

Distance-balancing of cube-connected cycles graphs

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Abstract

We show that the cube-connected cycles graphs CCC_n are distance-balanced via an investigation of the automorphisms of CCC_n . We then introduce a new diagrammatic way to represent minimal paths in CCC_n , termed interval-with-ends diagrams, and use this to prove that CCC_n is nicely distance-balanced if and only if n is even. We also observe that CCC_n fails to be distance-regular for all n .

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1. Introduction

The concept of distance-balanced graphs was implicitly introduced by Handa, where he conjectured that a bipartite distance-balanced graph which is not a cycle must be 3-connected [11]. However, this term itself is due to the work of Jerebic, Klavžar and Rall, who considered distance-balanced graphs in terms of different graph products [14]. Since then, much work has been done to find interesting properties of these graphs [2], [16], [18], including to see if a given family of graphs is distance-balanced or not. This line of inquiry has previously been fruitful for generalized Petersen graphs $GP(n, k)$ and Hamming graphs $H(n, q)$ [3], [15], [17]. For example, Jerebic, Klavžar and Rall originally conjectured that for any $k \geq 2$ there exists an N for which $GP(n, k)$ is not a distance-balanced graph for all $n \geq N$ [14]; this conjecture was proved by Yang et al. [22]. In addition, Frelih generalized the concept of distance-balanced graphs to ℓ -distance-balanced graphs where ℓ is a positive integer and a lower bound for the diameter of the graph [5]. Using this definition, Ma, Wang and Klavžar partially solved a conjecture of Miklavič and Sparl [20] by proving that for large n compared to k , the generalized Petersen graph $GP(n, k)$ is $\text{diam}(GP(n, k))$ -distance-balanced [19]. Interesting results have also been obtained by Ezadi for the edge-distance-balancing properties of generalized Petersen graphs, namely showing that $GP(6n + 8, 3)$ is not edge-distance-balanced for $n \geq 2$, and $GP(4n + 7, 2)$ is not edge-distanced-balanced for $n \geq 3$ [9], [10].

Our paper continues in this direction by studying a given family of graphs to see if it is distance-balanced or not, as well as an additional property, first introduced by Kutnar and Miklavič, of being nicely distance-balanced [18]. Specifically, we consider the distance-balanced and nicely distanced-balanced properties of cube-connected cycles graphs, denoted CCC_n . Cube-connected cycles are a family of graphs introduced by Preparata and Vuillemin for use as a network topology in parallel computing [21]. These graphs are 3-regular, connected and Cayley [1], and properties such as their diameter and the existence of cycles of all lengths have been studied [6], [7], [13].

In this note, we show that all cube-connected cycles graphs are distance-balanced, but nicely distanced-balanced if and only if they are derived from a hypercube in even dimensions. We do so by studying the automorphisms of the CCC_n , and developing a new diagrammatic way to represent minimal paths in CCC_n , termed *interval-with-ends diagrams*, which have potential for investigating other metric properties of CCC_n , including the enumeration of vertices equidistant from the ends of a given edge.

The structure of the paper is as follows. In Section 2 we review the definitions of our objects of interest, and also establish properties of the automorphism group of cube-connected cycles that allow us to conclude in Proposition 2.5 that CCC_n is distance-balanced for all n . In Section 3 we then introduce interval-with-ends diagrams as representations of minimal paths in CCC_n , which is then applied in Section 4 to prove our main Theorem 4.4 that CCC_n is nicely distance-balanced if and only if n is even. We conclude with a

brief Appendix A which explains why although some CCC_n are nicely distance-balanced, no CCC_n is distance-regular.

2. Distance-balancing in CCC_n

We begin with the definitions of the objects and properties of interest. Let G be a finite graph with vertices V and edges E . Given an edge $uv \in E$, let $W_{u,v} = \{x \in V : d(x,u) < d(x,v)\}$, and similarly $W_{v,u} = \{x \in V : d(x,v) < d(x,u)\}$. Also denote $W_u^v = \{x \in V : d(x,u) = d(x,v)\}$. Then G is said to be *distance-balanced* if for any edge $uv \in E$, $|W_{u,v}| = |W_{v,u}|$. G is said to be *nicely distance-balanced* if there exists a constant $k \in \mathbb{N}$ such that for any edge $uv \in E$, $|W_{u,v}| = |W_{v,u}| = k$.

