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The central local metric dimension of graphs with a single central vertex

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Abstract

A local metric set is a concept where the differences of two vertices on an edge of graph G can be identified. This concept is only relevant for a graph G, where all vertices is connected by an edge. Previous studies have examined the applications of local metric dimension, for instance applications in mathematical chemistry. A vertex in a graph is central if its greatest distance from any other vertex is the smallest as possible, a local metric set is even more interesting when it contains all the central vertices in the graph. This paper explores the development of a local metric set, namely the central local metric set. This concept also supports the affordable access to provide good services to the community in the placement of vital objects, such as health centers, education centers, and clean water stations, so that the community easy to access them. Let G with $|V(G)| = n$ and $W = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$ and $k \leq n$. In G, x is represented with regard to W as follows $r(x|W) = (d(x, x_1), d(x, x_2), \ldots, d(x, x_k))$. If $r(x\,|\,W) \neq r(y\,|\,W)$, $\forall x,y \in V(G)$, and xy is the edge of G, then W is a local metric set of G. Let $S(G)$ is a set of all central vertices of G. So, the central local metric set is W, if $S(G) \subseteq W$. Further, we called W with minimum cardinality as a central local basis set and the cardinality is called the central local metric dimension of G or $lmd_s(G)$. This study explores the central local metric set of graph having only one central vertex, that is $G \cong K_1 + H$, for some cases of H. The results show that $K_1 + H$, with $|V(H)| = m$, has only one central vertex if and only if no vertex in H has degree m-1. There are two possibilities of $lmd_s(K_1 + H)$, either the central set of $K_1 + H$ subset of its local basis set or no intersection between the central set and local basis set.

Subject Classification: *05C12.*

Keywords: *Central local metric dimension, Local metric set, Central vertex, A single central vertex graph, Affordable access.*

1. Introduction

A central vertex in graph theory is a concept that can represent some places in real life that are easy to access by society. A local metric set is a concept in graph theory that can represent two real objects that should not be similar. Let G be a graph that represents a district and the central vertex represents a public service place. Then, the government can use the central local metric dimension concept to optimize transportation infrastructure management. For example, by designing an efficient public transport network. Thus, people can easily reach the location of the public service. Take graph G. Since the metric dimension is related to distance, the vertices in the graph G must be connected. Henceforth, we use the notation of $V(G)$ as the set of the vertex in G and $E(G)$ as the set of edges in G, where the order of G is n. All graph notations and terminologies that we use in this paper refer to [1], [2]. While all concepts of local metric dimension of G, $lmd(G)$, and their characteristics refer to [3]. Recent studies have also focused specifically on the local metric dimension of some particular graphs that can see in [4], [5], [6], and [7]. Some essential properties of a local metric set of a graph G to support our main results are summarized in the following lemma.

Lemma 1.1: [3] *$V(G)$ has a subset U. Then U is a local metric set if there is a subset of U is a local metric set.*

Lemma 1.2: [8] *If W is a subset of $V(G)$, then for every two vertices v_i and v_j element of W applies $r(v_i|W) \neq r(v_j|W)$, for $i \neq j$.*

Several authors studied a new developmnet of local metric dimension concept, that we can see in [9] [10] [11] [12] [13] [14], and [15]. In addition, in its application, the members of the basis set in the metric dimension can be likened to sensors, where if one of the sensors does not work, then the navigation system on the use of sensors will not be sufficient in obtaining information and the solution to overcome this problem can be seen in [16]. Furthermore, Zubair et al. in [17] and [18], compared the results of the metric dimension $P(n,2)\odot K_1$ with its fault-tolerant metric dimension and it was concluded that the function in the metric dimension provides a limitation on the value of the fault-tolerant metric dimension.

In 2023, the new concept a central local metric dimension was introduced by Listiana et. al. [15]. The eccentricity of $u \in V(G)$, $e(v)$, is the largest $d(u,v)$ for all $v \in V(G)$, and the radius of G, $rad(G)$, is the smallest $e(v)$, $\forall v \in G$. A vertex $u \in V(G)$ is a central vertex if $e(u) = rad(G)$ and a central set of G is $S(G) = \{u | e(u) = rad(G)\}$ [15]. For W is a local metric set of G, W is a central local metric set if $S(G) \subseteq W$. A central local metric dimension of G, denoted by $lmd_s(G)$, is the minimum cardinality of W [15]. Bounds for $lmd_s(G)$ and some properties of it are also presented in [15] as shown in the following results.

