_i v_{i+1}) = \sum_{i=1}^{n-1} f(v_i) + F_{m+(i-1)}$. By the above operations the edge values are distinct and follows the same.

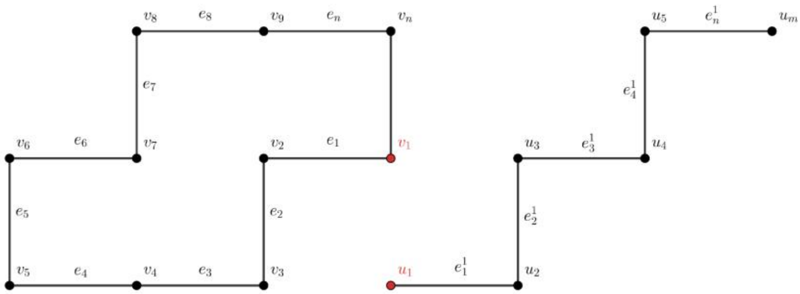

Figure 5
Generalized graph for $C_m \cup P_n$

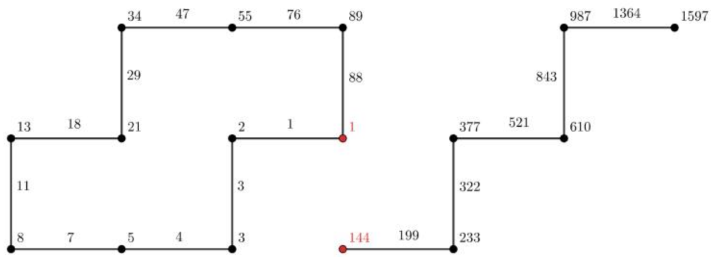

Figure 6
Fibonacci range graph for $C_{10} \cup P_6$

The calculated ratio of the induced edge values for $C_{10} \cup P_6$ (see Figure 6)

$R_t(e_i e_{i+1})$ = 3, 1.33, 1.750, 1.571,1.636, ...,

$R_t(e_j^1 e_{j+1}^1)$ = 1.620, 1.617, 1.618, 1.618, ...,

We observe from the calculated values converge to φ but restricted only for larger i. Therefore, we conclude that the graph $C_m \cup P_n$ is a φ-labeling.

Corollary 2.4: *Let (α,β) be the vertex label, then every cycle C_n, where $n \geq 3$ is a Fibonacci range labeling and every edge induced label follows the golden ratio.*

Theorem 2.4: *The graph $C_n \cup C_m$, for $m \geq 3$, $n \geq 3$ admits a φ-labeling.*

Proof: Consider $v_n v_1$ and $u_m u_1$ as the vertex set (see Figure 7) of the first cycle and second cycle C_m. Define the labeling function as in the case of **Theorem 2.3**. Label for $f(v_i) = \sum_{i=1}^{n} F_{i+1}$ and $f(u_i) = \sum_{i=1}^{m} F_{n+(i+1)}$, where $n = v_n$ of the 1^{st} cycle.

This simplifies the edge label to be distinct by the operation $f(v_1 v_2)$ = 1, and $f(v_i v_{i+1}) = \sum_{i=2}^{n-1} f(v_i) + F_{i-1}$, $f(v_n v_1) = f(v_n) - 1$. And for $f(u_i u_{i+1}) = \sum_{i=1}^{m-1} f(u_i) + F_{n+(i-1)}$, $f(u_m u_1) = f(u_m) - \{f(u_1) - 3\}$, where m, n represents the number of vertices. From the above relation the edge values are distinct and it follow the same.

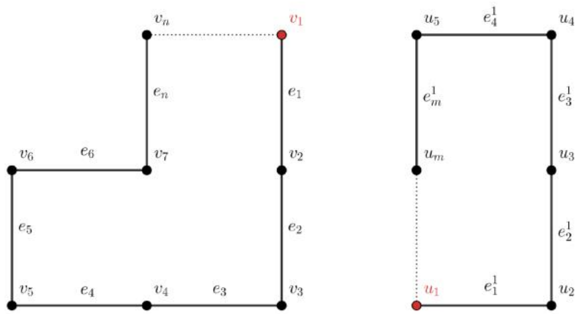

Figure 7
Generalized graph of $C_n \cup C_m$

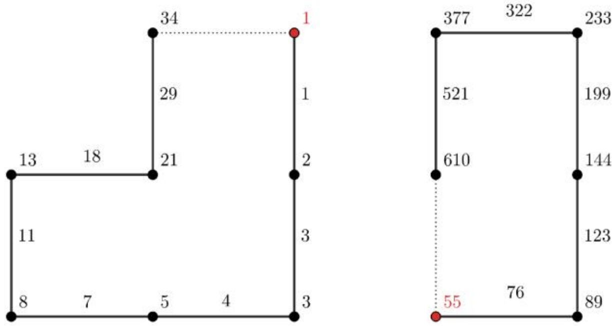

Figure 8
Fibonacci range graph of $C_8 \cup C_6$

The calculated ratio of the induced edge values for $C_8 \cup C_6$ (see Figure 8)

$R_t(e_ie_{i+1})$ = 3, 1.33 1.750 1.571 1.636 1.611, ...,

$R_t(e_j^1e_{j+1}^1)$ = 1.618, 1.617, 1.618, 1.618, 1.618, 1.618, ...,

For higher order, $R_t(e_ie_{i+1})$ or $R_t(e_j^1e_{j+1}^1)$ approaches to the value φ which determine that $C_n \cup C_m$ for $m \geq 3$, $n \geq 3$ is a φ-labeling.

Corollary 2.5: In any connected graphs to admit φ-labeling it can have at most two central vertex.

Corollary 2.6: *In any connected φ-labeling, graphs having at most two central vertex the vertex label is restricted only for 1 and 2.*

Summary

In this study, the we have proved for direct product and union of graphs based on the framework of Fibonacci range labeling. In order for graphs to admit such labeling it strongly depends only with Fibonacci numbers and it seems difficult to extend the method explicit by randomizing the vertex label having central vertex (say u_0 ≥3) then such type of graphs is a constraint in the case of Fibonacci range labeling.

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