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Congruence of order-decreasing and order-preserving on transformations with fixed point

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Abstract

Let C_n be the semigroup of all order-preserving and order-decreasing transformations on an n-element chain. Let ζ be an idempotent in C_n. The set $C_n(\zeta)$ of all $\alpha \in C_n$ such that the image of ζ is the set of all fixed points in α is a maximal nilpotent subsemigroup of C_n. For this semigroup, the cardinality and the rank have already determined by Yağci and Kormaz. In the present paper, we characterize all ideals and the maximal as well as minimal congruences on $C_n(\zeta)$. Moreover, we consider a class of congruences different from the Rees congruences.

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1. Introduction

Let T_n be the symmetric semigroup on the n-element set $X_n = \{1, 2, \ldots, n\}$. Under its natural order $\leq$, X_n forms a chain. A transformation $\alpha \in T_n$ is called *order-preserving* if $x < y$ implies $x\alpha \leq y\alpha$, for all $x, y \in X_n$, and order-decreasing if $x\alpha \leq x$ for all $x \in X_n$. The monoid of all order-preserving transformations in T_n is denoted by O_n, the monoid of all order-decreasing transformations is denoted by D_n. Both monoids O_n and D_n are already well-studied (see for examples [1, 2, 3]). The monoid rank of O_n is n [4] and the rank of D_n is $S(n, n-1)$, where $S(n,k)$ is the Stirling number of the second kind [5]. Congruences on O_n and D_n were already studied. It was proved by Aızenštat [6], and later by Lavers and Solomon [7], that the congruences of O_n are exactly the Rees congruences. An explicit characterisation of the minimum semilattice congruence on D_n is given by Umar [5]. In contrast with O_n, the monoid D_n is not regular. Also, combinatorial considerations were done for both monoids O_n and D_n. The cardinality of O_n is $\binom{2n-1}{n-1}$ [8] and $|D_n| = n!$ [5]. O_n is generated by the set E_1 of its idempotents of defect 1.

In the present paper, we study submonoids of the monoid $C_n = O_n \cap D_n$. The set of all fixed points in $\alpha \in T_n$ is defined by $Fix(\alpha) = \{x \in X_n : x\alpha = x\}$. An element e of a semigroup S is called *idempotent* if $e^2 = e$ and the set of all idempotents in S is denoted by $E(S)$. It is literally immediate that for $\alpha \in T_n$, α is an idempotent if and only if $Fix(\alpha) = im(\alpha) = \{x\alpha : x \in X_n\}$ ($im(\alpha)$ is called *the image* (*range*) of α). Moreover, an element a of a semigroup S with zero, denoted by 0, is called *m-nilpotent* (for short: nilpotent) if $a^m = 0$ for some positive integer m. The set of all nilpotents of S is denoted by $N(S)$. A semigroup S with zero is called *nilpotent* if there exists a positive integer m such that $S^m = 0$ (this is equivalent to that each element in S is nilpotent). It is well-known that $\alpha_0 \in T_n$ with $im(\alpha_0) = \{1\}$, the constant map to 1, is the zero in C_n. An $\alpha \in C_n$ is nilpotent if and only if $Fix(\alpha) = \{1\}$, i.e., C_n does not be nilpotent. The monoid C_n is well studied (see for examples [9, 10, 11]). The rank of C_n is $n-1$ [12] and the cardinality of C_n is the n^{th} Catalan number $\mathcal{C}_n$ (see for example [13]). C_n is also called Catalan monoid with the identity mapping id_{X_n}. There are many applications of C_n, especially in computer science. For example, it is given a specific connection between C_n and the

set of all binary trees on n source nodes (see for example in [8]). The ideals of C_n are studied in [14]. In [12], the authors give a formula for the nilpotent elements in C_n.

In [12], the authors study algebraic properties of nilpotent subsemigroups of C_n. Let $\zeta \in E(C_n)$ with $im(\zeta) = \{x_1 < x_2 < \ldots < x_k\}$ for $k = rank\zeta$ (note that $rank\zeta = |im(\zeta)|$). Then $x\zeta = x_i$ if $x \in \{x_i, x_i+1, \ldots, x_{i+1}-1\}$ for some $i \in \{1,2,\ldots,k\}$, where $x_{k+1} = n+1$. Conversely, for each set $Y \subseteq X_n$, there is a unique determined $\zeta_Y \in E(C_n)$ with $im(\zeta_Y) = Y$ and we put $C_{n,Y} = \{\alpha \in C_n : Fix(\alpha) = Y\}$. Clearly, $C_{n,Y}$ is a nilpotent semigroup with zero ζ_Y. Moreover, $C_{n,Y}$ is a maximal nilpotent subsemigroup of C_n. Note that the maximal nilpotent subsemigroups are the semigroups of the form $C_{n,Y}$, $Y \subseteq X_n$. In particular, $C_n = C_{n,\{1\}}$. We will study the semigroup $C_{n,Y}$ in the present paper. For this, we fix a set $Y = \{1 = x_1 < x_2 < \ldots < x_k\} \subseteq X_n$ for some $k \leq n$. In Theorem 3 in [12], the authors calculate the cardinality of $C_{n,Y}$, it is $\prod_{i=1}^{k} C_{(x_{i+1}-x_i)-1}$, give $\{\alpha \in C_{n,Y}$: there is $a \in X_n \setminus Y^+$ such that $a\alpha = a$–$1\}$, where $Y^+ = Y \cup \{x_1+1, x_2+1, \ldots, x_k+1\}$, as a minimal generating set for the case $X_n \setminus Y^+ \neq \varnothing$ and calculate the rank of $C_{n,Y}$, it is $(\prod_{i=1}^{k} C_{x_i-1})$ $(\prod_{i=1}^{k} C_{x_i-2})$. But there is no information about the ideals and congruences on $C_{n,Y}$. This gap will be filled by the present paper. We observe that $X_n \setminus Y^+ = \varnothing$ implies $C_{n,Y} = \{\zeta_Y\}$. Therefore, we restrict us for the case $X_n \setminus Y^+ \neq \varnothing$ in this paper. Note that each ideal I in a semigroup induces a congruence ρ_I by $(a,b) \in \rho_I$ if and only if $a,b \in I$ or $a = b$. Such a congruence is called *Rees congruence*. We will show that there are congruences different from the Rees congruences on $C_{n,Y}$. We characterise all ideals of $C_{n,Y}$ and determine the maximal as well as the minimal congruences. Moreover, we consider a semilattice of ideals and the congruences related to the corresponding Rees congruences.