Theorem 1.1: *Suppose that a connected graph G with order n,*

a. *If W is a local metric set of G, then*

$$\max\{|S(G)|, lmd\ (G)\} \leq lmd_s(G) \leq \min\{|V(G)|, |S(G) \cup W|\}$$

b. $diam(G) = rad(G)$ if and only if $S(G) = V(G)$

c. $lmd_s(G) = n$ if and only if $diam(G) = rad(G)$

There are some graphs have only one central vertex or we call it a single central vertex, Listiana et al. in [15] described that tree T with $diam(T) = k$ has a single central vertex for k even and grid graph $P_n \times P_m$ has a single central vertex for n,m odd. Both of Tree T and grid graph $P_n \times P_m$, the central set is also the local basis set, so that $lmd_s(T) = 1$ for k even and $lmd_s(P_n \times P_m) = 1$ for n,m odd. In this study, we explore graphs that have a single central vertex regarding the existence of the local metric sets in the graph, such as graph $G \cong K_1 + H$ and some graphs related to it. Graph $G \cong K_1 + H$ is interesting because when a vertex $c \in V(K_1)$ is adjacent to all other vertices in the graph, then $d(c,x) = 1$, for all x in $V(H)$, so that c is not always a member of the local basis set. Some specific graphs $G \cong K_1 + H$ with a single central vertex discussed in this paper are friendship graph $f_n \cong K_1 + 2P_n$, fan graph $F_n \cong K_1 +$

P_n, and wheel graph $W_n \cong K_1 + C_n$. The definitions related to these graphs are given below.

Definition 1.1: [19] A friendship graph, f_n, is a graph consisting of n copies C_3 that linked in one vertex called the center vertex.

Definition 1.2: [20] A fan graph, F_n, for $n \geq 2$, is a graph constructed by connecting trivial graph to all vertices of a path on n vertices P_n.

Definition 1.3: [21] A *wheel* graph, W_n, for $n \geq 3$, is a formed by taking C_n and adding a trivial graph that connecting to all vertices in the C_n.

Furthermore, the local metric dimension results of the previous graphs are presented in the following theorem.

Theorem 1.2: [4] $lmd(f_n) = n$, $n \geq 2$.

Theorem 1.3: [22] $lmd(F_n) = \begin{cases} 2, & 2 \leq n \leq 5 \\ \left\lceil \frac{n-1}{4} \right\rceil, & n \geq 6 \end{cases}$.

Theorem 1.4: [7] $lmd(W_n) = \left\lceil \frac{n}{4} \right\rceil$, $n \geq 5$

Before going to our main result, we explained the terminology of the twin vertex by referring to [3] and degree of graph by referring to [1]. If there is a vertex z so that x and y adjacent to it respectively, for $x, y, z \in V(G)$ where $xy \in E(G)$, then x and y are true twins. The degree of $u \in V(G)$, $\deg(u)$, is a number of edges that connected to u. Lemma 1.3 explains the lower bound of minimum degree $\delta(G)$ and upper bound of maximum degree $\Delta(G)$ of all vertices in G.

Lemma 1.3: [1] *If* $x \in V(G)$, *then* $0 \leq \delta(G) \leq deg(x) \leq \Delta(G) \leq |V(G)| - 1$.

2. Main Results

Let W be a local basis set of G, so $lmd(G) = |W|$. Lemma 2.1 describes $lmd_s(G)$ when G is a graph having a single central vertex.

Lemma 2.1: *Suppose that a connected graph* G *with* $|V(G)| = n$, $S(G) = \{c\}$, *and* W *is a local basis set of* G.

(i) If $S(G) \subseteq W$, then $lmd_s(G) = lmd(G)$

(ii) If $S(G) \cap W = \emptyset$ for every local basis set W of G, then $lmd_s(G) = lmd(G) + 1$.

Proof: Put $W = \{w_1, w_2, \dots, w_l\}$ as a local basis set of G, $l \leq n$, and $S(G) = \{c\}$.