The *hypercube graph* Q_n is the graph whose vertices and edges correspond combinatorially to those of the unit hypercube in $\mathbb{R}^n$. Notationally, the vertices of Q_n are binary strings of length n , and edges connect two vertices whose strings differ in just one digit. The labeling of vertices of Q_n is not unique; in particular, we have the following initial lemma which is common knowledge, but which nevertheless will be useful in what follows.

Lemma 2.1: *A choice of vertex u in Q_n as $00 \cdots 0$, along with a choice of labeling for the n vertices adjacent to u , determines the labeling of all vertices in Q_n .*

Proof: The automorphism group $\text{Aut}(Q_n)$ has order $2^n n!$ (see for example [12]), and each automorphism corresponds to a distinct labeling of the vertices of Q_n . Specifically, there are 2^n vertices in Q_n , and by Lemma 3.1.1 in [8] there is a subgroup H of $\text{Aut}(Q_n)$, where $|H| = 2^n$ and each element of H is translation by a fixed binary n -string; this establishes vertex-transitivity of Q_n , and allows us to label any vertex as $00 \cdots 0$. Moreover, the discussion following Lemma 3.1.1 in [8] identifies another subgroup K of $\text{Aut}(Q_n)$ which consists of permutations of digits in all n -strings, so that $|K| = n!$; this corresponds to the $n!$ permutations of the vertices $100 \cdots 00, 010 \cdots 00, \dots, 000 \cdots 01$ adjacent to $00 \cdots 0$. Since $H \cap K$ is just the identity element, it follows that $\text{Aut}(Q_n) = HK$, so that the choice of a vertex as $00 \cdots 0$ along with a choice of a labeling of its n neighboring vertices determines the labeling of the entire graph.

The *cube-connected cycles graph* CCC_n is the graph where each vertex of the hypercube graph Q_n is replaced by an n -cycle. We can label vertices in CCC_n by adding an extra digit to the binary strings of Q_n in order to keep track of the n vertices within each cycle. Specifically, in CCC_n vertices are $(n+1)$ -strings of the form $x_1 x_2 \dots x_n k$, where each $x_i \in \{0,1\}$, $k \in \{1, \dots, n\}$, and two vertices are equal if and only if each digit in their strings are equal. The x_i are referred to as *cube digits*, the k as *the cycle digit*. Edges are of two types: the first kind, termed a *cycle edge*, connects two vertices whose cube digits are equal but whose cycle digit differs by 1 (where we consider the cycle digits of 1 and n as differing by 1, cyclically). Thus cycle edges connect consecutive

vertices in the same cycle. The second kind of edge, termed a *cube edge*, connects two vertices whose cycle digits are equal to a common value k , and whose cube digits only differ in the k -th digit x_k . Thus cube edges connect adjacent cycles in such a way that CCC_n is 3-regular.

Important from the outset is the observation that the labeling of CCC_n is also not unique. In particular, we have the following lemma.

Lemma 2.2: *A choice of a vertex u in CCC_n as $000 \cdots 01$, along with a choice of orientation for the cycle containing u , determines the labeling of all vertices in CCC_n .*

Proof: Since CCC_n is a Cayley graph, it is vertex-transitive, so we may label any vertex u as $000 \cdots 01$. Once we then choose an orientation for the cycle containing u , that specifies the labeling of all other vertices in that cycle as $000 \cdots 02$ through $000 \cdots 0n$. Then by the cube edge relations, this determines the cube digits of all vertices in each neighboring cycle; specifically, the vertices in the cycle connected by a cube edge to $000 \cdots 0k$ will have a 1 in the k -th cube digit and 0's in the other cube digits. As a result, by Lemma 2.1, this in fact determines the cube digits for all cycles, hence cube digits for all vertices in CCC_n . Finally, given any vertex in CCC_n , it is then connected by a cube edge to another vertex, and they differ in some k -th cube digit, thus determining their cycle digit to be k ; hence all labels for all vertices in CCC_n are then fixed.

We then have two propositions which follow from Lemma 2.2.

Proposition 2.3: $|Aut(CCC_n)| = n2^{n+1}$.