2. The congruences

In this section, we characterize all maximal as well as all minimal congruences on $C_{n,Y}$. A congruence ρ is called *maximal* if for all congruences $\hat{\rho}$ on $C_{n,Y}$, it holds if $\rho \subseteq \hat{\rho}$ and $\rho \neq \hat{\rho}$ then $\hat{\rho}$ is the trivial congruence $C_{n,Y} \times C_{n,Y}$, ρ is called *minimal* if for all congruences $\hat{\rho}$ on $C_{n,Y}$, it holds if $\hat{\rho} \subseteq \rho$ and $\rho \neq \hat{\rho}$ then $\hat{\rho}$ is the trivial congruence $\{(a,a)$: $a \in C_{n,Y}\} = \Delta$. In order to determine all Rees congruences on $C_{n,Y}$, we

determine all ideals on $C_{n,Y}$. Recall that $(x+1)\alpha = x$ for all $x \in Y$, whenever $\alpha \in C_{n,Y}$. Further, $G_{n,Y}$ = $\{\alpha \in C_{n,Y}$: there is $a \in X_n \setminus \overline{Y^+}$ such that $a\alpha = a-1\}$ is a generating set of $C_{n,Y}$ [12] and we put $C^*_{n,Y} = \overline{G_{n,Y}} = \{\alpha \in C_{n,Y}$: $a - a\alpha \geq 2$ for all $a \in X_n \setminus Y^+\}$.

Proposition 2.1: Let $\rho \subseteq C_{n,Y} \times C_{n,Y}$ be an equivalence relation. Then ρ is a maximal congruence on $C_{n,Y}$ if and only if $C^*_{n,Y} \times C^*_{n,Y} \subseteq \rho$ and ρ corresponds to a partition in two blocks.

Proof: Suppose that $C^*_{n,Y} \times C^*_{n,Y} \subseteq \rho$ and ρ corresponds to a partition in two blocks. Let $\alpha_1, \alpha_2 \in C_{n,Y}$ and let $x \in X_n \setminus Y^+$. Then $x - x\alpha_1 \geq 2$ or $x - x\alpha_1 = 1$. If $x - x\alpha_1 = 1$ then $x\alpha_1 \notin Y$, i.e., $x\alpha_1 - x\alpha_1\alpha_2 \geq 1$ and $x - x\alpha_1\alpha_2 = (x - x\alpha_1) + (x\alpha_1 - x\alpha_1\alpha_2) \geq 1+1 = 2$. If $x - x\alpha_1 \geq 2$ then $x\alpha_1 - x\alpha_1\alpha_2 \geq 0$. This implies $(x\alpha_1 - x\alpha_1\alpha_2) + (x - x\alpha_1) \geq 0 + 2$, i.e., $x - x\alpha_1\alpha_2 \geq 2$. This shows $\alpha_1\alpha_2 \in C^*_{n,Y}$. Then for all $(\beta_1, \beta_2) \in \rho$ and for all $\alpha \in C_{n,Y}$, it holds $(\beta_1\alpha, \beta_2\alpha), (\alpha\beta_1, \alpha\beta_2) \in C^*_{n,Y} \times C^*_{n,Y} \subseteq \rho$. This shows that ρ is a congruence which is maximal since ρ corresponds to a partition in two blocks. Conversely, suppose that ρ is a maximal congruence on $C_{n,Y}$. Assume, for all $\alpha \in G_{n,Y}$, there is $\hat{\alpha} \in C^*_{n,Y}$ with $(\alpha, \hat{\alpha}) \in \rho$. Let now $\delta \in C^*_{n,Y} \setminus \{\zeta_Y\}$. Further, let $\alpha_{max} \in G_{n,Y}$ be the unique transformation defined by $x\alpha_{max} = x - 1$ for all $x \in X_n \setminus Y^+$. Then it is easy to verify that there is $\gamma \in C_{n,Y}$ such that $\gamma\alpha_{\max} = \delta$, i.e., $(\delta, \gamma\hat{\alpha}_{max}) \in \rho$. Note that $\gamma\hat{\alpha}_{max} = \{\zeta_Y\}$ or $x\delta \geq x\gamma\hat{\alpha}_{max}$ for all $x \in \{1, 2, \ldots, n\}$ and $y\delta > y\gamma\hat{\alpha}_{max}$ for some $y \in \{1, 2, \ldots, n\}$ as well as $\gamma\hat{\alpha}_{max} \in C^*_{n,Y}$. Continuing that way, we obtain (δ_0, δ_1), (δ_1, δ_2), …, $(\delta_p, \delta_{p+1}) \in \rho$ with $\delta_0 = \delta$ and $\delta_{p+1} = \zeta_Y$. This implies $(\delta, \zeta_Y) \in \rho$. Consequently, $C^*_{n,Y} \times C^*_{n,Y} \subseteq \rho$. Since for each $\alpha \in G_{n,Y}$, there is $\hat{\alpha} \in C^*_{n,Y}$ with $(\alpha, \hat{\alpha}) \in \rho$, we can conclude that $\rho = C_{n,Y} \times C_{n,Y}$, a contradiction. This shows that there is $\alpha \in G_{n,Y}$ with $(\alpha, \beta) \notin \rho$ for all $\beta \in C^*_{n,Y}$. Then it is easy to see that there is an equivalence relation $\hat{\rho} \subseteq C_{n,Y} \times C_{n,Y}$ with $C^*_{n,Y} \times C^*_{n,Y} \subseteq \hat{\rho}$ and $\hat{\rho}$ corresponds to a partition in two blocks such that $\rho \subseteq \hat{\rho}$. Since $\hat{\rho}$ is a congruence, as we have already shown in the first part of the proof, and since ρ is maximal, we have $\rho = \hat{\rho}$.