(i) Since $S(G) = \{c\} \subseteq W$, then $c \in W$. Let $c = w_k$, so $= \{w_1, w_2, \dots, w_{l-1}, c\}$.$W$ is a local metric set of G and $c \in W$, so W is a central local metric set of G. Since W is a local basis set of G, then W is a central local metric set of G with minimum cardinality, and $lmd_s(G) = |W| = lmd(G)$.

(ii) Let W be a local basis set of G. Since $S(G) \cap W = \emptyset$ and $S(G) = \{c\}$, then $c \notin W$. Let $U = S(G) \cup W = \{c\} \cup \{w_1, w_2, \dots, w_k\}$, so $|U| = k + 1$. W is a local metric set of G, then based on Lemma 1.1, U is also a local metric set of G. Besides that, $S(G) \subset U$, so U is a central local metric set of G. Then, $lmd_s(G) \le |U| = k + 1 = lmd(G) + 1$. Since $S(G) \cap W = \emptyset$ for every local basis set W of G, we have $lmd_s(G) > lmd(G)$. Hence $lmd_s(G) = lmd(G) + 1$.

Graph $K_1 + H$ is constructed by joining a vertex in the graph K_1 to every vertex in H. So, all vertices in H are adjacent to a single vertex in K_1. Let H be a graph with $V(H) = m$, if $H = K_m$, then $diam(K_m) = rad(K_m) = 1$ and $\deg(x) = m - 1$, for $\forall x \in V(K_m)$. Since $u \in V(K_1)$ adjacent to v, $\forall v \in V(K_m)$, then for every vertex $x \in V(K_1 + K_m)$, $\deg(x) = m$. Consequently, $diam(K_1 + K_m) = rad(K_1 + K_m) = 1$, by Theorem 1.1, $S(G) = V(K_1 + K_m)$ and $lmd_s(K_1 + K_m) = m$. A Graph $K_1 + K_m$ is one of the cases for $K_1 + H$ where $\deg(x) = m - 1$, $\forall x \in V(H)$, so that $S(H) = V(H)$. The same applies when one or more vertices of H have degree $m - 1$. Suppose that a graph H with $|V(H)| = m$ and $\deg(x_i) = m - 1$, $\forall x_i \in V(H)$, $1 \le i \le l \le m$. The vertex $x_i \in V(H)$ is adjacent to all other vertices in $K_1 + H$, so the eccentricity x_i in $K_1 + H$ is $e(x_i) = 1$. Besides that, let c be the only vertex in K_1, then c is also adjacent to all other vertices in $K_1 + H$ and the eccentricity c in $K_1 + H$ is $e(c) = 1$. Consequently, $S(K_1 + H) = \{c, x_i | x_i \in V(H),\ 1 \le i \le k \le m\}$.

Graphs $K_1 + K_m$ and $K_1 + H$, when one or more vertices of H have degree $m - 1$, are examples of graphs $K_1 + H$ with more than one central vertex. Both have in common that graph H has at least a vertex with degree $m - 1$. Now, we have found the constraint of H such that $G \cong K_1 + H$ has only one central vertex, as discussed in theorem 2.1.

Theorem 2.1: *Let $G \cong K_1 + H$ with $V(K_1) = \{c\}$ and $|V(H)| = m$. The central set G is $S(G) = \{c\}$ if and only if no vertex in H has degree $m - 1$.*

Proof: Let $G \cong K_1 + H$ with $V(K_1) = \{c\}$ and $|V(H)| = m$. There are two types of proof of theorem, such as:

(i) If the central set of G is $S(G) = \{c\}$ then there is no vertex in H has degree $m - 1$

Suppose there are some vertices $x \in V(H)$ have degree $m - 1$, consequently the eccentricity of x in H is $e(c) = 1$. Since $G \cong K_1 + H$ and $V(K_1) = \{c\}$, then c is adjacent to x, $\forall x \in V(H)$, and $d(c, x) = d(x, c) = 1$, so $e(c) = 1$. Finally, we get $rad(G) = e(x) = e(c) = 1$. Then $S(G) = \{c, x\}$, a contradiction with $S(G) = \{c\}$. Thus, its proved that there is no vertex in H has degree $m - 1$.