Proof: The automorphisms of CCC_n are in one-to-one correspondence with distinct vertex labelings of CCC_n . Since there are $n2^n$ vertices in CCC_n , there are $n2^n$ choices for the vertex $000 \cdots 01$, and then there are 2 choices of orientation for the cycle containing that vertex. Hence by Lemma 2.2 there are $n2^{n+1}$ different vertex labelings of CCC_n and the proposition follows.

Proposition 2.4: Given an edge uv in CCC_n , there is an automorphism $\varphi \in Aut(CCC_n)$ such that $\varphi(u) = v$ and $\varphi(v) = u$.

Proof: If uv is a cube edge, then by Lemma 2.2 there is a labeling of CCC_n (actually two) which labels $u = 000 \cdots 01$ and $v = 100 \cdots 01$; but there is also a labeling of CCC_n which reverses this, labeling $u = 100 \cdots 01$ and $v = 000 \cdots 01$. Thus there is an automorphism φ which matches up vertices with the same digits under these two labelings, so that $\varphi(u) = v$ and $\varphi(v) = u$.

On the other hand, if uv is a cycle edge, then by Lemma 2.2 there is a labeling of CCC_n which labels $u = 000 \cdots 01$ and $v = 000 \cdots 0n$; but there is also a labeling of CCC_n which reverses this, labeling $u = 000 \cdots 0n$ and $v = 000 \cdots 01$. Thus again there is an automorphism φ which matches up

vertices with the same digits under these two labelings, so that $\varphi(u) = v$ and $\varphi(v) = u$.

We now have our first major result.

Proposition 2.5: CCC_n is distance-balanced for all n .

Proof: By Proposition 2.4 in [14], if G is a graph such that for any edge uv in G , there is an automorphism $\varphi \in \text{Aut}(G)$ that results in $\varphi(u) = v$ and $\varphi(v) = u$, then G is distance-balanced. By Proposition 2.4 above, CCC_n is such a graph, and hence CCC_n is distance-balanced.

Before investigating the nice distance-balancing of CCC_n , in Section 3 we establish a new method of understanding minimum paths and distance in CCC_n .

3. Interval-with-ends diagrams for paths in CCC_n

A path in CCC_n between two vertices is a sequence of cube and cycle edges. Whenever an edge is traversed, exactly one digit changes in the corresponding vertex string; if it is a cycle edge, the cycle digit will change by 1, either increasing or decreasing (in the cyclic ordering of $1, \dots, n$); and if it is a cube edge, a cube digit will change from a 1 to 0, or vice versa. More specifically, if we are at a vertex $x_1x_2 \cdots x_nk$, we only have three directions we can go, namely we can either change the cycle digit k to $k - 1$ or $k + 1$ (in the cyclic ordering of $1, \dots, n$), and traverse a cycle edge; or we can change the k -th cube digit x_k and traverse that cube edge. With this in mind, our first goal in this section is to establish a new way to represent paths in CCC_n .

By vertex-transitivity, we may focus on paths from $u = 00 \cdots 01$ to another vertex $x = x_1x_2 \cdots x_nk$. We begin with an informal discussion of a path from u to x before formalizing our observations. In order to move from $u = 00 \cdots 01$ to $x = x_1x_2 \cdots x_nk$ via a path γ , we need to examine which cube digits in x are 1's, for these will need to be changed. For example, if $x_i = 1$, then we know that γ will need to traverse cycle edges within some cycle until the cycle digit changes to i ; then the changing of the i -th cube digit from 0 to 1 will result in the traversal of a cube edge, with the result that $x_i = 1$. We then have to do this for all such cube digits in x which are 1, and then finally move within the last cycle until the cycle digit changes to k .

As a result, we can represent paths γ in CCC_n symbolically using what we call an *interval-with-ends* (IWE) diagram, with four examples shown in Figure 1 for IWE diagrams from 00000001 to 01011006 in CCC_7 . To construct an IWE diagram from $u = 00 \cdots 01$ to $x = x_1x_2 \cdots x_nk$, we do the following:

- (1) Form a loop connecting the numbers 1 through n ; these represent the cycle digits for vertices in the path.
- (2) Circle all of the numbers i where the i -th cube digit needs to be changed from 0 to 1 in moving from u to x .