Let $X^*_k = \{r \in \{1, 2, 3, \ldots, k\} : x_r + 2 < x_{r+1}\}$. For $J \subseteq X^*_k$, we define a transformation α_J by $x\alpha_J = x_l$, whenever $x \in \{x_l, x_l + 1, \ldots, x_{l+1} - 2\}$, for some $l \in \{1, 2, \ldots, k\}$, and

$$(x_{l+1}-1)\alpha_J = \begin{cases} x_l & \text{if } l \notin J \\ x_l+1 & \text{if } l \in J \end{cases}$$

for $l \in \{1,2,\ldots,k\}$. It is clear that $\alpha_J \in C_{n,Y}$. Further, we put ρ_J as the congruence generated by $(\alpha_J, \alpha_\varnothing)$. Note that $\alpha_\varnothing = \zeta_Y$. It is easy to verify that $\beta\alpha_J = \alpha_J\beta = \alpha_\varnothing$ for all $\beta \in C_{n,Y}$. So, we have $\rho_J = \Delta \cup \{(\alpha_J, \alpha_\varnothing), (\alpha_\varnothing, \alpha_J)\}$ for $J \subseteq X_k^*$.

Proposition 2.2: $\{\rho_J : \varnothing \neq J \subseteq X_k^*\}$ is the set of all minimal congruences on $C_{n,Y}$.

Proof: Of course, ρ_J is minimal for $J \subseteq X_k^*$. Suppose that ρ is minimal congruence on $C_{n,Y}$. Assume that there is $(\alpha_1, \alpha_2) \in \rho \setminus \Delta$ with $(\alpha_1, \alpha_2) \notin \{(\alpha_\varnothing, \alpha_J), (\alpha_J, \alpha_\varnothing) : \varnothing \neq J \subseteq X_k^*\}$. Suppose there is $l \in X_k^*$ and $a \in \{x_l, x_l+1, \ldots, x_{l+1}-2\}$ such that $a\alpha_1 \neq a\alpha_2$. Without loss of generality, let $a\alpha_1 < a\alpha_2$. Then we define $\gamma \in C_{n,Y}$ by $x\gamma = x_i$, whenever $x \in \{x_i, x_i+1, \ldots, x_{i+1}-1\} \setminus \{x_{l+1}-1\}$ and $i \in \{1, 2, \ldots, k\}$ and $(x_{l+1}-1)\gamma = a$. It is easy to verify that $\gamma \in C_{n,Y}$. Then $(\gamma\alpha_1, \gamma\alpha_2) \in \rho$, where $(x_{l+1}-1)\gamma\alpha_1 = a\alpha_1 < a\alpha_2 = (x_{l+1}-1)\gamma\alpha_2$. If $a\alpha_2 = x_l + 1$ then $\gamma\alpha_1 = \alpha_\varnothing$ and $\gamma\alpha_2 = \alpha_{\{l\}}$, where $\{l\} \subseteq X_k^*$. Thus, $(\alpha_\varnothing, \alpha_{\{l\}}) \in \rho$. If $a\alpha_2 > x_l + 1$, then there is $\beta \in C_{n,Y}$ such that $\gamma\alpha_1\beta = \alpha_\varnothing$ and $\gamma\alpha_2\beta = \alpha_{\{l\}}$, i.e., $(\alpha_\varnothing, \alpha_{\{l\}}) \in \rho$. Consequently, $\rho_{\{l\}} \subsetneq \rho$, which contradicts the minimality of ρ. Suppose now that $x\alpha_1 = x\alpha_2$ for all $x \in \{1, 2, \ldots, n\} \setminus \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\}$. Since $(\alpha_1, \alpha_2) \notin \{(\alpha_\varnothing, \alpha_J), (\alpha_J, \alpha_\varnothing) : \varnothing \neq J \subseteq X_k^*\}$, there is $l \in X_k^*$ and $a \in \{x_l + 2, x_l + 3, \ldots, x_{l+1} - 1\}$ such that $a\alpha_1 \neq a\alpha_2$ and $\{a\alpha_1, a\alpha_2\} \neq \{x_l, x_l + 1\}$. Without loss generality, let $a\alpha_1 < a\alpha_2$. Then it is easy to verify that there is $\beta \in C_{n,Y}$ such that $\alpha_1\beta = \alpha_\varnothing$ and $\alpha_2\beta = \alpha_{\{l\}}$ and we obtain again a contradiction to the minimality of ρ. Hence, only elements of the form (α_0, α_J), $\varnothing \neq J \subseteq X_k^*$, belong to $\rho \setminus \Delta$. But the minimality of ρ provides that $\rho = \rho_J$ for some $\varnothing \neq J \subseteq X_k^*$.

Proposition 2.2 shows that there are exactly $2^{|X_k^*|} - 1$ minimal congruences on $C_{n,Y}$. We will introduce a partial order on $C_{n,Y}$, in order to characterize the ideals on $C_{n,Y}$. We recall that a transformation $\alpha \in T_n$ can also be written as $[a_1, a_2, \ldots, a_n]$, meaning $a_i = i\alpha$ for $i \in X_n$. If it is comfortable, we will use this notation. We can characterize the elements $[a_1, a_2, \ldots, a_n] \in C_{n,Y}$ as follows: $[a_1, a_2, \ldots, a_n] \in C_{n,Y}$ if and only if

(i) $a_i = i$ for all $i \in Y$;

(ii) $a_i < i$ for all $i \in X_n \setminus Y$;

(iii) $a_i \le a_{i+1}$ for all $i \in X_n \setminus \{n\}$.

Definition 1: Let $>$ be the relation on $C_{n,Y}$ such that for all $[a_1, a_2, \ldots, a_n], [b_1, b_2, \ldots, b_n] \in C_{n,Y}$ hold: $[a_1, a_2, \ldots, a_n] > [b_1, b_2, \ldots, b_n]$ if there are $p_i, q_i \in \{0,1, \ldots, n-2\}$ such that $b_i + p_i = a_{i-q_i}$ for all $i \in X_n$ with the following four properties:

(i) If there is $i \in X_n \setminus Y$ with $q_i = 0$ then $\{q_1, q_2, \ldots, q_n\} = \{0\}$.

(ii) If there is $i \in X_n \setminus Y$ with $p_i = 0$ and $a_i \notin Y$ then $\{p_1, p_2, \ldots, p_n\} = \{0\}$.

(iii) If $i, j \in X_n$ with $a_{i-q_i} = a_{j-q_j}$ then $p_i = p_j$.

(iv) If $i, j \in X_n$, then $i - q_i \le j - q_j$ if and only if $i \le j$.

