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Complex fuzzy ideals which coincide in Γ-semigroups

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Abstract

The thoughts of a complex fuzzy set were initiated by Ramot et al. [15] in 2000. It is a generalization of fuzzy sets. The main results of the paper, we discuss concepts of complex fuzzy ideals in Γ-semigroups and properties of a complex fuzzy ideal in Γ-semigroups with proven. Furthermore, we investigate essential conditions of coincidences complex fuzzy ideals in Γ-semigroups.

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1. Introduction

The complex fuzzy set is a method for solving data uncertainty problems and complex information. The Thoughts of fuzzy sets was published by Zadeh in 1965 [17]. In 1979, Kuroki [10] gave the concpets of defintions of semigorups. Later in 2003 Mordeson, Malik, and Kuroki [14] present the definition of fuzzy ideals in semigroups. In 2009. The operations of complex fuzzy sets were introduced by Ramot, Milo, Friedman and Kandel [15] in 2000. In 2020, Al-Husban and Davvaz, [2] introduced the concept of complex H_v subgroups and discussed various characteristics of these groups. The concept of complex fuzzy subsemigroup, complex fuzzy bi-ideal and complex fuzzy ideals in

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semigroups were introduced and studied by Majumder and Das [13]. Tamir, Jin and Kandel [11] studied the complex fuzzy set in structure cartesian by transforming the range from the unit circle to the complex plane. Al-Husban and Salleh [3] discussed complex fuzzy groups. Hu, Bi and Da [6] developed the complex fuzzy set in orthogonality and application to signal detection. The complex intuitionistic fuzzy soft sets introduced by Kumar and Bajaj [9]. In 2020 Al-Husban, Amourah and Jaber [1] present properties of bipolar complex fuzzy sets. In 2022, Rehman, Mahmood and Naeen [16] presented bipolar complex fuzzy sets and bipolar complex fuzzy ideals in semigroups. Later Mahmood, Rehman and Albaity [12] studied bipolar complex fuzzy Γ-ideals in Γ-semigroups. Recently, Khamrot, Iampan and Gaketem, [8] gave and proved the properties of the bipolar complex fuzzy interior ideals in semigroups. Mooreove, studing work of ideals like almost interior ideals [7], essential ideals [4] etc.

The remainder of this paper is organized as follows. In Section 2, we give some basic concepts and results of Γ-semigroup, fuzzy sets, and complex fuzzy sets. In section 3, we define complex fuzzy ideals and we discuss some basic properties of complex fuzzy ideals in Γ-semigroups. Section 4, we investigate essential conditions of coincidences of complex fuzzy ideals in Γ-semigroups.

2. Preliminaries

To compile this work self-sufficient, we briefly introduce a few definitions engaged in the remaining work.

A *Γ-semigroup* (Γ-SG) is an algebraic structure consisting of a non-empty set $\widetilde{\mathfrak{G}}$ together with an associative binary operation

Definition 2.1: Let $\widetilde{\mathfrak{M}} \neq \emptyset$ of an *Γ-SG* $\widetilde{\mathfrak{G}}$ is said to be

(1) A *Γ-subsemigroup* (Γ-SSG) if $\widetilde{\mathfrak{M}}\Gamma\widetilde{\mathcal{M}} \subseteq \widetilde{\mathfrak{M}}$.

(2) A *left Γ-ideal* (LΓ-ID) [right Γ-ideal (RΓ-ID)] of $\widetilde{\mathfrak{G}}$ if $\widetilde{\mathfrak{G}}\Gamma\widetilde{\mathfrak{M}} \subseteq \widetilde{\mathfrak{M}}$ $[\widetilde{\mathfrak{M}}\Gamma\widetilde{\mathfrak{G}} \subseteq \widetilde{\mathfrak{M}}]$.

(3) An *Γ-ideal* (Γ-ID) of $\widetilde{\mathfrak{G}}$ if it is both a LΓ-ID and a RΓ-ID of $\widetilde{\mathfrak{G}}$.

(4) A *generalized bi-Γ-ideal* (GB-Γ-ID) of $\widetilde{\mathfrak{G}}$ if $\widetilde{\mathfrak{M}}\Gamma\widetilde{\mathfrak{G}}\mathfrak{M} \subseteq \widetilde{\mathfrak{M}}$.

(5) A *bi-Γ-ideal (BΓ-ID) [interior Γ-ideal (IΓ-ID), (1,2)-Γ-ideal ((1,2)-Γ-ID)]* of $\widetilde{\mathfrak{G}}$ if $\widetilde{\mathfrak{M}}\Gamma\widetilde{\mathfrak{G}}\Gamma\widetilde{\mathfrak{M}} \subseteq \widetilde{\mathfrak{M}}$ $(\widetilde{\mathfrak{G}}\Gamma\widetilde{\mathfrak{M}}\Gamma\widetilde{\mathfrak{G}} \subseteq \widetilde{\mathfrak{M}},\ \widetilde{\mathfrak{M}}\Gamma\widetilde{\mathfrak{G}}\widetilde{\mathfrak{M}}^2 \subseteq \widetilde{\mathfrak{M}})$.

(5) A *quasi-Γ-ideal* (QΓ-ID) of $\widetilde{\mathfrak{G}}$ if $\widetilde{\mathfrak{M}}\Gamma\widetilde{\mathfrak{G}} \cap \widetilde{\mathfrak{G}}\Gamma\widetilde{\mathfrak{M}} \subseteq \widetilde{\mathfrak{M}}$.

A *regular* of an Γ-SG $\widetilde{\mathfrak{G}}$ if for each $\mathfrak{h} \in \widetilde{\mathfrak{G}}$, there exists $\mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{h}$. A *left* (right) regular of an SG $\widetilde{\mathfrak{G}}$ if for each $\mathfrak{h} \in \widetilde{\mathfrak{G}}$, there exists $\mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{r}\tilde{\alpha}\mathfrak{h}\tilde{\beta}\mathfrak{h}$ $(\mathfrak{h} = \mathfrak{h}\tilde{\alpha}\mathfrak{h}\tilde{\beta}\mathfrak{r})$. An *intra-regular* of an Γ-SG $\widetilde{\mathfrak{G}}$ if for each $\mathfrak{h} \in \widetilde{\mathfrak{G}}$, there exist $\mathfrak{r}, \mathfrak{t} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\omega} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{r}\tilde{\alpha}\mathfrak{h}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{t}$. A *semisimple* of an Γ-SG $\widetilde{\mathfrak{G}}$ if for every $\mathfrak{h} \in \widetilde{\mathfrak{G}}$, there exist $\mathfrak{r}, \mathfrak{t}, \mathfrak{e}, \mathfrak{p} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\omega}, \tilde{\lambda}, \tilde{\nu} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{r}\tilde{\alpha}\mathfrak{h}\tilde{\beta}\mathfrak{t}\tilde{\omega}\mathfrak{e}\tilde{\lambda}\mathfrak{h}\tilde{\nu}\mathfrak{p}$. A *weakly regular* of an SG $\widetilde{\mathfrak{G}}$ if for every $\mathfrak{h} \in \widetilde{\mathfrak{G}}$ there exist $\mathfrak{r}, \mathfrak{t} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha}, \tilde{\beta}, \tilde{\omega} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{t}$. A *left (right) quasi-regular* of an Γ-SG $\widetilde{\mathfrak{G}}$ if for every $\mathfrak{h} \in \widetilde{\mathfrak{G}}$, there exist $\mathfrak{r}, \mathfrak{t} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha}, \tilde{\beta}, \tilde{\omega} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{r}\tilde{\alpha}\mathfrak{h}\tilde{\beta}\mathfrak{t}\tilde{\omega}\mathfrak{h}$ $(\mathfrak{h} = \mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{t})$, Let CIN be the closed interval zero to one.

For any $\acute{\eta}_i \in CIN$ where $i \in \widetilde{\mathcal{M}}$ define

$$\bigvee_{i\in\widetilde{\mathcal{M}}} \acute{\eta}_i := \sup_{i\in\widetilde{\mathcal{M}}}\{\acute{\eta}_i\} \quad \text{and} \quad \bigwedge_{i\in\widetilde{\mathcal{M}}} \acute{\eta}_i := \inf_{i\in\widetilde{\mathcal{M}}}\{\acute{\eta}_i\}.$$

Definition 2.2: [17] A fuzzy set (FS) $\acute{\eta}$ of a non-empty set $\widetilde{\mathfrak{T}}$ is a function from $\widetilde{\mathfrak{T}}$ into the CIN, i.e, $\acute{\eta}: \widetilde{\mathfrak{T}} \to CIN$.

For any $\acute{\eta}_1, \acute{\eta}_2$ are FS of a non-empty set $\widetilde{\mathfrak{T}}$, we have

(1) $\acute{\eta}_1 \vee \eta'_2 = \max\{\acute{\eta}_1, \acute{\eta}_2\}$,

(2) $\acute{\eta}_1 \wedge \acute{\eta}_2 = \min\{\acute{\eta}_1, \acute{\eta}_2\}$,

(3) $\acute{\eta}_1 \subseteq \acute{\eta}_2 \Leftrightarrow \acute{\eta}_1(\mathfrak{h}) \leq \acute{\eta}_2(\mathfrak{h})$ for all $\mathfrak{h} \in \widetilde{\mathfrak{T}}$,

(4) $\acute{\eta}_1 = \acute{\eta}_2 \Leftrightarrow \acute{\eta}_1 \subseteq \acute{\eta}_2$ and $\acute{\eta}_2 \subseteq' \acute{\eta}_1$,

(5) $(\acute{\eta}_1 \wedge \acute{\eta}_1)(\mathfrak{h}) = \acute{\eta}_1(\mathfrak{h}) \wedge \acute{\eta}_1(\mathfrak{h})$ for all $\mathfrak{h} \in \widetilde{\mathfrak{T}}$.

Definition 2.3: [5] A subset $\widetilde{\mathfrak{K}} \neq \emptyset$ of set $\widetilde{\mathfrak{T}}$. The characteristic function $\lambda'_{\widetilde{\mathfrak{K}}}$ of $\widetilde{\mathfrak{K}}$ is defined to be a function $\lambda'_{\widetilde{\mathfrak{K}}}: \widetilde{\mathfrak{K}} \to CIN$ by

$$\lambda'_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) = \begin{cases} 1 & \text{if} \quad \mathfrak{h} \in \widetilde{\mathfrak{K}} \\ 0 & \text{if} \quad \mathfrak{h} \notin \widetilde{\mathfrak{K}} \end{cases}$$

for all $\mathfrak{h} \in \widetilde{\mathfrak{T}}$.

For two FSs $\acute{\eta}_1$ and $\acute{\eta}_2$ of an Γ-SG $\widetilde{\mathfrak{G}}$, define the product $\acute{\eta}_1 \circ \acute{\eta}_2$ as follows : for all $\mathfrak{h} \in \widetilde{\mathfrak{G}}$,

$$(\acute{\eta}_1 \circ_{\widetilde{\alpha}})(\mathfrak{h}) = \begin{cases} \bigvee_{(\mathfrak{r},\mathfrak{k})\in F_{\mathfrak{h}}} \{\acute{\eta}_1(\mathfrak{r}) \wedge \acute{\eta}_2(\mathfrak{r})\} & \text{if} \quad F_{\mathfrak{h}} \neq \emptyset, \\ 0 & \text{if} \quad F_{\mathfrak{h}} = \emptyset, \end{cases}$$

where $F_{\mathfrak{h}} := \{(\mathfrak{r}, \widetilde{\alpha}, \mathfrak{k}) \in \widetilde{\mathfrak{G}} \times \Gamma \times \widetilde{\mathfrak{G}} | \mathfrak{h} = \mathfrak{r}\widetilde{\alpha}\mathfrak{k}\}$.

Definition 2.4: [5] A FS $\acute{\eta}$ on an Γ-SG $\widetilde{\mathfrak{G}}$ is called

(1) A *fuzzy Γ-subsemigroup* (FΓ-SSG) on $\widetilde{\mathfrak{G}}$ if $\acute{\eta}(\mathfrak{h}\widetilde{\alpha}\mathfrak{r}) \geq \acute{\eta}(\mathfrak{h}) \wedge \acute{\eta}(\mathfrak{r})$ for all $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\widetilde{\alpha} \in \Gamma$.