- (3) Choose an interval along the loop that includes all the circled digits, has endpoints at circled digits, and has angular support less than 2π ; if there are m circled digits, we have m possible intervals.
- (4) Since we begin at u with cycle digit 1, draw an arrow from 1 to one of the endpoints of the interval, along the loop, with angular support less than 2π . We call this arrow the *initial end*; there are four possibilities for it, depending on which endpoint of the interval it connects to, and if it proceeds clockwise or counterclockwise.
- (5) Since we end at x with cycle digit k , draw an arrow from the other endpoint of the interval to k , along the loop, with angular support less than 2π . We call this arrow the *terminal end*; once the initial end is fixed, there are two possibilities for the terminal end, depending if it proceeds clockwise or counterclockwise.

We note that the interval may be an empty interval in the case where both u and x are in the same cycle in CCC_n , in which case we then have a single end connecting 1 to k as our IWE diagram.

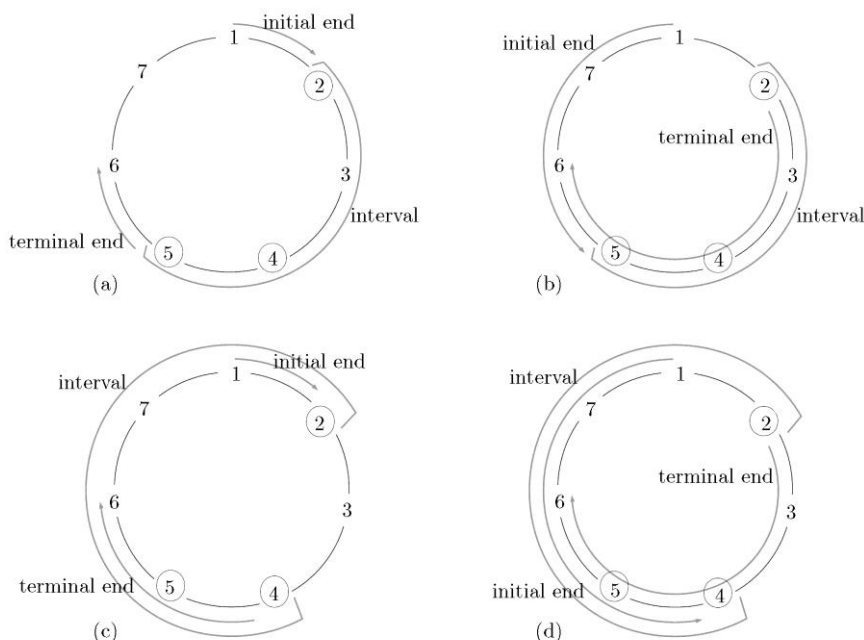


Figure 1

Four examples of interval-with-ends (IWE) diagrams from 00000001 to 01011006 in CCC_7 . Part (a) has length 8, since three circles are included in the interval, and five edges in the loop are included in the interval-with-ends; part (b) has length 13; part (c) has length 11; and part (d) has length 16.

There are multiple possible IWE diagrams from u to x ; yet for a given IWE diagram, there is a shortest path γ in CCC_n from u to x supported by that IWE diagram, namely the following: First, we proceed monotonically along the initial end from 1 to the first endpoint of the interval, and each edge we traverse in the initial end corresponds to a cycle edge in γ . Then, at the endpoint of the interval, we have a circled digit, and that circle corresponds to traversing a cube edge in γ . We then proceed monotonically along the interval, and every circle we encounter represents a cube edge in γ , with every edge in the interval a cycle edge in γ , until we reach the end of the interval with our final circle and cube edge. Then, we proceed monotonically along the terminal end to k , and every edge in the terminal end corresponds to a cycle edge in γ . Thus given an IWE diagram D , we will denote its length by $|D|$, with

$$|D| = (\# \text{ of edges in initial end}) + (\# \text{ of edges in terminal end}) \\ + (\# \text{ of edges in interval}) + (\# \text{ of circled digits})$$

As indicated in Figure 1, the length of γ for the IWE diagram in part (a) is 8, for part (b) is 13, for part (c) is 11, and for part (d) is 16. Note that if we do not move monotonically along the ends of the interval, but double back, or change cube digits other than the circled ones, this can give us other paths in CCC_n . But these will necessarily be longer than $|D|$. Furthermore, any path in CCC_n between $u = 00 \cdots 01$ and $x = x_1 x_2 \cdots x_n k$ can be mapped out in this cycle digit loop, where we record the edges traversed in the cycle digit loop, and circle the cube digits changed; although this mapping may not be an IWE itself, nevertheless an IWE diagram D will be a subset of it, and hence the length of that path will be at least $|D|$. Therefore, if $u = 00 \cdots 01$ and $x = x_1 x_2 \cdots x_n k$, we have that in CCC_n ,

$$d(u, x) = \min\{|D| : D \text{ is an IWE diagram from } u \text{ to } x\}.$$

Thus $d(u, x) = |D|$ for at least one IWE diagram D , which gives a shortest path γ in CCC_n from u to x .