Note that (iv) implies that if $a_{i-q_i} < a_{j-q_j}$ then $b_i \le b_j$. In fact, assume that $b_i > b_j$. Then $i > j$ and $i - q_i \ge j - q_j$, i.e., $a_{i-q_i} \ge a_{j-q_j}$, a contradiction. We will now characterize $\ge$, which will give us a tool for the characterization of all ideals of $C_{n,Y}$.

Lemma 2.1: *Let* $\alpha, \beta \in C_{n,Y}$. *Then there are* $\gamma_1, \gamma_2 \in C_{n,Y} \cup \{id_{X_n}\}$ *with* $\gamma_1 \alpha \gamma_2 = \beta$ *if and only if* $\alpha \ge \beta$.

Proof: Let $\alpha = [a_1, a_2, \ldots, a_n]$ and $\beta = [b_1, b_2, \ldots, b_n]$. Suppose that $\alpha \ge \beta$. Then there are $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \in \{0,1, \ldots, n-2\}$ such that $b_i + p_i = a_{i-q_i}$ for all $i \in X_n$ satisfying (i)-(iv) in Definition 1. We put $\gamma = id_{X_n}$ if there is $i \in X_n \setminus Y$ with $q_i = 0$ and we put $\delta = id_{X_n}$ if there is $i \in X_n \setminus Y$ with $a_{i-q_i} \notin Y$ and $p_i = 0$. Suppose that $q_i \ne 0$ for all $i \in X_n \setminus Y$. Then we define γ by

$$x\gamma = \begin{cases} x_i & \text{if } x = x_i \text{ for some } i \in \{1,2,\ldots,k\}; \\ x - q_x & \text{otherwise.} \end{cases}$$

If $x \in \{x_i + 1, x_i + 2, \ldots, x_{i+1} - 1\}$ for some $i \in \{1, 2, \ldots, k\}$, then $x_i \le b_x + p_x = a_{x-q_x} < x_{i+1}$, i.e., $x_i \le x - q_x < x_{i+1}$. Further, let $x < y \in \{x_i + 2, \ldots, x_{i+1} - 1\}$. Then $b_x < b_y$. This implies $a_{x-q_x} \le a_{y-q_y}$ by (iv) and thus, γ is order-preserving. Clearly, $Fix(\gamma) = Y$ and γ is order-decreasing. Thus $\gamma \in C_{n,Y}$. Suppose that $p_i \ne 0$ for all $i \in X_n \setminus Y$ with $a_{i-q_i} \notin Y$. Then we put $(a_{x-q_x})\delta^* = a_{x-q_x} - p_x$ for all $x \in X_n \setminus Y$ with $a_{x-q_x} \notin Y$ and $(a_{x-q_x})\delta^* = x_i$ for $x \in X_n$ with $a_{x-q_x} = x_i$ for some $i \in \{1, 2, \ldots, k\}$. Because of (iii), δ^* is well defined.

Note that $a_{x-q_x} = x_i$ for some $i \in \{1, 2, \ldots, k\}$ implies $p_x = 0$ because $x_i \le b_x \le b_x + p_x = a_{x-q_x} = x_i$. Then this, together with (iii) and (iv), gives that δ^* can be extended to a transformation $\delta \in C_{n,Y}$. Let $x \in Y$. Then $x\gamma\alpha\delta = x = x\beta$. Let $x = x_i + 1$ for some $i \in \{1, 2, \ldots, k\}$ then $x\gamma\alpha\delta = x_i\alpha\delta = x_i = x\beta$. Let $x \in X_n \setminus Y^+$. Then $x\gamma\alpha\delta = (x - q_x)\alpha\delta = (a_{x-q_x})\delta$. If $a_{x-q_x} \notin Y$ then $(a_{x-q_x})\delta = a_{x-q_x} - p_x = b_x = x\beta$. If $a_{x-q_x} = x_i$ for some $i \in \{1, 2, \ldots, k\}$ then $x\beta = a_{x-q_x} - p_x = (a_{x-q_x})\delta$. This shows that $\beta = \gamma\alpha\delta$. Suppose that there are γ, $\delta \in C_{n,Y} \cup \{id_{X_n}\}$ such that $\beta = \gamma\alpha\delta$. Let $i \in X_n$. Then $i\gamma\alpha\delta = i\beta$ and we put $q_i = i - i\gamma$ and $p_i = a_{i-q_i} - (a_{i-q_i})\delta$. We can calculate that $b_i = i\beta = i\gamma\alpha\delta = (i - q_i)\alpha\delta = (a_{i-q_i})\delta = a_{i-q_i} - p_i$, i.e., $a_{i-q_i} = b_i + p_i$. Now we show that (i)-(iv) are satisfied: (i) Suppose there is $i \in X_n \setminus Y$ with $q_i = 0$. Then $i = i\gamma$, i.e., $\gamma = id_{X_n}$. (ii) Suppose there is $i \in X_n \setminus Y$ with $a_{i-q_i} \notin Y$ such that $p_i = 0$. Then $a_{i-q_i} = (a_{i-q_i})\delta$, i.e., $\delta = id_{X_n}$. (iii) Suppose there are $i, j \in X_n$ with $a_{i-q_i} = a_{j-q_j}$. Then $(a_{i-q_i})\delta = (a_{j-q_j})\delta$, i.e., $p_i = p_j$. (iv) Let $i < j \in X_n$. If $i - p_i < j - p_j$ then we have $(a_{i-q_i})\delta \le (a_{j-q_j})\delta$ and $b_i = a_{i-q_i} - p_i = a_{i-q_i} - a_{i-q_i} + (a_{i-q_i})\delta = (a_{i-q_i})\delta \le (a_{j-q_j})\delta = a_{j-q_j} - p_j = b_j$, i.e., $i \le j$. Dually, we obtain if $a_{j-q_j} < a_{i-q_i}$ then $b_j \le b_i$. Together with (iii), we obtain (iv).

Lemma 2.2: $\le$ *is a partial order on* $C_{n,Y} \cup \{id_{X_n}\}$.