(2) A *fuzzy left Γ-ideal* (FLΓ-ID) [fuzzy right Γ-ideal (FRΓ-ID)] on $\widetilde{\mathfrak{G}}$ if $\acute{\eta}(\mathfrak{h}\widetilde{\alpha}\mathfrak{r}) \geq \acute{\eta}(\mathfrak{r})$ $(\acute{\eta}(\mathfrak{h}\widetilde{\alpha}\mathfrak{r}) \geq \acute{\eta}(\mathfrak{h}))$ for all $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\widetilde{\alpha} \in \Gamma$. A FS $\acute{\eta}$ on $\mathfrak{G}$ is called a *fuzzy Γ-ideal* (FΓ-ID) on $\widetilde{\mathfrak{G}}$ if it is both a FLΓ-ID and a FRΓ-ID of $\widetilde{\mathfrak{G}}$.

(3) A *fuzzy generalized bi-Γ-ideal* (FGBΓ-ID) on $\widetilde{\mathfrak{G}}$ if $\acute{\eta}(\mathfrak{h}\widetilde{\alpha}\mathfrak{r}\widetilde{\beta}\mathfrak{w}) \geq \acute{\eta}(\mathfrak{h}) \wedge \acute{\eta}(\mathfrak{w})$ for all $\mathfrak{h}\mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\widetilde{\alpha}, \widetilde{\beta} \in \Gamma$.

(4) A *fuzzy bi-Γ-ideal* (FBΓ-ID) on $\widetilde{\mathfrak{G}}$ if $\acute{\eta}$ is a FΓ-SSG and a FGBΓ-ID on $\widetilde{\mathfrak{G}}$.

(5) A *fuzzy interior Γ-ideal* (FIΓ-ID) on $\widetilde{\mathfrak{G}}$ if $\acute{\eta}$ is a FΓ-SSG on $\widetilde{\mathfrak{G}}$ and $\acute{\eta}(\mathfrak{h}\widetilde{\alpha}\mathfrak{r}\widetilde{\beta}\mathfrak{w}) \geq \acute{\eta}(\mathfrak{r})$ for all $\mathfrak{h}, \mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\widetilde{\alpha}, \widetilde{\beta} \in \Gamma$.

(6) A *fuzzy* (1,2) *Γ-ideal* (F (1,2) Γ-ID) on $\widetilde{\mathfrak{G}}$ if $\acute{\eta}$ is a FΓ-SSG on $\widetilde{\mathfrak{G}}$ and $\eta'(\mathfrak{h}\widetilde{\alpha}\mathfrak{r}\widetilde{\beta}(\mathfrak{w}\widetilde{\omega}\mathfrak{t})) \geq \acute{\eta}(\mathfrak{h}) \wedge \acute{\eta}(\mathfrak{w}) \wedge \acute{\eta}(\mathfrak{t})$ for all $\mathfrak{h}, \mathfrak{r}, \mathfrak{w}, \mathfrak{t} \in \widetilde{\mathfrak{G}}$ and $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\omega} \in \Gamma$.

(7) A *fuzzy quasi Γ-ideal* (FQΓ-ID) of $\widetilde{\mathfrak{G}}$ if $\acute{\eta}(\mathfrak{h}) \geq (\lambda'_{\widetilde{\mathfrak{G}}} \circ \eta')(\mathfrak{h}) \wedge (\acute{\eta} \circ \lambda'_{\widetilde{\mathfrak{G}}})(\mathfrak{h})$ for all $\mathfrak{h} \in \widetilde{\mathfrak{G}}$, where $\lambda'_{\widetilde{\mathfrak{G}}}$ is a FS such that mapping to 1.

Definition 2.5: [13] A complex fuzzy set (CF set) $\acute{\eta}^{RI}$ on $\widetilde{\mathfrak{T}}$ is an object having the form $\acute{\eta}^{RI} := \{(\widetilde{\mathfrak{T}}, \acute{\eta}^{RI}(\mathfrak{h}) = \acute{\eta}^{R}(\mathfrak{h}) + \acute{\eta}^{I}(\mathfrak{h})) | \mathfrak{h} \in \mathfrak{T}\}$, where $\acute{\eta}^{R} : \widetilde{\mathfrak{T}} \rightarrow CIN$ is a real number and $\acute{\eta}^{I} : \widetilde{\mathfrak{T}} \rightarrow CIN$ is an imaginary number.

3. Complex Fuzzy Ideal in Γ-Semigroups

In this section, we will study thoughts of CF set in an Γ-SG and we study properties of those.

Let $\emptyset \neq \widetilde{\mathfrak{K}} \subseteq \widetilde{\mathfrak{T}}$. The characteristic complex fuzzy set (shortly, CCF set) $\lambda'^{RI}_{\widetilde{\mathfrak{M}}} = (\widetilde{\mathfrak{M}}; \lambda'^{R}_{\widetilde{\mathfrak{M}}} + \lambda'^{I}_{\widetilde{\mathfrak{M}}})$ is defined as follows:

$$\lambda'^{R}_{\widetilde{\mathfrak{M}}} + \lambda'^{I}_{\widetilde{\mathfrak{M}}}(h) = \begin{cases} 1 + \iota 1 & \text{if} \quad \mathfrak{h} \in \widetilde{\mathfrak{M}} \\ 0 + \iota 0 & \text{if} \quad \mathfrak{h} \notin \widetilde{\mathfrak{M}}. \end{cases}$$

for all $\mathfrak{h} \in \widetilde{\mathfrak{T}}$.

Next, we study the intersection and product of CF set in an Γ-SG $\widetilde{\mathfrak{G}}$ as defined.

Let $\acute{\eta}^{RI} = (\widetilde{\mathfrak{G}}; \acute{\eta}^{R}, \acute{\eta}^{I}) = (\widetilde{\mathfrak{G}}; \acute{\eta}^{R} + \acute{\eta}^{I})$ and $\psi'^{RI} = (\widetilde{\mathfrak{G}}; \psi'^{P}, \psi'^{I}) = (\widetilde{\mathfrak{G}}; \psi'^{R} + \psi'^{I})$ are CF set in an SG $\widetilde{\mathfrak{G}}$. Define

(1) $(\acute{\eta}^{RI} \cap '\psi'^{RI})(\mathfrak{h}) = \acute{\eta}^{RI}(\mathfrak{h}) \wedge \psi'^{RI}(\mathfrak{h})$ for all $\mathfrak{h} \in \widetilde{\mathfrak{G}}$.

(2) $\acute{\eta}^{RI}(h) \precsim \psi'^{RI}(h) = \acute{\eta}^{RI}(\mathfrak{h}) \leq \psi'^{RI}(\mathfrak{h})$ for all $\mathfrak{h} \in \mathfrak{G}$.

(3) $\acute{\eta}^{RI} \circledcirc_{\tilde{\alpha}} \psi'^{RI} = (\mathfrak{G}; \acute{\eta}^{RI} \circledcirc_{\tilde{\alpha}} \psi'^{RI}) = (\mathfrak{G}; \acute{\eta}^{R} \circ_{\tilde{\alpha}} \psi'^{R} + \iota\acute{\eta}^{I} \circ_{\tilde{\alpha}} \iota\psi'^{I})$

where;

$$(\acute{\eta}^{R} \circ_{\tilde{\alpha}} \psi'^{R})(\mathfrak{h}) = \begin{cases} \bigvee_{(\mathfrak{k},\mathfrak{r}) \in F_{\mathfrak{h}}} \{\acute{\eta}^{R}(\mathfrak{k}) \wedge \psi'^{R}(\mathfrak{r})\} & \text{if} \quad F_{\mathfrak{h}} \neq \emptyset \\ 0 & \text{if} \quad F_{\mathfrak{h}} = \emptyset, \end{cases}$$

and

$$(\acute{\eta}^{I} \circ_{\tilde{\alpha}} \psi'^{I})(\mathfrak{h}) = \begin{cases} \bigvee_{(\mathfrak{k},\mathfrak{r}) \in F_{\mathfrak{h}}} \{\acute{\eta}^{I}(\mathfrak{k}) \wedge \psi'^{R}(\mathfrak{r})\} & \text{if} \quad F_{\mathfrak{h}} \neq \emptyset \\ 0 & \text{if} \quad F_{\mathfrak{h}} = \emptyset, \end{cases}$$

or we consider the product of CF set defined

$$(\acute{\eta}^{RI} \circledcirc_{\tilde{\alpha}} \psi'^{RI})(\mathfrak{h}) = \begin{cases} \bigvee_{(\mathfrak{k},\mathfrak{r}) \in F_{\mathfrak{h}}} \{\acute{\eta}^{RI}(\mathfrak{k}) \wedge \psi'^{RI}(\mathfrak{r})\} & \text{if} \quad F_{\mathfrak{h}} \neq \emptyset \\ 0 & \text{if} \quad F_{\mathfrak{h}} = \emptyset, \end{cases}$$

Obviously, the operation $\circledcirc$ is associative.

Definition 3.1: *A CF set $\acute{\eta}^{RI}$ of an Γ-SG $\widetilde{\mathfrak{G}}$ is called*

(1) A *complex fuzzy Γ-subsemigroup* (CFΓ-SSG) if $\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{r}) \Rightarrow \acute{\eta}^{R}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{R}(\mathfrak{h}) \wedge \acute{\eta}^{R}(\mathfrak{r})$ and $\acute{\eta}^{I}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{I}(\mathfrak{h}) \wedge \acute{\eta}^{I}(\mathfrak{r})$ for all $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha} \in \Gamma$.

(2) A *complex fuzzy left Γ-ideal* (CFL Γ-ID) if $\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{RI}(\mathfrak{r}) \Rightarrow \acute{\eta}^{R}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{R}(\mathfrak{r})$ and $\acute{\eta}^{I}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{I}(\mathfrak{r})$ for all $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha} \in \Gamma$.

(3) A *complex fuzzy right Γ-ideal* (CFR Γ-ID) if $\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{RI}(\mathfrak{h}) \Rightarrow \acute{\eta}^{R}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{R}(\mathfrak{h})$ and $\acute{\eta}^{I}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\eta}^{I}(\mathfrak{h})$ for all $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha} \in \Gamma$.

(4) A *complex fuzzy Γ-ideal* (CFΓ-ID) if it is both a CFLΓ-ID and CFRΓ-ID of $\widetilde{\mathfrak{G}}$.

(5) A *complex fuzzy quasi Γ-ideal* (CFQΓ-ID) if $\acute{\eta}^{RI}(\mathfrak{h}) \geq (\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \acute{\eta}^{RI})(\mathfrak{h}) \wedge (\acute{\eta}^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}})(\mathfrak{h}) \Rightarrow \eta'^{R}(\mathfrak{h}) \geq (\lambda'^{R}_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\acute{\eta}^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\widetilde{\mathfrak{G}}})(\mathfrak{h})$ and $\acute{\eta}^{I}(\mathfrak{h}) \geq (\lambda'^{I}_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) \wedge (\acute{\eta}^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\widetilde{\mathfrak{G}}})(\mathfrak{h})$ for all $\mathfrak{h} \in \widetilde{\mathfrak{G}}$ and $\alpha' \in \Gamma$, where $\lambda'^{RI}_{\widetilde{\mathfrak{G}}}$ is a CFS such that mapping to $1 + \iota 1$.

(6) A *complex fuzzy generalized bi-Γ-ideal* (CFGB-Γ-ID) if $\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{w}) \Rightarrow \acute{\eta}^{R}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \acute{\eta}^{R}(\mathfrak{h}) \wedge \acute{\eta}^{R}(\mathfrak{w})$ and $\acute{\eta}^{I}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \acute{\eta}^{I}(\mathfrak{h}) \wedge \acute{\eta}^{I}(\mathfrak{w})$ for all $\mathfrak{h}, \mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$.

(7) A *complex fuzzy bi-Γ-ideal* (CFB-Γ-ID) if $\acute{\eta}^{RI}$ is a CFΓ-SSG and a CFGB-Γ-ID of $\widetilde{\mathfrak{G}}$

(8) A *complex fuzzy interior-Γ-ideal* (CFI-Γ-ID) if $\acute{\eta}^{RI}$ is a CFΓ-SSG and $\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \acute{\eta}^{RI}(\mathfrak{r}) \Rightarrow \acute{\eta}^{R}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \acute{\eta}^{R}(\mathfrak{r})$ and $\acute{\eta}^{I}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \acute{\eta}^{I}(\mathfrak{r})$ for all $\mathfrak{h}, \mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$.