(ii) If There is no vertex in H has degree $m-1$, then $S(G)=\{c\}$

By lemma 1.3, for $x \in V(H)$, $\deg(x) \leq m-1$, since there is no vertex in H has degree $m-1$, then $\deg(x) \leq m-2$. Take $\deg(x)=m-2$, then there is only one vertex in $V(H)$ that not adjacent to x. So that, $e(x)=2$. Since $G \cong K_1+H$ and $V(K_1)=\{c\}$, then vertex $c \in V(K_1)$ adjacent to x, $\forall x \in H$, and $d(c,x)=d(x,c)=1$, for $x \in V(H)$, so that the eccentricity of vertex c and x in G are $e(c)=1$ and $e(x)=2$, respectively. Consequently, $rad(G)=e(c)=1$ and c is the only one central vertex in G. It is proved that $S(G)=\{c\}$.

The local central metric dimension of $G \cong K_1+H$ discussed in Theorem 2.2.

Theorem 2.2: *Let $S(G)$ and W are a central set and local basis set of $G \cong K_1+H$ with order H is m respectively, if there is no vertex in H has degree $m-1$, then:*

$$lmd_s(G)=\begin{cases} lmd(G), & S(G) \subset W \\ lmd(G)+1, & S(G) \cap W = \emptyset \end{cases}.$$

Proof: Let $G \cong K_1+H$ with $V(K_1)=\{c\}$. By Theorem 2.1, $S(G)=V(K_1)=\{c\}$. By lemma 2.1, it is proved that

$$lmd_s(G)=\begin{cases} lmd(G), & for\ S(G) \subset W \\ lmd(G)+1, & for\ S(G) \cap W = \emptyset \end{cases}.$$

Now, we are going to discussed the central local metric dimension of some cases of K_1+H graph, such as friendship graph, fan graph, and wheel graph.

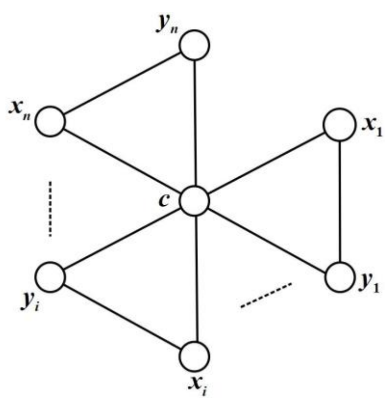

Figure 1
The illustration of $f_n \cong K_1+nP_2$

A friendship graph f_n is a graph that isomorphic with K_1+H for $H=nP_2$. The central set of $f_n \cong K_1+nP_2$, by the Theorem 2.1, is $S(f_n)=\{c\}$, where is $c \in V(K_1)$. Since c be a central vertex of $f_n \cong K_1 + nP_2$, then x_i and y_i are represented as two other vertices on P_2, where $1 \leq i \leq n$. Define, $V(f_n) = \{c\} \cup \{x_i, y_i | 1 \leq i \leq n\}$ and $E(f_n) = \{x_i c | 1 \leq i \leq n\} \cup \{y_i c | 1 \leq i \leq n\} \cup \{x_i y_i | 1 \leq i \leq n\}$. Thus, the graph $f_n \cong K_1 + nP_2$ has order $2n + 1$ and size $3n$. Figure 1 is an illustration of friendship graph $f_n \cong K_1 + nP_2$.

Figure 1 shows that c is a common vertex that is adjacent to all the other vertices in $f_n \cong K_1 + nP_2$ and the vertices x_i and y_i are true twin vertices for $1 \leq i \leq n$. In this regard, Observation 2.1 is obtained to simplify the proof process in the next theorem.

Observation 2.1: The vertex c is adjacent to $x \in V(f_n)$, $\forall x \in V(f_n) \backslash \{c\}$, so $\deg(c) = 2n$ and $d(x_i, c) = d(y_i, c) = 1$ for $1 \leq i \leq n$.

Next, we discuss the central local metric set of $f_n \cong K_1 + nP_2$ in theorem 2.3.

Theorem 2.3: *If* $f_n \cong K_1 + nP_2$ *is a friendship graph, for* $n \geq 2$, *then* $lmd_s(f_n) = n + 1$.