Proof: Let $\alpha \in C_{n,Y}$. Then $\alpha = id_{X_n}\,\alpha id_{X_n}$ and we get $\alpha \le \alpha$ by Lemma 2.1. Let $\alpha, \beta \in C_{n,Y}$ with $\alpha \le \beta$ and $\beta \le \alpha$. Then there are $\gamma_1, \delta_1, \gamma_2, \delta_2 \in C_{n,Y} \cup \{id_{X_n}\}$ such that $\alpha = \gamma_1\beta\delta_1$ and $\beta = \gamma_2\alpha\delta_2 = \gamma_2\gamma_1\alpha\delta_1\delta_2$ by Lemma 2.1. This implies $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = id_{X_n}$, i.e., $\alpha = \beta$. Let $\alpha, \beta, \gamma \in C_{n,Y}$ with $\alpha \le \beta$ and $\beta \le \gamma$. Then there are $\gamma_1, \delta_1, \gamma_2, \delta_2 \in C_{n,Y} \cup \{id_{X_n}\}$ such that $\beta = \gamma_1\gamma\delta_1$ and $\alpha = \gamma_2\beta\delta_2$. Then $\alpha = \gamma_2\gamma_1\gamma\delta_1\delta_2$, i.e., $\alpha \le \gamma$.

Let $P_{n,Y}$ be the set of all non-empty sets $S \subseteq C_{n,Y}$ with $\alpha \nparallel \beta$ for all $\alpha, \beta \in S$ with $\alpha \ne \beta$. Note that $\alpha \nparallel \beta$ means that α and β are not related with respect to $\le$. We will point out that there is a one-to-one corresponds between the ideals of $C_{n,Y}$ and the set $P_{n,Y}$. For $S \in P_{n,Y}$, let

$$(S] = \{\beta \in C_{n,Y} : \alpha > \beta \text{ for some } \alpha \in S\}.$$

Proposition 2.3: $\{(S] : S \in P_{n,Y}\}$ *is the set of all ideals of* $C_{n,Y}$ *and* $|\{(S] : S \in P_{n,Y}\}| = |P_{n,Y}|$.

Proof: Let $S \in P_{n,Y}$. We show that $(S]$ is an ideal. Let $\gamma_1, \gamma_2 \in C_{n,Y} \cup \{id_{X_n}\}$ and let $\alpha \in (S]$. Then there is $\beta \in S$ with $\beta \geq \alpha$. By Lemma 2.1, we obtain $\alpha \geq \gamma_1 \alpha \gamma_2$. Hence, $\beta \geq \gamma_1 \alpha \gamma_2$, i.e., $\gamma_1 \alpha \gamma_2 \in (S]$. Consequently, $(S]$ is an ideal.

Conversely, let I be an ideal. Then it is easy to verify that there is a set $I_0 \subseteq I$ with $I_0 \in P_{n,Y}$ and $I \subseteq [I_0)$. Let now $\alpha \in [I_0)$. Then there is $\beta \in I_0$ with $\beta \geq \alpha$, i.e., there are $\gamma_1, \gamma_2 \in C_{n,Y} \cup \{id_{X_n}\}$ such that $\gamma_1 \beta \gamma_2 = \alpha$, i.e., $\alpha \in I$ since I is an ideal. Thus, $[I_0) \subseteq I$ and we can conclude $I = [I_0)$. This shows that $\{(S] : S \in P_{n,Y}\}$ is the set of all ideals of $C_{n,Y}$.

It remains to show that $S_1, S_2 \in P_{n,Y}$ and $S_1 \neq S_2$ implies $[S_1) \neq [S_2)$. Let $S_1 \neq S_2 \in P_{n,Y}$. Without loss of generality there is $\eta \in S_1 \setminus S_2$. If $\eta \notin [S_2)$ the proof is completed. Suppose that $\eta \in [S_2)$. Then there is $\omega \in S_2$ with $\omega > \eta$. Assume that $\omega \in [S_1)$. Then there is $\gamma \in S_1$ with $\gamma \geq \omega$, i.e., $\gamma \geq \omega > \eta$. This implies $\eta \notin S_1 \in P_{n,Y}$, a contradiction. Hence, $\omega \in [S_2) \setminus [S_1)$, which shows that $[S_2) \neq [S_1)$.

It is easy to verify that ζ_Y is the least element in the partially ordered set $(C_{n,Y}, \leq)$. But all elements in $G_{n,Y}$ are maximal. In fact, for $\beta \in G_{n,Y}$, there is $i \in X_n \setminus Y^+$ such that $i\beta = i-1$. Assume, there is $\alpha \in C_{n,Y}$ with $\alpha > \beta$. Then there are $p_i, q_i \in \{0,1,2, \ldots, n-2\}$. such that $i\beta + p_i = (i - q_i)\alpha$. This implies $i - 1 + p_i = (i - q_i)\alpha < i - q_i$, i.e., $p_i + q_i < 1$ and thus $p_i = q_i = 0$. Hence, $\beta = \alpha$, a contradiction.

3. Semilattice of ideals

In this section, we will consider a set of ideals which forms a semilattice. For this, we consider, for each $i \in \{1, 2, \ldots, k\}$, a set $Y_i = \{x_i, x_i + 1, x_i + 2, \ldots, x_{i+1} - 2\}$ and define, for all $z = (z_1, z_2, \ldots, z_k) \in Y_1 \times Y_2 \times \ldots \times Y_k$, a set $A_z = \bigcup_{i=1}^{k} \{z_i + 1, z_i + 2, \ldots, x_{i+1} - 1\}$. Note that $\overline{A_z}$ means the set $X_n \setminus A_z$, the complement of A_z. First, we will verify that $C_{n,Y}^z = C_{n,Y} \cap T(X_n, \overline{A_z})$ is an ideal, where $T(X, \overline{A_z})$ is the semigroup of all transformations on X_n with restricted range $\overline{A_z}$ (see [15]). Moreover, we will show that $\{C_{n,Y}^z : z \in Y_1 \times Y_2 \times \ldots \times Y_n\}$ forms a semilattice of ideals under the usual set inclusion. Clearly, $C_{n,Y}^z$ is a semigroup. In Section 2, Proposition 2.3, we have characterised all ideals of $C_{n,Y}$ as sets of the form $(S]$, where $S \in P_{n,Y}$.

In order to show that $C^z_{n,Y}$ is an ideal, we have to show that $C^z_{n,Y} = (J]$ for some $J \in P_{n,Y}$.

Proposition 3.1: Let $z \in Y_1 \times Y_2 \times \ldots \times Y_k$ and let $G_z = \{[a_1, a_2, \ldots, a_n] \in C^z_{n,Y} : i - a_i = 1$ for some $i \in A_z \setminus \{x_1 + 1, x_2 + 1, \ldots, x_k + 1\}\}$. Then $(G_z] = C^z_{n,Y}$.