We can now proceed to Section 4, where IWE diagrams will allow us to write down specific equations that must hold for vertices in W_u^v for a given edge uv ; we will then use these equations to show that CCC_n is nicely-distanced balanced if and only if n is even.

4. Nice distance-balancing in CCC_n

In order to determine under what conditions CCC_n may be nicely distance-balanced, we first consider cube edges. We recall that in CCC_n there are a total of $n2^n$ vertices.

Proposition 4.1: If uv is a cube edge in CCC_n , then $|W_{u,v}| = |W_{v,u}| = n2^{n-1}$.

Proof: We may assume that $u = 00 \cdots 01$ and $v = 10 \cdots 01$. We claim that any vertex with a 0 as its first cube digit is in $W_{u,v}$, and any vertex with a 1 as its first cube digit is in $W_{v,u}$. To see that $x = 1x_2 \cdots x_n k \in W_{v,u}$, consider an IWE diagram D from u to x such that $|D| = d(u, x)$. Then observe that since u

and v differ only in the first cube digit, an IWE diagram D' from v to x can be obtained by using D , but deleting the circle around 1. Thus $d(x, v) \leq d(u, x) - 1 < d(u, x)$, so that $x \in W_{v,u}$. An entirely similar argument shows that if $x = 0x_2 \cdots x_n k$, then $x \in W_{u,v}$, and this proves the proposition.

Remark 4.2: Since $|W_{u,v}| = |W_{v,u}| = n2^{n-1}$ we also have that $W_u^v = \emptyset$.

We now turn to cycle edges for the following proposition.

Proposition 4.3: If uv is a cycle edge in CCC_n , then $W_u^v = \emptyset$ if and only if n is even.

Proof: We may assume throughout the proof that $u = 00 \cdots 01$ and $v = 00 \cdots 0n$.

When n is odd, consider the vertex $x = 00 \cdots 0 \left(\frac{n+1}{2}\right)$. It is in the same cycle as both u and v , and so the IWE diagram D_u that realizes $d(x, u)$ is a single end that connects 1 to $\left(\frac{n+1}{2}\right)$, with length $\frac{n-1}{2}$. Similarly, the IWE diagram D_v that realizes $d(x, v)$ is a single end that connects n to $\left(\frac{n+1}{2}\right)$, with length $\frac{n-1}{2}$. Thus $d(x, u) = d(x, v)$ in CCC_n , with $x \in W_u^v$. We therefore conclude that $W_u^v \neq \emptyset$ when n is odd.

In general, we now consider an arbitrary $x \in W_u^v$ in CCC_n , and we will show that n must be odd; this will conclude the proof. To set notation, we assume that $x = x_1 x_2 \cdots x_n k$. We first consider the case where $x_i = 0$ for all $i \in \{1, \dots, n\}$, so that x lies in the same cycle as both u and v . Then the IWE diagrams realizing $d(x, u)$ and $d(x, v)$ are just single ends connecting u to x and v to x , respectively. Since $x \in W_u^v$ with $d(x, u) = d(x, v)$, we must have $d(x, u) = k - 1 = d(x, v) = n - k$, so that $n + 1 = 2k$; thus n must be odd.

We now consider the case where x does not lie in the same cycle as u and v , and we observe that the cube digits that need to be changed to 1 in moving from u to x are the same cube digits that need to be changed in moving from v to x . We fix an IWE diagram D realizing $d(x, u)$, and observe that its interval I will have endpoints at numbers i and j which represent cube digits that need to be changed. An example is shown in Figure 2, where the specific cyclic ordering of i, j, k will not enter into the argument. Likewise we only assume that of $1, i, j, k$ and n , only 1 and n are necessarily distinct.