(9) A *complex fuzzy (1,2)-Γ-ideal* (CF (1,2)-Γ-ID) if $\acute{\eta}^{RI}$ is a CFΓ-SSG on $\widetilde{\mathfrak{G}}$ and $\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}(\mathfrak{w}\tilde{\omega}\mathfrak{t})) \geq \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{w}) \wedge \acute{\eta}^{RI}(\mathfrak{t}) \Rightarrow \acute{\eta}^{R}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}(\mathfrak{w}\tilde{\omega}\mathfrak{t})) \geq \acute{\eta}^{R}(\mathfrak{h}) \wedge \acute{\eta}^{R}(\mathfrak{w}) \wedge \acute{\eta}^{R}(\mathfrak{t})$ and $\acute{\eta}^{I}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}(\mathfrak{w}\tilde{\omega}\mathfrak{t})) \geq \acute{\eta}^{I}(\mathfrak{h}) \wedge \acute{\eta}^{I}(\mathfrak{w}) \wedge \acute{\eta}^{I}(\mathfrak{t})$ for all $\mathfrak{h}, \mathfrak{r}, \mathfrak{w}, \mathfrak{t} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\omega} \in \Gamma$.

In Definition 3.1 we have $(4) \Rightarrow (1)$, $(4) \Rightarrow (8)$ and $(7) \Rightarrow (6)$.
The following three examples are shown CFSs.

Example 3.2: Let $\widetilde{\mathfrak{G}} = \{\Xi, ♠, ♥, \Delta\}$ be an Γ-SG with the following Cayley table:

▶	Ξ	⟦	♥	Δ
Ξ	Ξ	Ξ	Ξ	Ξ
⟦	Ξ	Ξ	Ξ	Ξ
♥	Ξ	Ξ	⟦	♥
Δ	Ξ	Ξ	⟦	♥

Define CFS $\eta'^{RI}: \widetilde{\mathfrak{G}} \to \mathcal{C}$ by $\eta'^{RI}(\Xi) = 0.9 + \iota 0.8, \eta'^{RI}(⟦) = 0.7 + \iota 0.5, \eta'^{RI}(♥) = 0.5 + \iota 0.6, \eta'^{RI}(\Delta) = 0.5 + \iota 0.6$. Then η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

Example 3.3: Let $\widetilde{\mathfrak{G}} = \{\Xi, ⟦, ♥, \Delta\}$ be an Γ-SG with the following Cayley table:

▶	Ξ	⟦	♥	Δ
Ξ	Ξ	⟦	♥	Δ
⟦	⟦	Ξ	Δ	♥
♥	♥	Δ	⟦	Ξ
Δ	Δ	♥	⟦	Ξ

Define CFS $\acute{\eta}^{RI}: \tilde{\tilde{\mathfrak{G}}} \to CIN$ by $\acute{\eta}^{RI}(\Xi) = 0.3 + \iota 0.2, \acute{\eta}^{RI}(\llbracket) = 0.5 + \iota 0.3, \acute{\eta}^{RI}(\heartsuit) = 0.8 + \iota 0.7, \acute{\eta}^{RI}(\Delta) = 0.8 + \iota 0.5$. Then $\acute{\eta}^{RI}$ is a CFBΓ-ID of $\tilde{\tilde{\mathfrak{G}}}$.

Example 3.4: Let $\tilde{\tilde{\mathfrak{G}}} = \{\Xi, \llbracket, \heartsuit, \Delta\}$ be an Γ-SG with the following Cayley table:

▶	Ξ	⟦	♥	Δ
Ξ	Ξ	Ξ	Ξ	Ξ
⟦	Ξ	⟦	♥	Δ
♥	Ξ	♥	Ξ	♥
Δ	Ξ	♥	⟦	Ξ

Define CFS $\acute{\eta}^{RI}: \tilde{\tilde{\mathfrak{G}}} \to CIN$ by $\acute{\eta}^{RI}(\Xi) = 0.7 + \iota 0.8, \acute{\eta}^{RI}(\llbracket) = 0.4 + \iota 0.6, \acute{\eta}^{RI}(\heartsuit) = 0.6 + \iota 0.7, \acute{\eta}^{RI}(\Delta) = 0.3 + \iota 0.5$. Then $\acute{\eta}^{RI}$ is a CFIΓ-ID of $\tilde{\tilde{\mathfrak{G}}}$.

Next, we discuss some properties of CFΓ-IDs in SGs.

Theorem 3.5: *Let $\{\acute{\eta}_i^{RI} | i \in \mathcal{J}\}$ be a family of CFΓ-SSGs of an Γ-SG $\tilde{\tilde{\mathfrak{G}}}$. Then $\acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}$ is a CFΓ-SSG of $\tilde{\tilde{\mathfrak{G}}}$.*

Proof: Let $\mathfrak{h}, \mathfrak{r} \in \tilde{\tilde{\mathfrak{G}}}$ and $\tilde{\alpha} \in \Gamma$. Then,

$$\acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}(\mathfrak{h}\alpha\mathfrak{r}) \geq \acute{\bigcap}_{i\in\mathcal{J}} (\acute{\eta}_i^{RI}(\mathfrak{h}) \wedge \acute{\eta}_i^{RI}(\mathfrak{r})) = \wedge (\acute{\eta}_i^{RI}(\mathfrak{h}) \wedge \acute{\eta}_i^{RI}(\mathfrak{r}))$$
$$= (\wedge \acute{\eta}_i^{RI}(\mathfrak{h})) \wedge (\wedge \acute{\eta}_i^{RI}(\mathfrak{r})) = \acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}(\mathfrak{h}) \wedge \acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}(\mathfrak{r})$$

Thus, $\acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}$ is a CFΓ-SSG of $\tilde{\tilde{\mathfrak{G}}}$.

Theorem 3.6: *Let $\{\acute{\eta}_i^{RI} | i \in \mathcal{J}\}$ be a family of CFLΓ-IDs (CFRΓ-IDs) of an Γ-SG $\tilde{\tilde{\mathfrak{G}}}$. Then $\bigcap_{i\in\mathcal{J}} \acute{\eta}_i^{RI}$ is a CFLΓ-ID (CFRΓ-ID) of $\tilde{\tilde{\mathfrak{G}}}$.*

Proof: Let $\mathfrak{h}, \mathfrak{r} \in \tilde{\tilde{\mathfrak{G}}}$ and $\tilde{\alpha} \in \Gamma$. Then,

$$\acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}(\mathfrak{r})) = \wedge (\acute{\eta}_i^{RI}(\mathfrak{r})) = \wedge (\wedge \acute{\eta}_i^{RI}(\mathfrak{r})) = \acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}(\mathfrak{r}).$$

Thus, $\acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}$ is a CFLΓ-ID of $\tilde{\tilde{\mathfrak{G}}}$.

Theorem 3.7: *Let $\{\acute{\eta}_i^{RI} | i \in \mathcal{J}\}$ be a family of CFBΓ-IDs (CFIΓ-IDs, CF (1,2)-Γ-IDs) of an Γ-SG $\tilde{\tilde{\mathfrak{G}}}$. Then $\acute{\bigcap}_{i\in\mathcal{J}} \acute{\eta}_i^{RI}$ is a CFBΓ-ID (CFIΓ-ID, CF (1,2)-Γ-ID) of $\tilde{\tilde{\mathfrak{G}}}$.*

Proof: Let $\mathfrak{h},\mathfrak{r}\in\widetilde{\mathfrak{G}}$ and $\tilde{\alpha}\in\Gamma$. Then, By assumptions we have $\{\acute{\eta}_i^{RI}|i\in\mathcal{J}\}$ be a family of CFΓ-SSG of $\widetilde{\mathfrak{G}}$. Thus, by Theorem 3.5, $\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$. Let $\mathfrak{h},\mathfrak{r},\mathfrak{w}\in\widetilde{\mathfrak{G}}$ and $\tilde{\alpha},\tilde{\beta}\in\Gamma$. Then,

$$\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w})\geq\underset{i\in\mathcal{J}}{\acute{\cap}}(\acute{\eta}_i^{RI}(\mathfrak{h})\wedge\acute{\eta}_i^{RI}(\mathfrak{w}))=\wedge(\acute{\eta}_i^{RI}(\mathfrak{h})\wedge\acute{\eta}_i^{RI}(\mathfrak{w}))$$
$$=(\wedge\acute{\eta}_i^{RI}(\mathfrak{h}))\wedge(\wedge\acute{\eta}_i^{RI}(\mathfrak{w}))=\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}(\mathfrak{h})\wedge\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}(\mathfrak{w}).$$

Thus, $\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}$ is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that $\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}$ is a CFIΓ-ID, (CF (1,2)-Γ-ID) of $\widetilde{\mathfrak{G}}$.

Corollary 3.8: *Let $\{\acute{\eta}_i^{RI}|i\in\mathcal{J}\}$ be a family of CFGBΓ-IDs of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}$ is a CFGBΓ-ID of $\widetilde{\mathfrak{G}}$.*

Theorem 3.9: *Let $\{\acute{\eta}_i^{RI}|i\in\mathcal{J}\}$ be a family of CFQΓ-IDs of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}$ is a CFQΓ-IDQ of $\widetilde{\mathfrak{G}}$.*

Proof: Let $\mathfrak{h}\in\widetilde{\mathfrak{G}}$. Then for all $\tilde{\alpha}\in\Gamma$,

$$\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}(\mathfrak{h})\geq\underset{i\in\mathcal{J}}{\acute{\cap}}(\lambda'^{RI}_{\widetilde{\mathfrak{G}}}\odot_{\tilde{\alpha}}\acute{\eta}^{RI})\wedge(\acute{\eta}^{RI}\odot_{\tilde{\alpha}}\lambda'^{RI}_{\widetilde{\mathfrak{G}}})(\mathfrak{h})$$
$$=(\wedge(\lambda'^{RI}_{\widetilde{\mathfrak{G}}}\odot_{\tilde{\alpha}}\acute{\eta}^{RI})(\mathfrak{h}))\wedge(\wedge(\acute{\eta}^{RI}\odot_{\tilde{\alpha}}\lambda'^{RI}_{\widetilde{\mathfrak{G}}})(\mathfrak{h}))$$
$$=\underset{i\in\mathcal{J}}{\acute{\cap}}(\lambda'^{RI}_{\widetilde{\mathfrak{G}}}\odot_{\tilde{\alpha}}\acute{\eta}^{RI})(\mathfrak{h})\wedge(\acute{\eta}^{RI}\odot_{\tilde{\alpha}}\lambda'^{RI}_{\widetilde{\mathfrak{G}}})(\mathfrak{h}).$$

Thus, $\underset{i\in\mathcal{J}}{\acute{\cap}}\acute{\eta}_i^{RI}$ is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$.

The following theorems are study the CF set of types of SSGs on Γ-SGs.

Theorem 3.10: *Let $\widetilde{\mathfrak{K}}\neq\emptyset$ of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\widetilde{\mathfrak{K}}$ is an Γ-SSG of $\widetilde{\mathfrak{G}}$ if and only if the CCF function $\lambda'^{RI}_{\widetilde{\mathfrak{K}}}=(\widetilde{\mathfrak{K}};\lambda'^{R}_{\widetilde{\mathfrak{K}}}+\lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$.*

Proof: Suppose that $\widetilde{\mathfrak{K}}$ is an Γ-SSG of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h},\mathfrak{r}\in\widetilde{\mathfrak{G}}$, $\tilde{\alpha}\in\Gamma$. Then the following cases:

Case 1: If $\mathfrak{h},\mathfrak{r}\in\widetilde{\mathfrak{K}}$, then $\mathfrak{h}\tilde{\alpha}\mathfrak{r}\in\widetilde{\mathfrak{K}}$. Thus, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h})=1=\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})=\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h})=\iota 1=\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})=\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})$. Hence $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})\geq\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h})\wedge\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})\geq\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h})\wedge\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$.

Case 2: If $\mathfrak{h},\mathfrak{r}\notin\widetilde{\mathfrak{K}}$, then $\mathfrak{h}\tilde{\alpha}\mathfrak{r}\in\widetilde{\mathfrak{K}}$. Thus, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})\geq\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h})\wedge\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})\geq\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h})\wedge\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$.

Therefore, $(\widetilde{\mathfrak{K}};\lambda'^{R}_{\widetilde{\mathfrak{K}}}+\lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$.

Conversely, suppose that $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$ and $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{K}}$, $\tilde{\alpha} \in \Gamma$. If $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \notin \widetilde{\mathfrak{K}}$, then $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \leq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \leq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$. By hypothesis, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, which is a contradiction. Thus, $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \in \widetilde{\mathfrak{K}}$. Hence, $\widetilde{\mathfrak{K}}$ is an Γ-SSG of $\widetilde{\mathfrak{G}}$.