Proof: First, we show that $G_z \in P_{n,Y}$. Since $X_n \setminus Y^+ \neq \emptyset$, we get that $G_z \neq \emptyset$. Now, let $\alpha, \beta \in G_z$ with $\alpha \neq \beta$. Assume that $\alpha > \beta$. Then there are $\gamma, \delta \in C_{n,Y} \cup \{id_{X_n}\}$ with $\gamma \neq id_{X_n}$ or $\delta \neq id_{X_n}$ such that $\beta = \gamma\alpha\delta$. Suppose that $\alpha = [a_1, a_2, \ldots, a_n]$ and $\beta = [b_1, b_2, \ldots, b_n]$. Then there is $i \in \overline{A_z} \setminus \{x_1 + 1, x_2 + 1, \ldots, x_k + 1\}$ such that $b_i = i - 1$. On the other hand, we have $b_i = i\beta = i\gamma\alpha\delta \leq i - 2$, a contradiction. Similarly, we can show that $\beta > \alpha$ is impossible. Hence $G_z \in P_{n,Y}$. Let $\alpha \in [G_z)$. Then there are $\beta \in G_z$ and $\gamma, \delta \in C_{n,Y} \cup \{id_{X_n}\}$ such that $\alpha = \gamma\beta\delta$. Let $x \in X_n$. Then $x\alpha = x\gamma\beta\delta \leq x\gamma\beta$, where $x\gamma\beta \in \overline{A_z}$ since $\beta \in C^z_{n,Y} \subseteq T(X_n, \overline{A_z})$. This provides $x\alpha \in \overline{A_z}$. Consequently, we have shown $\alpha \in C^z_{n,Y}$. Conversely, let $\alpha \in C^z_{n,Y}$. Suppose $\alpha \notin G_z \subseteq [G_z)$. Then $a_i < i - 1$ for all $i \in A_z \setminus \{x_1 + 1, x_2 + 1, \ldots, x_k + 1\}$. Clearly, $\alpha_{max} \in G_z$. Then we consider $\beta \in C_{n,Y}$ defined by

$$x\beta = \begin{cases} a_{x+1} & \text{if } x \in X_n \setminus (Y \cup \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\}), \\ x - 1 & \text{if } x \in \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\}, \\ x & \text{if } x \in Y. \end{cases}$$

Clearly, β is order-preserving and order-decreasing. Let $x \in X_n \setminus (Y \cup \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\})$ then $x\beta = a_{x+1} < x + 1 - 1 = x$. Thus, we can conclude that $Fix(\beta) = Y$. So $\beta \in C_{n,Y}$ and we obtain

$$x\alpha_{max}\beta = \begin{cases} (x-1)\beta = a_x & \text{for } x \in X_n \setminus Y, \\ x\beta = x & \text{if } x \in Y. \end{cases}$$

This gives $\alpha = id_{X_n}\alpha_{max}\beta$, i.e., $\alpha_{max} > \alpha$ and thus, $\alpha \in (G_z]$. Altogether, we have shown that $C^z_{n,Y} = (G_z]$ with $G_z \in P_{n,Y}$.

Proposition 3.1 shows that $C^z_{n,Y}$ is an ideal for all $z \in Y_1 \times Y_2 \times \ldots \times Y_k$. Moreover, we have:

Proposition 3.2: $(\{C^z_{n,Y} : z \in Y_1 \times Y_2 \times \ldots \times Y_k\}, \cap)$ is a semilattice.

Proof: Let $a=(a_1, a_2, \ldots, a_k) \in Y_1 \times Y_2 \times \ldots \times Y_k$ and $b=(b_1, b_2, \ldots, b_k) \in Y_1 \times Y_2 \times \ldots \times Y_k$. Then let $a \cap b = (min\{a_1, b_1\}, min\{a_2, b_2\}, \ldots, min\{a_k, b_k\}) \in Y_1 \times Y_2 \times \ldots \times Y_k$. It is easy to verify that $C^a_{n,Y} \cap C^b_{n,Y} = C^{a \cap b}_{n,Y}$. In fact, let $\alpha \in C^a_{n,Y} \cap C^b_{n,Y}$. Then $\alpha \in C_{n,Y} \cap T(X_n, \overline{A_a}) \cap T(X_n, \overline{A_b})$, i.e., $\alpha \in C_{n,Y} \cap T(X_n, \overline{A_a} \cap \overline{A_b})$. Note that $\overline{A_a} \cap \overline{A_b} = \bigcup_{i=1}^k \{\{x_i, x_i+1, \ldots, a_i\} \cap \{x_i, x_i+1, \ldots, b_i\}\} = \bigcup_{i=1}^k \{\{x_i, x_i+1, \ldots, min\{a_i, b_i\}\} = \overline{A_{a \cap b}}$. This shows $C^a_{n,Y} \cap C^b_{n,Y} \subseteq C^{a \cap b}_{n,Y}$. The converse inclusion is clear since $T(X_n, \overline{A_a} \cap \overline{A_b}) \subseteq T(X_n, \overline{A_a})$ and $T(X_n, \overline{A_a} \cap \overline{A_b}) \subseteq T(X_n, \overline{A_b})$.

Now, for each ideal $C^z_{n,Y}$, $z \in Y_1 \times Y_2 \times \ldots Y_k$, we will consider a set of congruences containing the Rees congruence $\rho_{C^z_{n,Y}}$. For $z=(z_1, z_2, \ldots, z_k) \in Y_1 \times Y_2 \times \ldots \times Y_k$, let $Con(C^z_{n,Y})$ be the set of all congruences on $C_{n,Y}$ with $\rho_{C^z_{n,Y}} \subseteq \rho$ and $(\alpha, \beta) \in \rho$ implies $\alpha, \beta \in C^z_{n,Y}$ or $\alpha, \beta \notin C^z_{n,Y}$. We will determine the minimal elements and the largest element in $Con(C^z_{n,Y})$.