Proof: By Theorem 2.1 we get $S(f_n) = \{c\}$ and by Theorem 1.2, $lmd(f_n) = n$, for $n \geq 2$. In Observation 2.1, we have $d(x_i, c) = d(y_i, c) = 1$ for $1 \leq i \leq n$, then c isn't a member of any local basis set of f_n, otherwise if $c \in W$ where W is a local basis set of f_n, then $W - \{c\}$ is a local metric set of f_n, which means that $|W| = lmd(f_n) \leq |W - \{c\}| = |W| - 1$, which is impossible. Consequently $S \cap W = \emptyset$ for every local basis set W of f_n. By Theorem 2.2, it is proved that $lmd_s(f_n) = lmd(f_n) + 1 = n + 1$.

A fan graph, F_n, is isomorphic to $K_1 + H$ for $H = P_n$. By Definition 1.3, define $V(F_n) = \{c, x_1, \dots, x_n\}$ be the vertex set of F_n and $E(Fn) = \{cx_i | 1 \leq i \leq n\} \cup \{x_i x_{i+1} | 1 \leq i \leq n - 1\}$ be the edge set of F_n. Thus, $F_n \cong K_1 + P_n$ has $n + 1$ vertices and $2n - 1$ edges. Figure 2 is an illustration of $F_n \cong K_1 + P_n$.

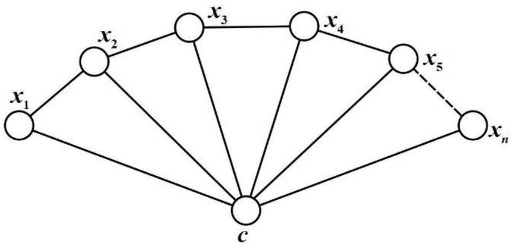

Figure 2
The illustration of graph $F_n \cong K_1 + P_n$

There is anomaly in the central set of $F_n \cong K_1 + P_n$, when $n = 3$. A path graph P_3 is a graph with order 3. Let $\{x_1, x_2, x_3\}$ be a vertex set of P_3 with $\deg(x_2) = 2$. So, this case is not related with Theorem 2.1. Since x_2 adjacent to all other vertices in P_3 and x_2 is also adjacent to c in $K_1 + P_3$, then the eccentricity of x_2 in K_1+P_3 is $e(x_2)=1$. Besides that, the vertex $c\in V(K_1)$ is also adjacent to all other vertices in K_1+P_3 and $e(c)=1$. Then, $rad(K_1+P_3)=e(x_2)=e(c)=1$ and the central set $F_3\cong K_1+P_3$, is $S(F_3)=\{c,x_2,\}$. Next, we focused to the central local metric dimension of fan graph $F_n\cong K_1+P_n$, for $n\geq 4$.

Lemma 2.2: *Let $F_n\cong K_1+P_n$ with $V(F_n)=\{c,x_1,x_2,\ldots,x_n\}$ and $deg(c)=n$. If $W\subseteq V(F_n)$ is a local basis set of F_n, then $c\notin W$ for $n\geq 6$.*

Proof: Let W be a local basis of $F_n\cong K_1+P_n$, for $n\geq 6$, with $V(F_n)=\{c,x_1,x_2,\ldots,x_n\}$ where $\deg(c)=n$. Suppose that $c\in W$. For any two adjacent vertices $u,v\in V(F_n)$ it holds that $r(u|W)\neq r(v|W)$. Since $deg(c)=n$, for every $x\in V(F_n)$ we have $d(x,c)=1$. So, $\forall u,v\in V(F_n)$, $uv\in E(F_n)$, $d(u,c)=d(v,c)=1$. Hence $W-\{c\}$ is still a local metric set of F_n for $n\geq 6$, which means that $|W|=lmd(F_n)\leq|W-\{c\}|=|W|-1$, a contradiction. Thus, W does not contain c.

Theorem 2.3: *If $F_n\cong K_1+P_n$ is a fan graph, for $n\geq 4$, then $lmd_s(F_n)=\left\lceil\frac{n-1}{4}\right\rceil+1$*

Proof: By Theorem 2.1, $S(F_n)=\{c\}$, for $n\geq 4$. Theorem 1.3 shows that $lmd(F_n)=2$ for $2\leq n\leq 5$ and $lmd(F_n)=\left\lceil\frac{n-1}{4}\right\rceil$ for $n\geq 6$. Then, referring to Theorem 1.1.a. we obtain $lmd_s(F_n)\geq lmd(F_n)$. Further, we divided the proof into two cases.