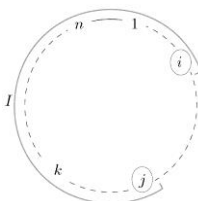


Figure 2

An interval I for a possible IWE diagram realizing $d(x, u)$.

Our first observation is that no end emanating from 1 to either i or j can proceed counterclockwise, for if it did, this IWE diagram could work for v , but with one less edge, hence resulting in $d(x, v) < d(x, u)$; but this is not true, since we are assuming $x \in W_u^v$. So any end emanating from 1 for an IWE diagram realizing $d(x, u)$ must proceed clockwise. Likewise, any end emanating from n for an IWE diagram realizing $d(x, v)$ must proceed counterclockwise.

We now consider the case where the IWE diagram D_u realizing $d(x, u)$ has the same interval as the IWE diagram D_v realizing $d(x, v)$. In this case, denote the number of edges in the initial and terminal ends for D_u , with an endpoint at 1 and k respectively, as α_u^1 and α_u^k . Likewise denote the number of edges in the initial and terminal ends for D_v , with an endpoint at n and k respectively, as β_v^n and β_v^k . Since D_u and D_v share the same interval with the same circled cube digits, and we are assuming $d(x, u) = d(x, v)$, we must have

$$\alpha_u^1 + \alpha_u^k = \beta_v^n + \beta_v^k. \quad (1)$$

Observe that each of these quantities has limited possibilities. Specifically, from the reasoning in the preceding paragraph, α_u^1 must be either $i - 1$ or $j - 1$, and likewise β_v^n must be $n - i$ or $n - j$. The possibilities for α_u^k and β_v^k are more numerous, and so we take a moment to focus on α_u^k and observe that the terminal end with endpoint at k has four options:

- if it joins k to i without passing through n , then $\alpha_u^k = |k - i|$ and $\alpha_u^1 = j - 1$;
- if it joins k to i and passes through n , then $\alpha_u^k = n - k + i$ and $\alpha_u^1 = j - 1$;
- if it joins k to j without passing through n , then $\alpha_u^k = |k - j|$ and $\alpha_u^1 = i - 1$;
- if it joins k to j and passes through n , then $\alpha_u^k = n - k + j$ and $\alpha_u^1 = i - 1$.

Likewise considering β_v^k , the options for its terminal end with endpoint at k will be:

- if it joins k to i without passing through n , then $\beta_v^k = |k - i|$ and $\beta_v^n = n - j$;
- if it joins k to i and passes through n , then $\beta_v^k = n - k + i$ and $\beta_v^n = n - j$;
- if it joins k to j without passing through n , then $\beta_v^k = |k - j|$ and $\beta_v^n = n - i$;
- if it joins k to j and passes through n , then $\beta_v^k = n - k + j$ and $\beta_v^n = n - i$.

The key observation is that regardless of which possibility occurs, the left hand side of Equation 1 will contain a -1 term, along with a $\pm i$, $\pm j$, and $\pm k$ term, and possibly one n term; likewise the right hand side of Equation 1 will

contain n , along with a $\pm i$, $\pm j$, and $\pm k$ term, and possibly one more n term. The result is that when we isolate the n 's and 1 on the right hand side, we obtain an even left hand side with two terms each of i, j and k . Therefore, in order for n to have an integer solution, n must be odd.

We conclude by considering the case where the IWE diagram D_u realizing $d(x, u)$ does not use the same interval as the IWE diagram D_v realizing $d(x, v)$. This case does not apply when there is just one circled cube digit that needs to be changed, and so from here on we may assume that $i \neq j$. For this case denote the endpoints of the interval for D_v as i' and j' , and the interval itself as I' , where we likewise necessarily have $i' \neq j'$. An example is shown in Figure 3, where the two intervals are colored in gray and black, respectively.

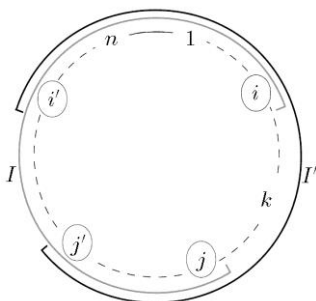


Figure 3
Intervals I and I' for possible IWE diagrams realizing $d(x, u)$ and $d(x, v)$, respectively.