Theorem 3.11: *Let $\widetilde{\mathfrak{K}} \neq \emptyset$ of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\widetilde{\mathfrak{K}}$ is an LΓ-ID (RΓ-ID) of $\widetilde{\mathfrak{G}}$ if and only if the CCF function $\lambda'^{RI}_{\widetilde{\mathfrak{K}}} = (\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFLΓ-ID (CFRΓ-ID) of $\widetilde{\mathfrak{G}}$.*

Proof: Suppose that $\widetilde{\mathfrak{K}}$ is an LIΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$ $\tilde{\alpha} \in \Gamma$. Then the following cases:

Case 1: If $\mathfrak{r} \in \widetilde{\mathfrak{K}}$, then $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \in \widetilde{\mathfrak{K}}$. Thus, $1 = \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r}) = \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})$ and $\iota 1 = \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r}) = \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r})$. Hence, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$.

Case 2: If $\mathfrak{r} \notin \widetilde{\mathfrak{K}}$, then $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \in \widetilde{\mathfrak{K}}$. Thus, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$.

Therefore, $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$.

Conversely, suppose that $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$ and $\mathfrak{r} \in \widetilde{\mathfrak{K}}$, $\tilde{\alpha} \in \Gamma$. If $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \notin \widetilde{\mathfrak{K}}$, then $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \leq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \leq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$. By hypothesis, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\mathfrak{r}) \geq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{r})$, which is a contradiction. Thus, $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \in \widetilde{\mathfrak{K}}$. Hence, $\widetilde{\mathfrak{K}}$ is an LΓ-ID of $\widetilde{\mathfrak{G}}$.

Corollary 3.12: Let $\widetilde{\mathfrak{K}} \neq \emptyset$ of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\widetilde{\mathfrak{K}}$ is an Γ-ID of $\widetilde{\mathfrak{G}}$ if and only if the CCF function $\lambda'^{RI}_{\widetilde{\mathfrak{K}}} = (\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

Theorem 3.13: *Let $\widetilde{\mathfrak{K}} \neq \emptyset$ of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\widetilde{\mathfrak{K}}$ is a BΓ-ID (IΓ-ID, (1,2)-Γ-ID) of $\widetilde{\mathfrak{G}}$ if and only if the CCF function $\lambda'^{RI}_{\widetilde{\mathfrak{K}}} = (\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFBΓ-ID (CFIΓ-ID, CF (1, 2)-Γ-ID) of $\widetilde{\mathfrak{G}}$.*

Proof: Suppose that $\widetilde{\mathfrak{K}}$ is a BΓ-ID of $\widetilde{\mathfrak{G}}$. Then $\widetilde{\mathfrak{K}}$ is an Γ-SSG of $\widetilde{\mathfrak{G}}$. Thus, by Theorem 3.10, $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$. Let $\mathfrak{h}, \mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$. Then the following cases:

Case 1: If $\mathfrak{h}, \mathfrak{w} \in \widetilde{\mathfrak{K}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$, then $\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w} \in \widetilde{\mathfrak{K}}$. Thus, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) = 1 = \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{w}) = \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) = \iota 1 = \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{w}) = \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w})$. Hence, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$.

Case 2: If $\mathfrak{h}, \mathfrak{w} \notin \widetilde{\mathfrak{K}}$, then $\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w} \in \widetilde{\mathfrak{K}}$. Thus, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$.

Therefore, $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFIΓ-ID (CF (1, 2)-Γ-ID) of $\widetilde{\mathfrak{G}}$.

Conversely suppose that $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$. Then $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$. Thus by Theorem 3.10, $\widetilde{\mathfrak{K}}$ is an Γ-SSG of $\widetilde{\mathfrak{G}}$.

Let $\mathfrak{h}, \mathfrak{w} \in \widetilde{\mathfrak{K}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$. If $\mathfrak{h}\tilde{\beta}\mathfrak{r}\tilde{\beta}\mathfrak{w} \notin \widetilde{\mathfrak{K}}$, then $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \leq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \leq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$. By hypothesis, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$, and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \wedge \lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{w})$, which is a contradiction. Thus, $\mathfrak{h}\mathfrak{r}\mathfrak{w} \in \widetilde{\mathfrak{K}}$. Hence, $\widetilde{\mathfrak{K}}$ is a BI of $\widetilde{\mathfrak{G}}$. Similarly, we can show that $\widetilde{\mathfrak{K}}$ is an IΓ-ID ((1,2)-Γ-ID) of $\widetilde{\mathfrak{G}}$.

Corollary 3.14: *Let $\widetilde{\mathfrak{K}} \neq \emptyset$ of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\widetilde{\mathfrak{K}}$ is a GBΓ-ID of $\widetilde{\mathfrak{G}}$ if and only if the CCF function $\lambda'^{RI}_{\widetilde{\mathfrak{K}}} = (\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFGBΓ-ID of $\widetilde{\mathfrak{G}}$.*

Theorem 3.15: *Let $\widetilde{\mathfrak{K}} \neq \emptyset$ of an Γ-SG $\widetilde{\mathfrak{G}}$. Then $\widetilde{\mathfrak{K}}$ is a QΓ-ID of $\widetilde{\mathfrak{G}}$ if and only if the CF function $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$.*

Proof: Suppose that $\widetilde{\mathfrak{K}}$ is a QΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h} \in \widetilde{\mathfrak{G}}$. Then the following cases:

Case 1: If $\mathfrak{h} \in \widetilde{\mathfrak{K}}$, then for all $\tilde{\alpha} \in \Gamma$ such that $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{R}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{I}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) \wedge (\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$.

Case 2: If $\mathfrak{h} \notin \widetilde{\mathfrak{K}}$, then $\mathfrak{h}$ is expressible $\mathfrak{h} = \mathfrak{y}\tilde{\alpha}\mathfrak{z}$ for all $\tilde{\alpha} \in \Gamma$ or not.

Suppose $\mathfrak{h}$ is expressible as $\mathfrak{h} = \mathfrak{y}\tilde{\alpha}\mathfrak{z}$. Since $\mathfrak{h} \notin \widetilde{\mathfrak{K}}$ either $\mathfrak{y} \in \widetilde{\mathfrak{K}}$ or $\mathfrak{z} \notin \widetilde{\mathfrak{K}}$. If $\mathfrak{y} \in \widetilde{\mathfrak{K}}$ and $\mathfrak{z} \notin \widetilde{\mathfrak{K}}$, then there cannot be another exprssion of the form $\mathfrak{x} = \mathfrak{a}\tilde{\alpha}\mathfrak{b}$ for all $\tilde{\alpha} \in \Gamma$, where $\mathfrak{a} \in \widetilde{\mathfrak{K}}$ and $\mathfrak{b} \in \widetilde{\mathfrak{K}}$. Then $\mathfrak{x} \in \widetilde{\mathfrak{K}}\Gamma\widetilde{\mathfrak{G}} \cap \widetilde{\mathfrak{G}}\Gamma\widetilde{\mathfrak{K}} \subseteq \widetilde{\mathfrak{K}}$. Thus, $(\lambda'^{R}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) = 0$ or $(\eta'^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\widetilde{\mathfrak{K}}})(\mathfrak{h}) = 0$ and $(\lambda'^{I}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) = \iota 0$ or $(\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\widetilde{\mathfrak{K}}})(\mathfrak{h}) = \iota 0$. Hence, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{R}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{I}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) \wedge (\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$. Therefore, $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$.

Conversely suppose that $(\widetilde{\mathfrak{K}}; \lambda'^{R}_{\widetilde{\mathfrak{K}}} + \lambda'^{I}_{\widetilde{\mathfrak{K}}})$ is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$ and $\mathfrak{h} \in \widetilde{\mathfrak{K}}\Gamma\widetilde{\mathfrak{G}} \cap \widetilde{\mathfrak{G}}\Gamma\widetilde{\mathfrak{K}}$. If $\mathfrak{h} \notin \widetilde{\mathfrak{K}}$, then for all $\tilde{\alpha} \in \Gamma$ such that $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \leq (\lambda'^{R}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \circ \lambda'^{R}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \leq (\lambda'^{I}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) \wedge (\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$. By hypothesis, Then, $\lambda'^{R}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{R}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$ and $\lambda'^{I}_{\widetilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{I}_{\widetilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) \wedge (\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\widetilde{\mathfrak{K}}})(\mathfrak{h})$, which is a contradiction. Thus, $\mathfrak{h} \in \widetilde{\mathfrak{K}}$. Hence, $\widetilde{\mathfrak{K}}$ is a QΓ-ID of $\widetilde{\mathfrak{G}}$.

Definition 3.16: A CF set $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\widetilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ on $\widetilde{\mathfrak{G}}$ with $\pi', \omega' \in CIN$. Define the set

$\mathcal{P}(\eta'^{R} + \iota\eta'^{I}, (\pi', \omega')) = \{\mathfrak{h} \in \widetilde{\mathfrak{G}} | \eta'^{R}(\mathfrak{h}) \geq \pi', \eta'^{I}(\mathfrak{h}) \geq \omega'\}$ is called $(\boldsymbol{\pi}', \boldsymbol{\omega}')$-cut of a CF set of $\widetilde{\mathfrak{G}}$.

In the following theorems, we give a relationship between a Γ-SSGs and the (π', ω')-cut of a CF set.

Theorem 3.17: A CF set $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CF Γ-SSG (CFLΓ-ID, CFRΓ-ID, CFΓ-ID,) of an Γ-SG $\widetilde{\mathfrak{G}}$ if and only if $\emptyset \neq \mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega')) \subseteq \widetilde{\mathfrak{G}}$ is a Γ-SSG (LΓ-ID, RΓ-ID, Γ-ID) for all $\pi', \omega' \in CIN$.

Proof: Let $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CF Γ-SSG of $\widetilde{\mathfrak{G}}$ and $\mathfrak{h}, \mathfrak{r} \in \eta'^{RI}$, $\tilde{\alpha} \in \Gamma$ such that $\pi', \omega' \in CIN$. Then $\eta'^R(\mathfrak{h}) = \eta'^R(\mathfrak{r}) \geq \pi'$ and $\eta'^I(\mathfrak{h}) = \eta'^I(\mathfrak{r}) \geq \omega'$. If $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \notin \mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$, then $\eta'^R(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) < \pi'$ and $\eta'^I(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) < \omega'$. Since $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CF Γ-SSG we have $\eta'^R(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^R(\mathfrak{h}) \wedge \eta'^R(\mathfrak{r})$ and $\eta'^I(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^I(\mathfrak{h}) \wedge \eta'^I(\mathfrak{r})$. Which is a contradiction so, $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \in \mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$. Hence, $\mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$ is a Γ-SSG of $\widetilde{\mathfrak{G}}$.

For the converse, assume that $\mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$ is a Γ-SSG of $\widetilde{\mathfrak{G}}$ and $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$.

If $\mathfrak{h}, \mathfrak{r} \in \eta'^{RI}$, then $\eta'^R(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \eta'^R(\mathfrak{h}) = \eta'^R(\mathfrak{r}) \geq \pi'$ and $\eta'^I(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \eta'^I(\mathfrak{h}) = \eta'^I(\mathfrak{r}) \geq \omega'$. Thus, $\eta'^R(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^R(\mathfrak{h}) \wedge \eta'^R(\mathfrak{r})$ and $\eta'^I(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^I(\mathfrak{h}) \wedge \eta'^I(\mathfrak{r})$.

If $\mathfrak{h}, \mathfrak{r} \notin \eta'^{RI}$, then $\eta'^R(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^R(\mathfrak{h}) \wedge \eta'^R(\mathfrak{r})$ and $\eta'^I(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^I(\mathfrak{h}) \wedge \eta'^I(\mathfrak{r})$.

Hence, $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CFΓ-SSG.

Theorem 3.18: *A CF set $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CFBΓ-ID (CFIΓ-ID, CF (1,2)-Γ-ID) of an Γ-SG $\widetilde{\mathfrak{G}}$ if and only if $\emptyset \neq \mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega')) \subseteq \widetilde{\mathfrak{G}}$ is a BΓ-ID (IΓ-ID, (1,2)-Γ-ID) of $\widetilde{\mathfrak{G}}$ for all $\pi', \omega' \in CIN$.*

Proof: Let $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$. Then $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CFΓ-SSG. Thus by Theorem 3.17, $\mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$ is a Γ-SSG of $\widetilde{\mathfrak{G}}$. Let $\mathfrak{h}, \mathfrak{r}, \mathfrak{w} \in \eta'^{RI}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$ such that $\pi', \omega' \in CIN$. Then $\eta'^R(\mathfrak{h}) = \eta'^R(\mathfrak{w}) \geq \pi'$ and $\eta'^I(\mathfrak{h}) = \eta'^I(\mathfrak{w}) \geq \omega'$.