Case 1: $4\leq n\leq 5$

By Theorem 1.3, $lmd(F_n)=2$ for $2\leq n\leq 5$. Let $W=\{c,x_3\}\subseteq V(F_n)$. W is a basic local set of F_n, $n=4{,}5$ because there are two adjacent vertices $x,y\in V(F_n)-W$ with $d(x,x_3)\neq d(y,x_3)$, so that $r(x|W)\neq r(y|W)$. For $n=4$, then the only two adjacent vertices in F_n that are not in the set W are x_1 and x_2. Let $x=x_1$ and $y=x_2$, then $r(x_1|W)=(1,2)$ and $r(x_2|W)=(1,1)$, so $r(x_1|W)\neq r(x_2|W)$. The same is true for $n=5$, Let $x=x_4$ and $y=x_5$, because $d(x,x_3)\neq d(y,x_3)$ then $r(x_4|W)\neq r(x_5|W)$.Since $(F_n)\subset W$, by Theorem 2.1 it is proved that $lmd_s(F_n)=lmd(F_n)=2$, $n=4{,}5$.

Case 2: $n\geq 6$

For any local basis set W of F_n, $n\geq 6$, based on Lemma 2.2, we get the central vertex $c\notin W$. Since $S(F_n)=\{c\}$ is a central set of F_n, for $n\geq 6$, we have $S(F_n)\cap W=\emptyset$ for any local basis W of F_n. So, by Theorem 2.1 it follows that $lmd_s(F_n)=lmd(F_n)+1=\left\lceil\frac{n-1}{4}\right\rceil+1$, $n\geq 6$.

From the first case, we get $lmd_s(F_n)=2$, for $n=4{,}5$. Then, it is easy to see that $\left\lceil\frac{n-1}{4}\right\rceil+1=2$ for $4\leq n\leq 5$. So according to the second case, it can be concluded that $lmd_s(F_n)=\left\lceil\frac{n-1}{4}\right\rceil+1$, for all $n\geq 4$.

A Wheel graph, W_n, is defined as a graph that isomorphic to $K_1 + H$ for $H = C_n$. By Definition 1.3, let $c \in V(K_1)$ be a center vertex and the other n vertices are denoted by x_i, for $1 \leq i \leq n$. Define, $V(W_n) = \{c\} \cup \{x_i | 1 \leq i \leq n\}$ and $E(W_n) = \{x_i c | 1 \leq i \leq n\} \cup \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \{x_n x_1\}$, then $|V(W_n)| = n + 1$. Figure 3 is an illustration of a wheel graph $W_n \cong K_1 + C_n$.

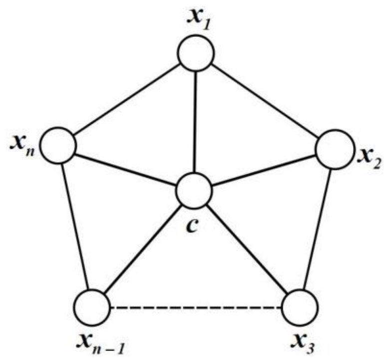

Figure 3
The illustration of $W_n \cong K_1 + C_n$

A wheel graph $W_n \cong K_1 + C_n$ when $n = 3$, also have anomaly in its central set. A graph C_3 is a regular graph where $\deg(x) = 2$, $\forall x \in C_3$. This case is not relevant with Theorem 2.1, since every vertex in C_3 adjacent each other. The consequence is every vertex in $K_1 + C_3$ also adjacent each other and $diam(K_1 + C_3) = rad(K_1 + C_3) = 1$. Thus, by Theorem 1.1 $S(W_3) = V(W_3)$ and $lmd_s(W_3) = |V(W_3)|$.

Lemma 2.3 describes the local basis set of W_n, for $n > 5$. This lemma is used to support the proof of theorem 2.4.

Lemma 2.3: *Let $W_n \cong K_1 + C_n$ with $V(W_n) = \{c, x_1, x_2, \ldots, x_n\}$ and $deg(c) = n$. For any local basis set $W \subseteq V(W_n)$ of W_n, for $n \geq 5$, the vertex c is not a member of W, i.e., $c \notin W$.*

Proof: Let W be a local basis set of $W_n \cong K_1 + C_n$, for $n \geq 5$. Suppose that W contain of c or $c \in W$, then for $\forall u, v \in V(W_n)$ where $uv \in E(W_n)$, $r(u|W) \neq r(v|W)$. Though $deg(c) = n$, so for every $x \in V(W_n)$, $d(x, c) = 1$ holds true. In other words, for $\forall u, v \in V(W_n)$ where $uv \in E(W_n)$, $d(u, c) = d(v, c)$ applies. So, $W - \{c\}$ is also a local metric set of W_n for $n \geq 5$. This is in contradiction with W being a basis local metric set. Which proves that $c \notin W$.