We now observe that the total number of edges in I will be $n - |i - j|$, and the total number of edges in I' will be $n - |i' - j'|$. As above, denote the number of edges in the initial and terminal ends for D_u as α_u^1 and α_u^k , respectively. Likewise denote the number of edges in the initial and terminal ends for D_v as β_v^n and β_v^k , respectively. Since D_u and D_v share the same circled cube digits, and we are assuming $d(x, u) = d(x, v)$, we must have

$$n - |i - j| + \alpha_u^1 + \alpha_u^k = n - |i' - j'| + \beta_v^n + \beta_v^k$$

or

$$-|i - j| + \alpha_u^1 + \alpha_u^k = -|i' - j'| + \beta_v^n + \beta_v^k. \quad (2)$$

As above, the $\alpha_u^1 + \alpha_u^k$ terms will contain a -1 term, along with a $\pm i$, $\pm j$, and $\pm k$ term, and possibly one n term; similarly, the $\beta_v^n + \beta_v^k$ terms will contain n , along with a $\pm i'$, $\pm j'$, and $\pm k$ term, and possibly one more n term. As a result, now the left hand side of Equation 2 has two terms each of i and j , and one term of k , and the right hand side has two terms each of i' and j' , and one term of k . Moving all these terms to the left hand side, and all n 's and the 1 to the right hand side, we see the left hand side is even with two terms each of i, i', j, j' and k . Thus, again in order for n to have an integer solution, n must be odd.

Our main theorem then follows.

Theorem 4.4: CCC_n is nicely distance-balanced if and only if n is even.

Proof: We know by Proposition 4.1 that if uv is a cube edge, $|W_{u,v}| = |W_{v,u}| = n2^{n-1}$, and by Remark 4.2 we know $W_u^v = \emptyset$. But if uv is a cycle edge and n is odd, by Proposition 4.3 we know that $W_u^v \neq \emptyset$. Applying now Proposition 2.5, we must have $|W_{u,v}| = |W_{v,u}| < n2^{n-1}$. Hence for n odd, CCC_n is not nicely distance-balanced.

On the other hand, if n is even and uv is a cycle edge, then by Proposition 4.3 we know that $W_u^v = \emptyset$, so by Proposition 2.5 we must have $|W_{u,v}| = |W_{v,u}| = n2^{n-1}$. Hence for n even, CCC_n is nicely distance-balanced.

Appendix A: CCC_n are not distance-regular

For a graph G with a vertex u , we define $G_k(u) = \{w \in V \mid d(u, w) = k\}$. Recall from [4] that a graph G is *distance-regular* if given an integer $i \geq 0$, there are integers b_i, c_i such that for any ordered pair of vertices (u, v) satisfying $i = d(u, v)$, then there are precisely c_i neighbors of v in $G_{i-1}(u)$ and b_i neighbors of v in $G_{i+1}(u)$. Here we prove the following:

Proposition A.1: CCC_n is not distance-regular for any n .

Proof: As distance-regular graphs are nicely distance-balanced, Theorem 4.4 proves that the graphs CCC_n , where n is odd, cannot be distance-regular.

When n is even, we can identify an example that shows there is not a uniform c_i as prescribed by the definition of distance-regularity. Specifically, in CCC_n for n even, consider the ordered pair (u, v) of vertices where $u = 0 \cdots 00 \frac{n}{2}$ and $v = 0 \cdots 01n$. Minimal paths from u to v must first pass through $\frac{n}{2}$ cycle edges in u 's cycle until the cycle digit changes to n , and then the cube edge is traversed to change the n -th cube digit from 0 to 1. It is evident that since n is even, there are two such minimal paths, one which proceeds clockwise in the cycle from u , and the other which proceeds counterclockwise in the cycle.

The example is then evident. Observe that there is only one neighbor of v , namely $0 \cdots 00n$, which is in $G_{i-1}(u)$, adjacent along the cube edge. However, now consider the ordered pair (u, v) , but swapping the roles of u and v from the previous paragraph, so that $u = \cdots 01n$ and $v = 0 \cdots 00 \frac{n}{2}$. Then the minimal paths are the same, but there are now two neighbors of v , namely $0 \cdots 00 \left(\frac{n}{2} + 1\right)$ and $0 \cdots 00 \left(\frac{n}{2} - 1\right)$, which are in $G_{i-1}(u)$, now adjacent along two different cycle edges. Therefore there is not a uniform c_i as prescribed in the definition of distance-regularity.

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