If $\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w} \notin \mathcal{P}(\eta''^R + \iota\eta'^I, (\pi', \omega'))$, then $\eta'^R(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) < \pi'$ and $\eta'^I(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) < \omega'$. Since $\eta'^{RI} = (\widetilde{\mathfrak{G}}; \eta'^R, \eta'^I) = (\widetilde{\mathfrak{G}}; \eta'^R + \iota\eta'^I)$ is a CFGBΓ-ID we have $\eta'^R(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \eta'^R(\mathfrak{h}) \wedge \eta'^R(\mathfrak{r})$ and $\eta'^I(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \eta'^I(\mathfrak{h}) \wedge \eta'^I(\mathfrak{w})$.. Which is a contradiction so, $\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w} \in \mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$. Hence, $\mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$ is a BΓ-ID of $\widetilde{\mathfrak{G}}$.

For the converse, assume that $\mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$ is a BΓ-ID of $\widetilde{\mathfrak{G}}$. Then $\mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$ is an Γ-SSG of $\widetilde{\mathfrak{G}}$. Thus by Theorem 3.17, $\mathfrak{h}\alpha\mathfrak{r} \in \mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$. Hence, $\mathcal{P}(\eta'^R + \iota\eta'^I, (\pi', \omega'))$ is an Γ-SSG of $\widetilde{\mathfrak{G}}$.

Let $\mathfrak{h}, \mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\alpha, \tilde{\beta} \in \Gamma$.

If $\mathfrak{h}, \mathfrak{w} \in \eta'^{RI}$, then $\eta'^{R}(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) = \eta'^{R}(\mathfrak{h}) = \eta'^{R}(\mathfrak{w}) \geq \pi'$ and $\eta'^{I}(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) = \eta'^{I}(\mathfrak{h}) = \eta'^{I}(\mathfrak{w}) \geq \omega'$. Thus, $\eta'^{R}(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \eta'^{R}(\mathfrak{h}) \wedge \eta'^{R}(\mathfrak{w})$ and $\eta'^{I}(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \eta'^{I}(\mathfrak{h}) \wedge \eta'^{I}(\mathfrak{w})$.

If $\mathfrak{h}, \mathfrak{w} \notin \eta'^{RI}$, then $\eta'^{R}(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \eta'^{R}(\mathfrak{h}) \wedge \eta'^{R}(\mathfrak{w})$ and $\eta'^{I}(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \eta'^{I}(\mathfrak{h}) \wedge \eta'^{I}(\mathfrak{w})$.

Hence, $\eta'^{RI} = (\tilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\tilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ is a CFBΓ-ID of $\tilde{\mathfrak{G}}$.

Corollary 3.19: *A CF set $\eta'^{RI} = (\tilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\tilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ is a CFGBΓ-ID of an Γ-SG $\mathfrak{G}$ if and only if $\emptyset \neq \mathcal{P}(\eta'^{R} + \iota\eta'^{I}, (\pi', \omega')) \subseteq \tilde{\mathfrak{G}}$ is a GBΓ-ID of $\tilde{\mathfrak{G}}$ for all $\pi', \omega' \in CIN$.*

Theorem 3.20: *A CF set $\eta'^{RI} = (\tilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\tilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ is a CFQΓ-ID of an Γ-SG $\mathfrak{G}$ if and only if $\emptyset \neq \mathcal{P}(\eta'^{R} + \iota\eta'^{I}, (\pi', \omega')) \subseteq \tilde{\mathfrak{G}}$ is a QΓ-ID of $\tilde{\mathfrak{G}}$ for all $\pi', \omega' \in [0,1]$.*

Proof: Let $\eta'^{RI} = (\tilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\tilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ is a CFQ Γ-ID of $\tilde{\mathfrak{G}}$ and $\mathfrak{h} \in \eta'^{RI}$, $\tilde{\alpha} \in \Gamma$ such that $\pi', \omega' \in CIN$. Then $(\lambda'^{R}_{\tilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^{R})$ $(\mathfrak{h}) \wedge (\eta'^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\tilde{\mathfrak{G}}})$ $(\mathfrak{h}) \geq \pi'$ and $(\lambda'^{I}_{\tilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) \wedge (\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\tilde{\mathfrak{G}}})(\mathfrak{h}) \geq \omega'$. If $\mathfrak{h} \notin \mathcal{P}(\eta'^{R} + \iota\eta'^{I}, (\pi', \omega'))$, then $\lambda'^{R}_{\tilde{\mathfrak{K}}}(\mathfrak{h}) \leq (\lambda'^{R}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \odot_{\tilde{\alpha}} \lambda'^{R}_{\tilde{\mathfrak{K}}})(\mathfrak{h})$ and $\lambda'^{I}_{\tilde{\mathfrak{K}}}(\mathfrak{h}) \leq (\lambda'^{I}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})$ $(\mathfrak{h}) \wedge (\eta'^{I} \odot_{\tilde{\alpha}} \lambda'^{I}_{\tilde{\mathfrak{K}}})(\mathfrak{h})$. Since $\eta'^{RI} = (\tilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\tilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ is a CFQ Γ-ID of $\tilde{\mathfrak{G}}$ we have $\lambda'^{R}_{\tilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{R}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \odot_{\tilde{\alpha}} \lambda'^{R}_{\tilde{\mathfrak{K}}})(\mathfrak{h})$ and $\lambda'^{I}_{\tilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{I}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})$ $(\mathfrak{h}) \wedge (\eta'^{I} \odot_{\tilde{\alpha}} \lambda'^{I}_{\tilde{\mathfrak{K}}})(\mathfrak{h})$, which is a contradiction. Thus, $\mathfrak{h} \in \mathcal{P}(\eta'^{R} + \iota\eta'^{I}, (\pi', \omega'))$. Hence, $\mathcal{P}(\eta'^{R} + \iota\eta'^{I}, (\pi', \omega'))$ is a QΓ-ID on $\tilde{\mathfrak{G}}$.

Suppose that $\eta'^{RI} = (\tilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\tilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ is a QΓ-ID of $\tilde{\mathfrak{G}}$ and let $\mathfrak{h} \in \tilde{\mathfrak{G}}$ and $\tilde{\alpha} \in \Gamma$ Then the following cases:

Case 1: If $\mathfrak{h} \in \tilde{\mathfrak{K}}$, then $(\lambda'^{R}_{\tilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\tilde{\mathfrak{G}}})(\mathfrak{h}) \geq \pi'$ and $(\lambda'^{I}_{\tilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^{I})$ $(\mathfrak{h}) \wedge (\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\tilde{\mathfrak{G}}})(\mathfrak{h}) \geq \omega'$.

Case 2: If $\mathfrak{h} \notin \tilde{\mathfrak{K}}$, then $\mathfrak{h}$ is expressible $\mathfrak{h} = \mathfrak{y}\alpha\mathfrak{z}$ or not.

Suppose $\mathfrak{h}$ is expressible as $\mathfrak{h} = \mathfrak{y}\tilde{\alpha}\mathfrak{z}$. Since $\mathfrak{h} \notin \tilde{\mathfrak{K}}$ either $\mathfrak{y} \in \tilde{\mathfrak{K}}$ or $\mathfrak{z} \notin \tilde{\mathfrak{K}}$. If $\mathfrak{y} \in \tilde{\mathfrak{K}}$ and $\mathfrak{z} \notin \tilde{\mathfrak{K}}$, then there cannot be another exprssion of the form $\mathfrak{x} = \mathfrak{a}\alpha\mathfrak{b}$ where $\mathfrak{a}, \mathfrak{b} \in \tilde{\mathfrak{K}}$ and $\tilde{\alpha} \in \Gamma$. Then $\mathfrak{x} \in \tilde{\mathfrak{F}}\mathfrak{K} \cap \tilde{\mathfrak{K}}\mathfrak{F} \subseteq \tilde{\mathfrak{K}}$. Thus, $(\lambda'^{R}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) < \pi'$ or $(\eta'^{R} \odot_{\tilde{\alpha}} \lambda'^{R}_{\tilde{\mathfrak{K}}})(\mathfrak{h}) < \pi'$ and $(\lambda'^{I}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})(\mathfrak{h}) < \omega'$ or $(\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\tilde{\mathfrak{K}}})(\mathfrak{h}) < \omega'$. Hence, $\lambda'^{R}_{\tilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{R}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{R})(\mathfrak{h}) \wedge (\eta'^{R} \circ_{\tilde{\alpha}} \lambda'^{R}_{\tilde{\mathfrak{K}}})(\mathfrak{h})$ and $\lambda'^{I}_{\tilde{\mathfrak{K}}}(\mathfrak{h}) \geq (\lambda'^{I}_{\tilde{\mathfrak{K}}} \circ_{\tilde{\alpha}} \eta'^{I})$ $(\mathfrak{h}) \wedge (\eta'^{I} \circ_{\tilde{\alpha}} \lambda'^{I}_{\tilde{\mathfrak{K}}})(\mathfrak{h})$. Therefore, $\eta'^{RI} = (\tilde{\mathfrak{G}}; \eta'^{R}, \eta'^{I}) = (\tilde{\mathfrak{G}}; \eta'^{R} + \iota\eta'^{I})$ is a CFQΓ-ID of $\tilde{\mathfrak{G}}$.

Theorem 3.21: *Let η'^{RI} be CF set of an Γ-SG $\tilde{\mathfrak{G}}$. Then η'^{RI} is a CFΓ-SSG of $\tilde{\mathfrak{G}}$ if and on ly if $\eta'^{RI} \odot \eta'^{RI} \precsim \eta'^{RI}$.*

Proof: Assume that η'^{RI} is a CFΓ-SSG of $\tilde{\mathfrak{G}}$ and let $\mathfrak{h} \in \tilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$

If $F_{\mathfrak{h}} = \emptyset$. Then $(\eta'^R \circ_{\tilde{\alpha}} \eta'^R)(\mathfrak{h}) = 0 \leq \eta'^R(\mathfrak{h})$ and $(\eta'^I \circ_{\tilde{\alpha}} \eta'^I)(\mathfrak{h}) = 0 \leq \eta'^I(\mathfrak{h})$.

If $F_{\mathfrak{h}} \neq \emptyset$. Then $(\eta'^R \circ_{\tilde{\alpha}} \eta'^R)(\mathfrak{h}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \eta'^R(\mathfrak{i}) \wedge \eta'^R(\mathfrak{j}) \leq \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \eta'^R(\mathfrak{i}\tilde{\alpha}\mathfrak{j}) = \eta'^R(\mathfrak{h})$ and $(\eta'^I \circ_{\tilde{\alpha}} \eta'^I)(\mathfrak{h}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \eta'^I(\mathfrak{i}) \wedge \eta'^I(\mathfrak{j}) \leq \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \eta'^I(\mathfrak{i}\tilde{\alpha}\mathfrak{j}) = \eta'^I(\mathfrak{h})$.

Hence, $\eta'^{RI} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$.

Conversely, assume that $\eta'^{RI} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$ and $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Then

$\eta'^R(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \leq (\eta'^R \circ_{\tilde{\alpha}} \eta'^R)(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\alpha\mathfrak{r}}} \eta'^R(\mathfrak{i}) \wedge \eta'^R(\mathfrak{j}) \leq \eta'^R(\mathfrak{h}) \wedge \eta'^R(\mathfrak{r})$,

$$\eta'^I(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \leq (\eta'^I \circ_{\tilde{\alpha}} \eta'^I)(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} \eta'^I(\mathfrak{i}) \wedge \eta'^I(\mathfrak{j}) \leq \eta'^I(\mathfrak{h}) \wedge \eta'^I(\mathfrak{r}).$$

Thus, η'^{RI} is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$.

Theorem 3.22: *Let η'^{RI} be CFS of an Γ-SG $\widetilde{\mathfrak{G}}$. Then η'^{RI} is a CFLΓ-ID (CFRΓ-ID) of $\widetilde{\mathfrak{G}}$ if and on ly if $\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}(\eta'^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}} \precsim \lambda'^{RI}_{\widetilde{\mathfrak{G}}})$.*

Proof: Assume that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h} \in \widetilde{\mathfrak{G}}$.

If $F_{\mathfrak{h}} = \emptyset$. Then $(\lambda'^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^R)(\mathfrak{h}) = 0 \leq \eta'^R(\mathfrak{h})$ and $(\lambda'^I_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^I)(\mathfrak{h}) = 0 \leq \eta'^I(\mathfrak{h})$.