For this, we define a relation $\sim_z$ on $C_{n,Y}$ by $\alpha \sim_z \beta$ if $\alpha, \beta \in C^z_{n,Y}$ or $\alpha, \beta \notin C^z_{n,Y}$ and for all $x \in X_n$ one of the following conditions hold: (i) $x\alpha = x\beta \in A_z$ or (ii) $x\alpha, x\beta \in \overline{A_z}$ or (iii) $x \in \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\}$ and $x\alpha, x\beta \in \overline{A_z} \cup \{z_1+1, z_2+1, \ldots, z_k+1\}$.

Proposition 3.3: Let $z=(z_1, z_2, \ldots, z_k) \in Y_1 \times Y_2 \times \ldots Y_k$. Then $\sim_z$ is a congruence on $C_{n,Y}$ and it is the largest element on $Con(C^z_{n,Y})$.

Proof: Clearly, $\sim_z$ is reflexive and symmetric. Let $(\alpha, \beta), (\beta, \gamma) \in \sim_z$. Then $\alpha, \beta, \gamma \in C^z_{n,Y}$ or $\alpha, \beta, \gamma \notin C^z_{n,Y}$. For later case let $x \in X_n$. Suppose $x\alpha = x\beta \in A_z$. If $x\beta = x\gamma$ then $x\alpha = x\gamma$. If $x \in \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\}$ and $x\beta, x\gamma \in \{z_1+1, z_2+1, \ldots, z_k+1\} \cup \overline{A_z}$ then $x\alpha, x\gamma \in \{z_1+1, z_2+1, \ldots, z_k+1\} \cup \overline{A_z}$. Suppose $x\alpha, x\beta \in \overline{A_z}$. If $x\beta, x\gamma \in \overline{A_z}$ then $x\alpha, x\gamma \in \overline{A_z}$. If $x \in \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\}$ and $x\beta, x\gamma \in \{z_1+1, z_2+1, \ldots, z_k+1\} \cup \overline{A_z}$ then $x\alpha, x\gamma \in \overline{A_z} \cup \{z_1+1, z_2+1, \ldots, z_k+1\}$. Suppose $x\alpha, x\beta \in \overline{A_z} \cup \{z_1+1, z_2+1, \ldots, z_k+1\}$ and $x \in \{x_2 - 1, x_3 - 1, \ldots, x_k - 1\}$. Then, of course, $x\alpha, x\gamma \in \overline{A_z} \cup \{z_1+1, z_2+1, \ldots, z_k+1\}$. This shows that $\sim_z$ is transitive. Let $(\alpha, \beta) \in \sim_z$ and $\gamma \in C_{n,Y}$. If $\alpha, \beta \in C^z_{n,Y}$ then it is easy to verify that $\gamma\alpha, \gamma\beta, \alpha\gamma, \beta\gamma \in C^z_{n,Y}$. Suppose now that $\alpha, \beta \notin C^z_{n,Y}$. For this case let $x \in X_n$. Then $x\gamma\alpha = x\gamma\beta \in A_z$ or $x\gamma\alpha, x\gamma\beta \in \overline{A_z}$ since $x\gamma \notin \{x_2 - 1, x_3 - 1, \ldots, x_k - 1, n\}$. Hence, $(\gamma\alpha, \gamma\beta) \in \sim_z$. Suppose that $x\alpha = x\beta \in A_z$. Then $x\alpha\gamma$

$= x\beta\gamma \in A_z$ or $x\alpha\gamma, x\beta\gamma \in \overline{A_z}$. If $x\alpha, x\beta \in \overline{A_z}$ then there is $l \in \{1, 2, \ldots, k\}$ such that $x\alpha \in \{x_l, x_l+1, \ldots, z_l\}$. Then $x\alpha\gamma \in \{x_l, x_l+1, \ldots, z_l-1\}$, i.e., $x\alpha\gamma \in \overline{A_z}$. Dually, we can show that $x\beta\gamma \in \overline{A_z}$. If $x\alpha, x\beta \in \overline{A_z} \cup \{z_1+1, z_2+1, \ldots, z_k+1\}$ and $x \in \{x_2-1, x_3-1, \ldots, x_k-1, n\}$, then $x\alpha\gamma, x\beta\gamma \in \overline{A_z}$. Hence $(\alpha\gamma, \beta\gamma) \in \sim_z$. Consequently, $\sim_z$ is a congruence on $C_{n,Y}$. Note that $\alpha \sim_z \beta$, whenever $\alpha, \beta \in \overline{C^z_{n,Y}}$. Hence, $\rho_{C^z_{n,Y}} \subseteq \sim_z$. Since $\alpha \sim_z \beta$ implies $\alpha, \beta \in C^z_{n,Y}$ or $\alpha, \beta \in \overline{C^z_{n,Y}}$, we have shown that $\sim_z \in Con(C_{n,Y})$. It remains to show that $\sim_z$ is the greatest element on $Con(C^*_{n,Y})$. For this let $\rho \in Con(C^*_{n,Y})$. We will show that $\rho \subseteq \sim_z$. Let $(\alpha, \beta) \in \rho$. Then $\alpha, \beta \in C^z_{n,Y}$ or $\alpha, \beta \notin C^z_{n,Y}$. We have to consider the later case. Assume that there is $a \in X_n$ such that $a\alpha \in A_z$ or $a\beta \in A_z$ and $a\alpha \neq a\beta$ and $a \notin \{x_2-1, x_3-1, \ldots, x_k-1, n\}$ or $a\alpha \notin \overline{A_z} \cup \{z_2+1, z_3+1, \ldots, z_k+1\}$ or $a\beta \notin \overline{A_z} \cup \{z_2+1, z_3+1, \ldots, z_k+1\}$. Then there is $l \in \{1, 2, \ldots, k\}$ with $a \in \{z_l+1, \ldots, x_{l+1}-1\}$, $a\alpha \in \{z_l, z_l+1, \ldots, x_{l+1}-2\}$, and $a\beta \in \{z_l, z_l+1, \ldots, x_{l+1}-2\}$. Without loss of generality, we assume that $a\alpha < a\beta$. If $a < x_{l+1}-1$ then we define $\gamma \in C_{n,Y}$ by $x\gamma = x_i$ if $x \in \{x_i, x_i+1, \ldots, x_{i+1}-1\} \setminus \{x_{l+1}-1\}$ as well as $i \in \{1, 2, \ldots, k\}$ and $(x_{l+1}-1)\ \gamma = a$. Clearly, $\gamma \in C_{n,Y}$. Then $(\gamma\alpha, \gamma\beta) \in \rho$ with $x\gamma\alpha, x\gamma\beta \in \{x_1, x_2, \ldots, x_k\}$ for all $x \in \{1, 2, \ldots, n\} \setminus \{x_{l+1}-1\}$ and $(x_{l+1}-1)\ \gamma\alpha = a\alpha$ as well as $(x_{l+1}-1)\ \gamma\beta = a\beta$. If $a\alpha \in \overline{A_z}$ then $a\beta \in A_z$. Thus, $\gamma\alpha \in C^z_{n,Y}$ and $\gamma\beta \in \overline{C^z_{n,Y}}$, a contradiction to $\rho \in Con(C^z_{n,Y})$. If $a\alpha \in A_z$ then $a\beta - 1 \in A_z$ and we define $\delta \in C_{n,Y}$ by