Theorem 2.4: *Let $W_n \cong K_1 + C_n$ be a wheel graph, for $n \geq 4$, then* $lmd_s(W_n) = \left\lceil \frac{n}{4} \right\rceil + 1$

Proof: Based on Theorem 2.1, we get $S(W_n) = \{c\}$. To proof this theorem, we divide it into two cases.

Case 1: $n = 4$

Take $W = \{c, x_1\}$, then $S \subset W \subset V(W_4)$. Based on Lemma 1.2, for two adjacent vertices c and x_1, then $r(c|W) \neq r(x_1|W)$. Since the vertex $cx_2, cx_3, cx_4 \in E(W_4)$, then $r(c|W) \neq r(x_2|W)$, $r(c|W) \neq r(x_3|W)$, and $r(c|W) \neq r(x_4|W)$, respectively. Similarly, vertex $x_1x_2, x_1x_4 \in E(W_4)$, so $r(x_1|W) \neq r(x_2|W)$ and $r(x_1|W) \neq r(x_4|W)$, respectively. There are vertices x_2, x_3, and x_4 in $V(W_4) - W$, where $x_2x_3, x_3x_4 \in E(W_4)$. It is easy to see that $d(x_2, x_1) \neq d(x_3, x_1)$ and $d(x_3, x_1) \neq d(x_4, x_1)$, so $r(x_2|W) \neq r(x_3|W)$ and $r(x_3|W) \neq r(x_4|W)$. Consequently, we concluded that, for $\forall u, v \in V(W_4)$ where $uv \in E(W_4)$, $r(u|W) \neq r(v|W)$. Then, W is a central local metric set. Since $d(c, x_i) = 1$, for $1 \leq i \leq 4$, then $S(W_4) = \{c\}$ is not a central local metric set. So, W is a basis local set of W_4, hence $lmd_s(W_4) = |W| = 2$.

Case 2: $n \geq 5$

By Theorem 1.4 we obtain $lmd(W_n) = \left\lceil \frac{n}{4} \right\rceil$ for $n \geq 5$. Referring to the Theorem 1.1.a, we obtain $lmd_s(W_n) \geq lmd(W_n)$. Let W be a local basis set of W_n. By Lemma 2.3, $c \notin W$, so $S \cap W = \emptyset$. By Theorem 2.1 it is proved that $lmd_s(W_n) = lmd(W_n) + 1 = \left\lceil \frac{n}{4} \right\rceil + 1$, for $n \geq 5$.

From case 1, we obtain $lmd_s(W_4) = 2$, which still corresponds to $lmd_s(W_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ when $n = 4$, that is $lmd_s(W_4) = \left\lceil \frac{4}{4} \right\rceil + 1$ =2. So, from cases 1 and 2, it is proved that $lmd_s(W_n) = \left\lceil \frac{n}{4} \right\rceil + 1$, for $n \geq 4$.

3. Conclusions

In this paper, we obtain $lmd_s(G)$ of $G \cong K_1 + H$, with $V(K_1) = \{c\}$ and order H is m. Uniquely, when no vertex in H has degree $m - 1$, graph $G \cong K_1 + H$ has a single central vertex c. We have a result that in some cases a local basis set of $G \cong K_1 + H$ is not contain of c. So, there are two possibilities of $lmd_s(G)$, i.e., $lmd_s(G) = \begin{cases} lmd(G), & S(G) \subset W \\ lmd(G) + 1, & S(G) \cap W = \emptyset \end{cases}$, where W is a local basis set of G. There are some cases of graph $G \cong K_1 + H$ discussed in this paper, such as friendship graph $f_n \cong K_1 + 2P_n$, fan graph $F_n \cong K_1 + P_n$, and wheel graph $W_n \cong K_1 + C_n$. The central local metric dimension on graph G, for G is a graph having a single central vertex, can still be developed by finding other properties of a graph that generally satisfy the condition.

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