If $F_{\mathfrak{h}} \neq \emptyset$. Then $(\lambda'^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^R)(\mathfrak{h}) \leq \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \lambda'^R_{\widetilde{\mathfrak{G}}}(\mathfrak{i}) \wedge \eta'^R(\mathfrak{j}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} 1 \wedge \eta'^R(\mathfrak{j}) = \eta'^R(\mathfrak{j}) = \eta'^R(\mathfrak{h})$
and $(\lambda'^I_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^I)(\mathfrak{h}) \leq \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \lambda'^I_{\widetilde{\mathfrak{G}}}(\mathfrak{i}) \wedge \eta'^I(\mathfrak{j}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \iota 1 \wedge \eta'^I(\mathfrak{j}) = \eta'^I(\mathfrak{j}) = \eta'^I(\mathfrak{h})$. Hence, $\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$.

Conversely, assume that $\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$ and $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Then

$$\begin{aligned} \acute{\eta}^R(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) &\leq (\lambda'^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\alpha}} \eta'^R)(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} \lambda'^R_{\widetilde{\mathfrak{G}}}(\mathfrak{i}) \wedge \eta'^R(\mathfrak{j}) \\ &= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} 1 \wedge \eta'^R(\mathfrak{j}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} \eta'^R(\mathfrak{j}) = \eta'^R(\mathfrak{r}) \end{aligned}$$

and

$$\begin{aligned} \acute{\eta}^I(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) &\leq (\lambda'^I_{\widetilde{\mathfrak{G}}} \circ \eta'^I)(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} \lambda'^I_{\widetilde{\mathfrak{G}}}(\mathfrak{i}) \wedge \eta'^I(\mathfrak{j}) \\ &= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} 1 \wedge \eta'^I(\mathfrak{j}) = \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} \eta'^I(\mathfrak{j}) = \eta'^I(\mathfrak{r}) \end{aligned}$$

Thus, η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$.

Corollary 3.23: *Let η'^{RI} be CFS of an Γ-SG $\widetilde{\mathfrak{G}}$. Then η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$ if and on ly if $\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$ and $\eta'^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}} \precsim \eta'^{RI}$*

Theorem 3.24: *Let η'^{RI} be CFS of an Γ-SG $\widetilde{\mathfrak{G}}$. Then η'^{RI} is a CFBΓ-ID (CFIΓ-ID, CF (1,2)-Γ-ID) of $\widetilde{\mathfrak{G}}$ if and only if $\eta'^{RI} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$ and $\eta'^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\beta}} \eta'^{RI} \precsim \eta'^{RI}$.*

Proof: Assume that η'^{RI} is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$. Then η'^{RI} is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$. Thus, $\eta'^{RI} \circ_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$. Let $\mathfrak{h} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$.

If $F_{\mathfrak{h}} = \emptyset$. Then $(\eta'^R \circ_{\tilde{\alpha}} \lambda'^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \eta'^R)(\mathfrak{h}) = 0 \leq \eta'^R(\mathfrak{h})$ and $(\eta'^I \circ_{\tilde{\alpha}} \lambda'^I_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \eta'^R)(\mathfrak{h}) = 0 \leq \eta'^I(\mathfrak{h})$.

If $F_{\mathfrak{h}} \neq \emptyset$. Then

$$\begin{aligned}
\left(\acute{\eta}^R \circ_{\tilde{\alpha}} \acute{\lambda}^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^R\right)(\mathfrak{h}) &= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \acute{\eta}^R(\mathfrak{i}) \wedge (\acute{\lambda}^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^R)(\mathfrak{j}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \acute{\eta}^R(\mathfrak{i}) \wedge \Big(\bigvee_{(\mathfrak{m},\mathfrak{e}) \in F_{\mathfrak{j}}} \acute{\lambda}^R_{\widetilde{\mathfrak{G}}}(\mathfrak{m}) \wedge \acute{\eta}^R)(\mathfrak{e})\Big) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \acute{\eta}^R(\mathfrak{i}) \wedge \Big(\bigvee_{(\mathfrak{m},\mathfrak{e}) \in F_{\mathfrak{j}}} 1 \wedge \acute{\eta}^R)(\mathfrak{e})\Big) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{e}) \in F_{\mathfrak{h}}} \acute{\eta}^R(\mathfrak{i}) \wedge \acute{\eta}^R(\mathfrak{e}) = \acute{\eta}^R)(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}
\left(\acute{\eta}^I \circ_{\tilde{\alpha}} \acute{\lambda}^I_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^I\right)(\mathfrak{h}) &= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \acute{\eta}^I(\mathfrak{i}) \wedge (\acute{\lambda}^I_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^I)(\mathfrak{j}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \acute{\eta}^I(\mathfrak{i}) \wedge \Big(\bigvee_{(\mathfrak{m},\mathfrak{e}) \in F_{\mathfrak{j}}} \acute{\lambda}^I_{\widetilde{\mathfrak{G}}}(\mathfrak{m}) \wedge \acute{\eta}^I)(\mathfrak{e})\Big) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}}} \acute{\eta}^I(\mathfrak{i}) \wedge \Big(\bigvee_{(\mathfrak{m},\mathfrak{e}) \in F_{\mathfrak{j}}} 1 \wedge \acute{\eta}^I)(\mathfrak{e})\Big) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{e}) \in F_{\mathfrak{h}}} \acute{\eta}^I(\mathfrak{i}) \wedge \acute{\eta}^I(\mathfrak{e}) = \acute{\eta}^I)(\mathfrak{h}),
\end{aligned}$$

Hence, $\acute{\eta}^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\beta}} \acute{\eta}^{RI} \precsim \acute{\eta}^{RI}$.

Conversely, assume that $\acute{\eta}^{RI} \odot_{\tilde{\alpha}} \acute{\eta}^{RI} \precsim \acute{\eta}^{RI}$ and $\acute{\eta}^{RI} \odot_{\tilde{\alpha}} \acute{\lambda}^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\beta}} \acute{\eta}^{RI} \precsim \acute{\eta}^{RI}$. Let $\mathfrak{h}, \mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$. Since $\acute{\eta}^{RI} \circ_{\tilde{\alpha}} \acute{\eta}^{RI} \precsim \acute{\eta}^{RI}$ we have $\acute{\eta}^{RI}$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$ by Theorem 3.21.

$$\begin{aligned}
\acute{\eta}^R\big(\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}\big) \leq \acute{\eta}^R \circ_{\tilde{\alpha}} \big(\acute{\lambda}^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^R\big)(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) &= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^R(\mathfrak{i}) \wedge (\acute{\lambda}^R_{\widetilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^R)(\mathfrak{j}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j}) \in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^R(\mathfrak{i}) \wedge \Big(\bigvee_{(\mathfrak{m},\mathfrak{e}) \in F_{\mathfrak{j}}} \acute{\lambda}^R_{\widetilde{\mathfrak{G}}}(\mathfrak{m}) \wedge \acute{\eta}^R)(\mathfrak{e})\Big)
\end{aligned}$$

$$= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^R(\mathfrak{i}) \wedge (\bigvee_{(\mathfrak{m},\mathfrak{e})\in F_{\mathfrak{j}}} 1 \wedge \acute{\eta}^R)(\mathfrak{e})$$

$$\leq \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^R(\mathfrak{i}) \wedge (\bigvee_{(\mathfrak{m},\mathfrak{e})\in F_{\mathfrak{j}}} \acute{\eta}^R(\mathfrak{e})) = \acute{\eta}^R(\mathfrak{h}) \wedge \acute{\eta}^R)(\mathfrak{w})$$

and

$$\acute{\eta}^I\left(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}\right) \leq \acute{\eta}^I \circ_{\tilde{\alpha}} \left(\acute{\lambda}^I_{\tilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^I\right)(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) = \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^I(\mathfrak{i}) \wedge (\acute{\lambda}^i_{\tilde{\mathfrak{G}}} \circ_{\tilde{\beta}} \acute{\eta}^I)(\mathfrak{j})$$

$$= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^I(\mathfrak{i}) \wedge (\bigvee_{(\mathfrak{m},\mathfrak{e})\in F_{\mathfrak{j}}} \acute{\lambda}^I_{\tilde{\mathfrak{G}}}(\mathfrak{m}) \wedge \acute{\eta}^I)(\mathfrak{e}))$$

$$= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^I(\mathfrak{i}) \wedge (\bigvee_{(\mathfrak{m},\mathfrak{e})\in F_{\mathfrak{j}}} 1 \wedge \acute{\eta}^I)(\mathfrak{e})$$

$$\leq \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{h}\alpha\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \acute{\eta}^I(\mathfrak{i}) \wedge (\bigvee_{(\mathfrak{m},\mathfrak{e})\in F_{\mathfrak{j}}} \acute{\eta}^R(\mathfrak{e})) = \acute{\eta}^I(\mathfrak{h}) \wedge \acute{\eta}^I)(\mathfrak{w}).$$

Thus, η'^{RI} is a CFBΓ-ID of $\tilde{\mathfrak{G}}$.

Corollary 3.25: *Let η'^{RI} be CF set of an Γ-SG $\tilde{\mathfrak{G}}$. Then η'^{RI} is a CFGΓ-ID of $\tilde{\mathfrak{G}}$ if and on ly if $\eta'^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\tilde{\mathfrak{G}}} \odot_{\tilde{\beta}} \eta'^{RI} \precsim \eta'^{RI}$.*

Theorem 3.26: Let η'^{RI} be CFS of an Γ-SG $\tilde{\mathfrak{G}}$. Then η'^{RI} is a CFQΓ-ID of $\tilde{\mathfrak{G}}$ if and on ly if $\lambda'^{RI}_{\tilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \cap' \eta'^{RI} \odot_{\tilde{\beta}} \lambda'^{RI}_{\tilde{\mathfrak{G}}} \precsim \eta'^{RI}$.

Proof: Assume that η'^{RI} is a CFQΓ-ID of $\tilde{\mathfrak{G}}$ and let $\mathfrak{h} \in \tilde{\mathfrak{G}}$. It follows Theorem 3.2**Error! Reference source not found.**, $\lambda'^{RI}_{\tilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$ and $\eta'^{RI} \odot_{\tilde{\alpha}} \overline{\overline{\tilde{\mathfrak{G}}}} \precsim \eta'^{RI}$. Thus, $\lambda'^{RI}_{\tilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \cap' \eta'^{RI} \odot_{\tilde{\beta}} \lambda'^{RI}_{\tilde{\mathfrak{G}}} \precsim \eta'^{RI}$.

Conversely, assume that $\lambda'^{RI}_{\tilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \cap' \eta'^{RI} \odot_{\tilde{\beta}} \lambda'^{RI}_{\tilde{\mathfrak{G}}} \precsim \eta'^{RI}$. Then $\lambda'^{RI}_{\tilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI} \precsim \eta'^{RI}$ and $\eta'^{RI} \odot_{\tilde{\alpha}}, \overline{\overline{\tilde{\mathfrak{G}}}} \precsim \eta'^{RI}$. It follows Theorem 3.22, η'^{RI} is a CFLΓ-ID and η'^{RI} is a CFRΓ-ID of $\tilde{\mathfrak{G}}$. Thus, by Theorem 4.19, η'^{RI} is a CFQΓ-ID of $\tilde{\mathfrak{G}}$.

4. Essential conditions of coincidences of complex fuzzy ideals in semigroups

In this section, we investigate essential conditions of coincidences of complex fuzzy ideals in semigroups.

The following lemma shows that every CFΓ-ID is a CFBΓ-ID of an Γ-SG.

Lemma 4.1: *Every CFΓ-ID of an Γ-SG $\tilde{\mathfrak{G}}$ is a CFBΓ-ID of $\tilde{\mathfrak{G}}$.*

Proof: Suppose that η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h},\mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$, we have that η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h})$ and so $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{r})$ Hence, η'^{RI} is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$.Let $\mathfrak{h},\mathfrak{r},\mathfrak{k} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha},\tilde{\beta} \in \Gamma$. Since η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$, we have that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{k}) = \eta'^{RI}((\mathfrak{h}\mathfrak{r}\mathfrak{k})) \geq \eta'^{RI}(\mathfrak{k})$ and so, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{k}) \geq \eta'^{RI}(\mathfrak{k}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{k})$. Hence, η'^{RI} is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$.

In order to consider the converse of Lemma 4.1, we need to reinforce the condition of an Γ-SG.