$$x\delta = \begin{cases} x_i & \text{if } x \in \{x_i, x_i+1, \ldots, x_{i+1}-1\} \text{ and } i \in \{1,2,\ldots,k\} \setminus \{l\}, \\ x_l & \text{if } x \in \{x_l, x_l+1, \ldots, a\beta-1\}, \\ a\beta-1 & \text{if } x \in \{a\beta, a\beta+1, \ldots, x_{l+1}-1\}. \end{cases}$$

It is routine to check that $\delta \in C_{n,Y}$. Then $(x_{l+1}-1)\ \gamma\alpha\delta = x_l$ and $(x_{l+1}-1)\ \gamma\beta\delta = a\beta - 1 \geq z_l$ and $x\gamma\alpha\delta = x\gamma\beta\delta \in Y$ for all $x \in X_n \setminus \{x_{l+1}-1\}$. Thus, $(\gamma\alpha\delta, \gamma\beta\delta) \in \rho$ with $\gamma\alpha\delta \in C^z_{n,Y}$ and $\gamma\beta\delta \in \overline{C^z_{n,Y}}$, which contradicts that $\rho \in Con(C^z_{n,Y})$. If $a = x_{l+1}-1$ then $a\beta > z_l + 1$ and we can conclude $(\alpha\delta, \beta\delta) \in \rho$ with $x\alpha\delta \in \overline{A_z}$ for all $x \in X_n$ and $(x_{l+1}-1)\beta\delta = a\beta - 1 > z_l$, i.e., $a\beta \in A_z$. Thus, $\alpha\delta \in C^z_{n,Y}$ and $\beta\delta \in \overline{C^z_{n,Y}}$, a contradiction to $\rho \in Con(C^z_{n,Y})$. Hence $x\alpha = x\beta \in A_z$ or $x\alpha, x\beta \in \overline{A_z}$ or $x \in \{x_2-1, x_3-1, \ldots, x_k-1, n\}$ and $x\alpha, x\beta \in \{z_1+1, z_2+1, \ldots, z_k+1\} \cup \overline{A_z}$ for all $x \in \{1, 2, \ldots,$

$n\}$. This shows that $\alpha \sim_z \beta$. Consequently, $\rho \subseteq \sim_z$. This shows that $\sim_z$ is the largest congruence in $Con(C^z_{n,Y})$.

For $z=(z_1, z_2, \ldots, z_k) \in Y_1 \times Y_2 \times \ldots \times Y_k$, let Ω_z be the set of all $(\alpha,\beta) \in \sim_z \setminus \rho_{C^z_{n,Y}}$ such that $\gamma \in C_{n,Y}$ and $\gamma < \alpha$ implies $\gamma \in C^z_{n,Y}$. For $(\alpha,\beta) \in \Omega_z$, let $C^z_{n,Y}(\alpha,\beta)$ be the set $\rho_{C^z_{n,Y}} \cup \{(\alpha,\beta), (\beta,\alpha)\}$. By Lemma 2.2, it is easy to verify:

Lemma 3.1: *$C^z_{n,Y}(\alpha,\beta)$ is a congruence in $Con(C^z_{n,Y})$ for each $z \in Y_1 \times Y_2 \times \ldots \times Y_k$ and all $(\alpha,\beta) \in \Omega_z$.*

A congruence ρ is called minimal in $Con(C^z_{n,Y})$, $z \in Y_1 \times Y_2 \times \cdots \times Y_k$, if there is no $\delta \in Con(C^z_{n,Y})$ with $\rho \subset \delta \subset \rho_{C^z_{n,Y}}$.

Proposition 3.4: Let $z \in Y_1 \times Y_2 \times \ldots \times Y_k$ and let $\rho \in Con(C^z_{n,Y})$. Then ρ is minimal in $Con(C^z_{n,Y})$ if and only if there is $(\alpha,\beta) \in \Omega_z$ with $\rho = C^z_{n,Y}(\alpha,\beta)$.

Proof: One direction is clear by Lemma 3.1. For the converse direction, we suppose that ρ is minimal in $Con(C^z_{n,Y})$. Assume that there are $(\gamma,\delta) \in \rho$ with $\gamma,\delta \in \overline{C^z_{n,Y}}$ and $\eta \in C_{n,Y}$ such that $\gamma\eta \in \overline{C^z_{n,Y}}$ or $\eta\gamma \in \overline{C^z_{n,Y}}$. Then it is easy to verify that there are $\beta_1, \beta_2 \in C_{n,Y} \cup \{id_{X_n}\}$ such that $\beta_1\gamma\beta_2, \beta_1\delta\beta_2 \in \overline{C^z_{n,Y}}$ and $\varepsilon \in C_{n,Y}$ with $\varepsilon < \beta_1\gamma\beta_2$ implies $\varepsilon \in C^z_{n,Y}$. Then the congruence $C^z_{n,Y}(\beta_1\gamma\beta_2, \beta_2\delta\beta_2)$ is contained properly in ρ. This contradicts the minimality. This shows that $\emptyset \neq \rho \setminus \rho_{C^z_{n,Y}} \subseteq \Omega_z$ and there is $(\varepsilon_1,\varepsilon_2) \in \rho \setminus \rho_{C^z_{n,Y}} \subseteq \Omega_z$. By Lemma 3.1, ρ contains the congruence $C^z_{n,Y}(\varepsilon_1,\varepsilon_2)$. Because of the minimality of ρ, we can calculate that $\rho = C^z_{n,Y}(\varepsilon_1,\varepsilon_2)$.

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