Theorem 4.2: *In a regular Γ-SG of $\widetilde{\mathfrak{G}}$, the CFBΓ-IDs and the CFΓ-IDs coincide.*

Proof: Suppose that η'^{RI} is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h},\mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is regular, we have $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \in (\mathfrak{h}\Gamma\widetilde{\mathfrak{G}}\Gamma\mathfrak{h})\widetilde{\mathfrak{G}} \subseteq \mathfrak{h}\Gamma\widetilde{\mathfrak{G}}$ which implies that $\mathfrak{h}\tilde{\alpha}\mathfrak{r} = \mathfrak{h}\tilde{\alpha}\mathfrak{k}\tilde{\beta}\mathfrak{h}$ for some $\mathfrak{k} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha},\tilde{\beta}$ Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}\tilde{\beta}\mathfrak{h}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{h}) = \eta'^{RI}(\mathfrak{h})$. Hence η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

The following lemma shows that every CFBΓ-ID is a CF (1,2)-Γ-ID on an Γ-SG.

Lemma 4.3: *Every CFBΓ-ID of an Γ-SG of $\widetilde{\mathfrak{G}}$ is a CF (1,2)-Γ-ID of $\widetilde{\mathfrak{G}}$.*

Proof: Suppose that η'^{RI} is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h},\mathfrak{r},\mathfrak{k},\mathfrak{w} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha},\tilde{\beta},\tilde{\gamma} \in \Gamma$. Then

$$\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}(\mathfrak{k}\tilde{\gamma}\mathfrak{w})) = \acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{k})\tilde{\gamma}\mathfrak{w}) \geq (\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}) \wedge \acute{\eta}^{RI}(\mathfrak{w}))$$
$$\geq ((\acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{k})) \wedge \acute{\eta}^{RI}(\mathfrak{w})) = \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{k}) \wedge \acute{\eta}^{RI}(\mathfrak{w}).$$

Thus, $\acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}(\mathfrak{k}\gamma\mathfrak{w})) \geq \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{k}) \wedge \acute{\eta}^{RI}(\mathfrak{w})$. Hence, $\acute{\eta}^{RI}$ is a CF (1,2)-Γ-ID of $\widetilde{\mathfrak{G}}$.

To consider the converse of Lemma 4.3 we need to reinforce the condition of an Γ-SG.

Theorem 4.4: *In a regular Γ-SG of $\widetilde{\mathfrak{G}}$, the CF (1,2)-Γ-IDs and the CFBΓ-IDs coincide.*

Proof: Assume that $\acute{\eta}^{RI}$ is a CF (1,2)-Γ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h},\mathfrak{r},\mathfrak{k} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha},\tilde{\beta} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is regular, we have $\mathfrak{h}\tilde{\alpha}\mathfrak{r} \in (\mathfrak{h}\Gamma\widetilde{\mathfrak{G}}\Gamma\mathfrak{h})\Gamma\widetilde{\mathfrak{G}} \subseteq \mathfrak{h}\Gamma\widetilde{\mathfrak{G}}\Gamma\mathfrak{h}$, which implies that $\mathfrak{h}\tilde{\alpha}\mathfrak{r} = \mathfrak{h}\tilde{\alpha}\mathfrak{s}\tilde{\gamma}\mathfrak{h}$ for some $\mathfrak{s} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha},\tilde{\gamma} \in \Gamma$. Thus,

$$\acute{\eta}^{RI}\big(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{k}\big) = \acute{\eta}^{RI}((\mathfrak{h}\tilde{\alpha}\mathfrak{s}\gamma\mathfrak{h})\tilde{\beta}\mathfrak{k}) = \acute{\eta}^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{s}\gamma(\mathfrak{h}\tilde{\beta}\mathfrak{k}))$$
$$\geq \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{k}) = \acute{\eta}^{RI}(\mathfrak{h}) \wedge \acute{\eta}^{RI}(\mathfrak{k}).$$

Hence, $\acute{\eta}^{RI}$ is a CFIB of $\widetilde{\mathfrak{G}}$.

The following theorem shows that every CFΓ-ID is a CF $(1,2)$-Γ-ID on an Γ-SG.

Theorem 4.5: *Every CFΓ-ID of an Γ-SG of $\widetilde{\mathfrak{G}}$ is a CF $(1,2)$-Γ-ID of $\widetilde{\mathfrak{G}}$.*

Proof: Suppose that η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$, we have that η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, $\overline{\eta}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \overline{\eta}(\mathfrak{h})$ and so, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{r})$. Hence, η'^{RI} is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$.

Let $\mathfrak{h}, \mathfrak{r}, \mathfrak{k}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$. Since η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$, we have that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Thus,

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) = \eta'^{RI}((\mathfrak{h}\tilde{\alpha}\mathfrak{r})\tilde{\beta}\mathfrak{w}) \geq \eta'^{RI}(\mathfrak{w})$$

and so,

$$\eta'^{RI}((\mathfrak{h}\tilde{\alpha}\mathfrak{r})\tilde{\beta}\mathfrak{w}) \geq \eta'^{RI}(\mathfrak{w}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{k}) \wedge \eta'^{RI}(\mathfrak{w}).$$

Hence, η'^{RI} is a CF $(1,2)$-Γ-ID of $\widetilde{\mathfrak{G}}$.

In the following results, we show that CFGIBs and CFBΓ-IDs on types of SGs.

Lemma 4.6: *In a regular Γ-SG of $\widetilde{\mathfrak{G}}$, the CFGBΓ-ID and the CFBΓ-ID coincide.*

Proof: Suppose that η'^{RI} is a CFGBΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is regular, there exist $\mathfrak{k} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega} \in \Gamma$ such that $\mathfrak{r} = \mathfrak{r}\tilde{\beta}\mathfrak{k}\tilde{\omega}\mathfrak{r}$. Thus,

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}(\mathfrak{r}\tilde{\beta}\mathfrak{k}\tilde{\omega}\mathfrak{r})) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}(\mathfrak{r}\tilde{\beta}\mathfrak{k})\tilde{\omega}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{r}).$$

Hence, η'^{RI} is a CFSG of $\widetilde{\mathfrak{G}}$. By Definition 3.1(6) we have η'^{RI} is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$.

Lemma 4.7: *In left (right) regular SG of $\widetilde{\mathfrak{G}}$, the CFGBΓ-ID and the CFBΓ-ID coincide.*

Proof: Suppose that η'^{RI} is a CFGBΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is left regular, there exist $\mathfrak{k} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega} \in \Gamma$ such that $\mathfrak{r} = \mathfrak{k}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{r}$. Thus,

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}(\mathfrak{k}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{r})) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}(\mathfrak{k}\tilde{\beta}\mathfrak{r})\tilde{\omega}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{v}).$$

Hence, η'^{RI} is a CFSG of $\widetilde{\mathfrak{G}}$. By Definition 3.1(6) we have η'^{RI} is a CFBΓ-ID of $\widetilde{\mathfrak{G}}$.

Lemma 4.8: *In left (right) quasi-regular Γ-SG $\widetilde{\mathfrak{G}}$, the CFGBΓ-ID and the CFBΓ-ID coincide.*

Proof: Suppose that η'^{RI} is a CFGBΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is left quasi-regular, there exist $\mathfrak{k}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega}, \gamma \in \Gamma$ such that $\mathfrak{r} =$

$\mathfrak{k}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{w}\gamma\mathfrak{r}$. Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \eta'^{RI}((\mathfrak{h}\tilde{\alpha}(\mathfrak{k}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{w}\gamma\mathfrak{r})) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}(\mathfrak{k}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{w})\gamma\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{r})$. Hence, η'^{RI} is a CFSG of $\tilde{\mathfrak{G}}$. By Definition 3.1(6) we have η'^{RI} is a CFBΓ-ID of $\tilde{\mathfrak{G}}$.

Lemma 4.9: *In weakly regular SG of $\tilde{\mathfrak{G}}$, the CFGBΓ-ID and the CFBΓ-ID coincide.*

Proof: Suppose that η'^{RI} is a CFGBΓ-ID of $\tilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{r} \in \tilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\tilde{\mathfrak{G}}$ is weakly regular, there exists $\mathfrak{k}, \mathfrak{w} \in \tilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega}, \gamma \in \Gamma$ such that $\mathfrak{h} = \mathfrak{h}\tilde{\beta}\mathfrak{k}\tilde{\omega}\mathfrak{h}\gamma\mathfrak{w}$. Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) = \eta'^{RI}((\mathfrak{h}\tilde{\beta}\mathfrak{k}\tilde{\omega}\mathfrak{h}\gamma\mathfrak{w})\tilde{\alpha}\mathfrak{r}) = \eta'^{RI}(\mathfrak{h}\tilde{\beta}(\mathfrak{k}\tilde{\omega}\mathfrak{h}\gamma\mathfrak{w})\tilde{\alpha}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{r})$. Hence, η'^{RI} is a CFSG of $\tilde{\mathfrak{G}}$. By Definition 3.1(6) we have η'^{RI} is a CFBΓ-ID of $\tilde{\mathfrak{G}}$.

By Lemma 4.6, 4.7, 4.8 and 4.9, we have Theorem 4.10.

Theorem 4.10: *Let $\tilde{\mathfrak{G}}$ be an Γ-SG. If $\tilde{\mathfrak{G}}$ is regular, left (right) regular, left (right) quasi-regular or weakly regular, then the CFGBΓ-ID and the CFBΓ-ID coincide.*

The following lemma shows that every CFΓ-ID is a CFIΓ-ID on an Γ-SG.

Lemma 4.11: *Every CFΓ-ID of an Γ-SG of $\tilde{\mathfrak{G}}$ is a CFIΓ-ID of $\tilde{\mathfrak{G}}$.*

Proof: Suppose that η'^{RI} is a CFΓ-ID of $\tilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{k} \in \tilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since η'^{RI} is a CFΓ-ID of $\tilde{\mathfrak{G}}$, we have that η'^{RI} is a CFRΓ-ID of $\tilde{\mathfrak{G}}$. Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) \geq \eta'^{RI}(\mathfrak{h})$ and so, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) \geq \eta'^{RI}(\mathfrak{h}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{k})$. Hence, η'^{RI} is a CF Γ -SSG of $\tilde{\mathfrak{G}}$. Let $\mathfrak{h}, \mathfrak{k}, \mathfrak{r} \in \tilde{\mathfrak{G}}$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma$. Then, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}\tilde{\beta}\mathfrak{r}) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}(\mathfrak{k}\tilde{\beta}\mathfrak{r})) \geq \eta'^{RI}(\mathfrak{k}\tilde{\beta}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{k})$ Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}\tilde{\beta}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{k})$. Hence, η'^{RI} is a CFIΓ-ID of $\tilde{\mathfrak{G}}$.

To consider the converse of Lemma 4.11, we need to reinforce the condition of an Γ-SG.

Lemma 4.12: *In a regular Γ-SG of $\tilde{\mathfrak{G}}$, the CFIΓ-IDs and the CFΓ-IDs coincide.*

Proof: Suppose that η'^{RI} is a CFIΓ-ID of $\tilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{k} \in \tilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\tilde{\mathfrak{G}}$ is regular, there exist $\mathfrak{r} \in \tilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{h}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{h}$. Thus,

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}((\mathfrak{h}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{h})\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}((\mathfrak{h}\tilde{\beta}\mathfrak{r})\tilde{\omega}\mathfrak{h})\tilde{\alpha}\mathfrak{k} \geq \eta'^{RI}(\mathfrak{h}).$$

Hence, η'^{RI} is a CFRΓ-ID of $\tilde{\mathfrak{G}}$. Similarly, we can show that η'^{RI} is a CFLΓ-ID of $\tilde{\mathfrak{G}}$. Thus, η'^{RI} is a CFRΓ-ID of $\tilde{\mathfrak{G}}$.

Lemma 4.13: *In a left (right) regular Γ-SG of $\tilde{\mathfrak{G}}$, the CFIΓ-IDs and the CFΓ-IDs coincide.*

Proof: Suppose that η'^{RI} is a CFIΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{k} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is left regular, there exist $\mathfrak{r} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{h}$. Thus,

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}((\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{h})\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}((\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{h})\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}((\mathfrak{r}\tilde{\beta}\mathfrak{h})\tilde{\omega}\mathfrak{h}\tilde{\alpha}\mathfrak{k}) \geq \eta'^{RI}(\mathfrak{h}).$$

Hence η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

Lemma 4.14: *In an intra-regular Γ-SG of $\widetilde{\mathfrak{G}}$, the CFIΓ-IDs and the CFΓ-IDs coincide.*

Proof: Suppose that η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{k} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is intra-regular, there exist $\mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega}, \tilde{\gamma} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{h}\tilde{\gamma}\mathfrak{w}$. Thus,

$$\begin{aligned}\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) &= \eta'^{RI}((\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{h}\gamma\mathfrak{w})\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}(((\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{h})\gamma\mathfrak{w})\tilde{\alpha}\mathfrak{k}) \\ &= \eta'^{RI}((\mathfrak{r}\tilde{\beta}\mathfrak{h})\tilde{\omega}\mathfrak{h}\gamma(\mathfrak{w}\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}(\mathfrak{h}).\end{aligned}$$

Hence, η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

Lemma 4.15: *In a semisimple Γ-SG of $\widetilde{\mathfrak{G}}$, the CFIΓ-IDs and the CFΓ-IDs coincide.*

Proof: Suppose that η'^{RI} is a CFIΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{k} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is semisimple, there exist $\mathfrak{r}, \mathfrak{w}, \mathfrak{t} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega}, \tilde{\gamma}, \tilde{\tau} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{w}\tilde{\gamma}\mathfrak{h}\tilde{\tau}\mathfrak{t}$. Thus,

$$\begin{aligned}\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) &= \eta'^{RI}((\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{w}\tilde{\gamma}\mathfrak{h}\tilde{\tau}\mathfrak{t})\tilde{\alpha}\mathfrak{k}) \\ &= \eta'^{RI}((\mathfrak{r}\tilde{\beta}\mathfrak{h}\tilde{\omega}\mathfrak{w})\tilde{\gamma}\mathfrak{h}\tilde{\tau}(\mathfrak{t}\tilde{\alpha}\mathfrak{k}) \geq \eta'^{RI}(\mathfrak{h})\end{aligned}$$

Hence, η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

Lemma 4.16: *In a left (right) quasi-regular Γ-SG of $\widetilde{\mathfrak{G}}$, the CFIΓ-IDs and the CFΓ-IDs coincide.*

Proof: Suppose that η'^{RI} is a CFIΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{k} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is left quasi-regular, there exist $\mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega}, \tilde{\gamma} \in \Gamma$ such that $\mathfrak{k} = \mathfrak{r}\tilde{\beta}\mathfrak{k}\tilde{\omega}\mathfrak{w}\tilde{\gamma}\mathfrak{k}$. Thus,

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}(\mathfrak{h}\tilde{\alpha}(\mathfrak{r}\tilde{\beta}\mathfrak{k}\tilde{\omega}\mathfrak{w}\tilde{\gamma}\mathfrak{k})) = \eta'^{RI}((\mathfrak{h}\tilde{\alpha}\mathfrak{r})\tilde{\beta}\mathfrak{k}\tilde{\omega}(\mathfrak{w}\tilde{\gamma}\mathfrak{k})) \geq \eta'^{RI}(\mathfrak{k}).$$

Hence, η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

Lemma 4.17: *In a weakly regular Γ-SG of $\widetilde{\mathfrak{G}}$, the CFIΓ-IDs and the CFΓ-IDs coincide.*

Proof: Suppose that η'^{RI} is a CFIΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{k} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Since $\widetilde{\mathfrak{G}}$ is weakly regular, there exist $\mathfrak{r}, \mathfrak{w} \in \widetilde{\mathfrak{G}}$ and $\tilde{\beta}, \tilde{\omega}, \tilde{\gamma}, \tilde{\tau} \in \Gamma$ such that $\mathfrak{h} = \mathfrak{h}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{h}\tilde{\tau}\mathfrak{w}$. Thus,

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}((\mathfrak{h}\tilde{\beta}\mathfrak{r}\tilde{\omega}\mathfrak{h}\tilde{\tau}\mathfrak{w})\tilde{\alpha}\mathfrak{k}) = \eta'^{RI}((\mathfrak{h}\tilde{\beta}\mathfrak{r})\tilde{\omega}\mathfrak{h}\tilde{\tau}(\mathfrak{w}\tilde{\alpha}\mathfrak{k})) \geq \eta'^{RI}(\mathfrak{h}).$$

Hence, η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, we can show that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$. Thus, η'^{RI} is a CFΓ-ID of $\widetilde{\mathfrak{G}}$.

By Lemma 4.12, 4.13, 4.14, 4.15, 4.16 and 4.17, we have Theorem 4.18.

Theorem 4.18: *Let $\widetilde{\mathfrak{G}}$ be an Γ-SG. If $\widetilde{\mathfrak{G}}$ is regular, left (right) regular, intra-regular, semisimple, left (right) quasi-regular or weakly regular, then the CFIΓ-IDs and the CFΓ-IDs coincide.*

The following theorem shows that every CFΓ-ID is a CFQΓ-ID of an Γ-SG.

Theorem 4.19: *Every CFLΓ-ID (CFRΓ-ID) of an Γ-SG $\widetilde{\mathfrak{G}}$ is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$.*

Proof: Suppose that η'^{RI} is a CFLΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h} \in \widetilde{\mathfrak{G}}$ and $\tilde{\alpha} \in \Gamma$.

If $F_{\mathfrak{h}} = \emptyset$, then it is easy to verify that $\eta'^{RI}(\mathfrak{h}) \geq (\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI})(\mathfrak{h}) \wedge (\eta'^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}})(\mathfrak{h})$.

If $F_h \neq \emptyset$, then

$$\begin{aligned}(\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI})(\mathfrak{h}) &= \bigvee_{(\mathrm{i},\mathrm{j}) \in F_{\mathfrak{h}}} \{\lambda'^{RI}_{\widetilde{\mathfrak{G}}}(\mathrm{j}) \wedge \eta'^{RI}(\mathrm{j})\} = \bigvee_{(\mathrm{i},\mathrm{j}) \in F_{\mathfrak{h}}} \{(1 + \iota 1) \wedge (\eta'^{R} + \eta'^{I})(\mathrm{j})\} \\ &\leq \bigvee_{(\mathrm{i},\mathrm{j}) \in F_{\mathfrak{h}}} (\eta'^{R} + \eta'^{I})(\mathrm{j})\} = \bigvee_{(\mathrm{i},\mathrm{j}) \in F_{\mathfrak{h}}} \{\eta'^{RI}(\mathrm{j})\} \\ &\leq \bigvee_{(\mathrm{i},\mathrm{j}) \in F_{\mathfrak{h}}} \{\eta'^{RI}(\mathrm{i}\tilde{\alpha}\mathrm{j})\} \leq \eta'^{RI}(\mathfrak{h}).\end{aligned}$$

Thus, $\eta'^{RI}(\mathfrak{h}) \geq (\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI})(\mathfrak{h})$ and so, $\eta'^{RI}(\mathfrak{h}) \geq (\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI})(\mathfrak{h}) \wedge (\eta'^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}})(\mathfrak{h})$.

Hence, η'^{RI} is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$. Similarly, if η'^{RI} is a CFRΓ-ID of $\widetilde{\mathfrak{G}}$, then η'^{RI} is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$.

The following theorem shows that every CFQΓ-ID is a CFΓ-SSG on an Γ-SG.

Theorem 4.20: *Every CFQΓ-ID of an Γ-SG $\widetilde{\mathfrak{G}}$ is a CFΓ-SSG of $\widetilde{\mathfrak{G}}$.*

Proof: Assume that η'^{RI} is a CFQΓ-ID of $\widetilde{\mathfrak{G}}$ and let $\mathfrak{h}, \mathfrak{r} \in \widetilde{\mathfrak{G}}$, $\tilde{\alpha} \in \Gamma$. Then

$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq (\lambda'^{RI}_{\widetilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI})(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \wedge (\eta'^{RI} \odot_{\tilde{\alpha}} \lambda'^{RI}_{\widetilde{\mathfrak{G}}})(\mathfrak{h}\tilde{\alpha}\mathfrak{r})$

$$= \bigvee_{(\mathfrak{p},\mathfrak{q})\in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} \{\lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{p}) \wedge \eta'^{RI}(\mathfrak{q})\} \wedge \bigvee_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}}} \{\eta'^{RI}(\mathfrak{a}) \wedge \lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{b})\}$$
$$\geq (\lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{r})) \wedge (\eta'^{RI}(\mathfrak{h}) \wedge \lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{r}))$$
$$= (1+\iota 1) \wedge (\eta'^{R} + \eta'^{I})(\mathfrak{r}) \wedge (\eta'^{R} + \eta'^{I})(\mathfrak{h}) \wedge (1+\iota 1)$$
$$\geq (\eta'^{R} + \eta'^{I})(\mathfrak{r}) \wedge (\eta'^{R} + \eta'^{I})(\mathfrak{h}) = \wedge\, \eta'^{RI}(\mathfrak{r}) \wedge \eta'^{RI}(\mathfrak{h}).$$

Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{r})$. Hence, η'^{RI} is a CFΓ-SSG of $\tilde{\mathfrak{G}}$.

The following theorem shows that every CFQΓ-ID is a CFBΓ-ID on an Γ-SG.

Theorem 4.21: *Every CFQΓID of an Γ-SG $\tilde{\mathfrak{G}}$ is a CFBΓ-ID of $\tilde{\mathfrak{G}}$.*

Proof: Assume that η'^{RI} is a CFQΓ-ID of $\tilde{\mathfrak{G}}$ and let $\mathfrak{h},\mathfrak{r} \in \tilde{\mathfrak{G}}$, $\widetilde{\alpha'} \in \Gamma$. Then by Theorem 4.20, η'^{RI} is a CFΓ-SSG of $\tilde{\mathfrak{G}}$. Let $\mathfrak{h},\mathfrak{r},\mathfrak{w}$ and $\tilde{\alpha},\tilde{\beta} \in \Gamma$. Then

$$\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq (\lambda'^{RI}_{\tilde{\mathfrak{G}}} \odot_{\tilde{\alpha}} \eta'^{RI})(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \wedge (\eta'^{RI} \odot_{\tilde{\beta}} \lambda'^{RI}_{\tilde{\mathfrak{G}}})(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w})$$
$$= \bigvee_{(\mathfrak{p},\mathfrak{q})\in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \{\lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{p}) \wedge \eta'^{RI}(\mathfrak{q})\} \wedge \bigvee_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}}} \{\eta'^{RI}(\mathfrak{a}) \wedge \lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{b})\}$$
$$\geq (\lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}) \wedge \eta'^{RI}(\mathfrak{w})) \wedge (\eta'^{RI}(\mathfrak{h}) \wedge \lambda'^{RI}_{\tilde{\mathfrak{G}}}(\mathfrak{r}\tilde{\beta}\mathfrak{w}))$$
$$= (1+\iota 1) \wedge (\eta'^{R} + \acute{\eta}^{I})(\mathfrak{w}) \wedge (\eta'^{R} + \acute{\eta}^{I})(\mathfrak{h}) \wedge (1+\iota 1)$$
$$\geq (\acute{\eta}^{R} + \acute{\eta}^{I})(\mathfrak{w}) \wedge (\eta'^{R} + \iota\acute{\eta}^{I})(\mathfrak{h}) = \wedge\, \eta'^{RI}(\mathfrak{w}) \wedge \eta'^{RI}(\mathfrak{h}).$$

Thus, $\eta'^{RI}(\mathfrak{h}\tilde{\alpha}\mathfrak{r}\tilde{\beta}\mathfrak{w}) \geq \eta'^{RI}(\mathfrak{h}) \wedge \eta'^{RI}(\mathfrak{w})$. Hence, η'^{RI} is a CFBΓ-ID of $\tilde{\mathfrak{G}}$.

The following result is an immediate consequence of Theorem 4.21 and Lemma 4.3.

Corollary 4.22: *Every CFQΓ-ID of an Γ-SG $\tilde{\mathfrak{G}}$ is a CF (1,2)-Γ-ID of $\tilde{\mathfrak{G}}$.*

5. Conclusion

The concept of a complex fuzzy set is a generalization of the fuzzy set which solves the problem of uncertainty in data in the form of complex numbers. In this paper, we introduced a new type of complex fuzzy structure characterized by the notion of complex fuzzy ideals (generalized bi-ideals, bi-ideals, interior ideals, (1,2)-ideals) in Γ-semigroups. We considered the properties of complex fuzzy ideals. We characterized essential conditions of coincidences complex fuzzy ideals related to regular, and intra-regular, semisimple, quasi-regular, and weakly regular in Γ-semigroups. In the continuity of this paper, we will investigate the characterized essentially characterized conditions of coincidences of the complex interval valued fuzzy ideals (hesitant, intuitionistic, etc.) of a hypersemigroup and their algebraic properties.

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