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# Super graceful labeling of co-maximal subgroup graphs of some cyclic groups

Usha Kumari *
Seema Mehra †
Vishal Garg §
*Department of Mathematics*
*Maharshi Dayanand University*
*Rohtak 124001*
*Haryana*
*India*

Amit Sehgal ‡
*Department of Mathematics*
*Pandit Neki Ram Sharma Govt. College*
*Rohtak 124001*
*Haryana*
*India*

## Abstract

The co-maximal subgroup graph $\Gamma(G)$, of a finite group $G$, is a special kind of undirected graph, in which the set of vertices contains non-trivial proper subgroups of $G$ and two distinct vertices $A$ and $U$ are adjacent iff $AU = G$. In this paper, we discuss the structures of co-maximal subgroup graphs of finite cyclic groups $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p_1^n p_2^m}$, $\mathbb{Z}_{p_1^n p_2 p_3}$, $\mathbb{Z}_{p_1^n p_2^m p_3}$ and $\mathbb{Z}_{p_1^n p_2^m p_3^l}$ for $n$, $m$, $l \in N$, where $p_1$, $p_2$ and $p_3$ are distinct prime numbers. Also, we establish the super graceful labeling of these graphs.

***Subject Classification:*** *05C25, 05C78.*

***Keywords:*** *Super graceful labeling, Non-trivial proper subgroup of a group, Co-maximal subgroup graph, Graceful graph.*

* *ORCID-ID:* `0009-0009-7912-5124`
† *ORCID-ID:* `0000-0001-7219-9387`
§ *ORCID-ID:* `0000-0001-9851-8550`
‡ *ORCID-ID:* `0000-0001-7820-8037`

* *E-mail:* `usha.rs.maths@mdurohtak.ac.in (Corresponding Author)`
† *E-mail:* `sberwal2007.math@mdurohtak.ac.in`
§ *E-mail:* `vishalgarg6145@gmail.com`
‡ *E-mail:* `amit_sehgal_iit@yahoo.com`

## 1. Introduction

For a finite group $G$, Akbari, Miraftab and Nikandish [10] introduced the co-maximal subgroup graph $\Gamma(G)$. More results on the co-maximal subgroup graph are available in [1] and [4]. Saha, Biswas and Das [8] studied various properties of the co-maximal subgroup graph of $\mathbb{Z}_n$ and characterized the value of $n$ for which the graph is Hamiltonian, Eulerian, perfect, etc. and characterized various properties like diameter, domination number, etc. The co-maximal subgroup graph of a finite group $G$, denoted by $\Gamma(G)$ is defined as a graph in which vertex set is the collection of all non-trivial proper subgroups of $G$ and two distinct vertices $A$ and $U$ are adjacent iff $AU = G$.

The graph labeling and its different techniques were introduced by Rosa [2] in his work. Graceful labeling of a graph $G$ is defined as an injective function $f: V(G) \to \{0,1,2,\dots,|E(G)|\}$ such that the induced function $f^*: E(G) \to \{1,2,3,\dots,|E(G)|\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is a bijective function. A graph that allows for graceful labeling is called a graceful graph. The initial term used for this kind of labeling in graphs was $\beta$-valuation and it was used as an effective method to decompose a complete graph into isomorphic subgraphs. Sehgal et al. [3] established that the power graph of the finite abelian group $\mathbb{Z}_2^{k-1} \times \mathbb{Z}_4$ is a graceful graph without any condition on k. Recently, Renu et al. [9] established the gracefulness of prime index graph of groups $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ and $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$.

Super graceful labeling is a bijection $f: V(G) \cup E(G) \to \{1,2,3,\dots,p+q\}$, where $|V(G)| = p$ and $|E(G)| = q$, such that $f(uv) = |f(u) - f(v)|$ for every $uv \in E(G)$. A graph is called a super graceful graph if it admits super graceful labeling. Lau, Shiu and Ng [5] established results about the super graceful labeling of disjoint union of specific star graphs, complete graphs, complete tripartite graphs $K_{1,1,n}$ and various families of trees. Lau et al. [6] proved the existence of k-super gracefulness of graphs $G$ with $\Delta(G) \in \{1,2\}$, where $\Delta(G)$ denotes the maximum degree of a vertex in $G$.

In this paper, we establish the super gracefulness of co-maximal subgroup graphs of finite cyclic groups $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p_1^n p_2^m}$, $\mathbb{Z}_{p_1^n p_2 p_3}$, $\mathbb{Z}_{p_1^n p_2^m p_3}$ and $\mathbb{Z}_{p_1^n p_2^m p_3^l}$ for $n, m, l \in \mathbb{N}$, where $p_1, p_2$ and $p_3$ are distinct prime numbers. The reader may refer to [7] for the basic results of group theory used in the paper.

## 2. Notations

This section provides an overview of the notations used throughout the paper. The following notations are used:

| | |
|---|---|
| $deg(u)$ | degree of vertex $u$ in co-maximal subgroup graph |
| $N_i^j$ | a set $\{i, i+1, i+2, \dots, j\}$ |
| $\lvert G\rvert$ | The cardinality of set $G$ |
| $\mathbb{Z}_n$ | Cyclic group under addition modulo $n$, of order $n$ |

## 3. Main Results

**Theorem 1:** *The co-maximal subgroup graph of the group $\mathbb{Z}_{p^n}$, where $p$ is any prime number, and $n \geq 2$, is a super graceful graph.*

***Proof:*** Firstly, we discuss the structure of the co-maximal subgroup graph of $\mathbb{Z}_{p^n}$, where $n \geq 2$ (Figure 1).

Since the group $\mathbb{Z}_{p^n}$ has $n-1$ nontrivial proper subgroups, the vertex set of this graph contains $n-1$ vertices.

Let the vertex set be $V(\Gamma(\mathbb{Z}_{p^n})) = \{H_i \mid i \in N_1^{n-1}\}$, where $|H_i| = p^i$. Now, $H_iH_j = \mathbb{Z}_{p^n}$ for $i, j \in N_1^{n-1}$ iff $lcm(|H_i|, |H_j|) = p^n$, which is not possible for any $i, j \in N_1^{n-1}$. Consequently, we have $E(\Gamma(\mathbb{Z}_{p^n})) = \phi$. Here we get $deg(H_i) = 0$, $\forall\, i \in N_1^{n-1}$.

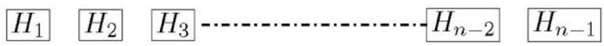

**Figure 1**
**Co-maximal Subgroup Graph of Group $\mathbb{Z}_{p^n}$**

We define a mapping $f: V(\Gamma(\mathbb{Z}_{p^n})) \cup E(\Gamma(\mathbb{Z}_{p^n})) \to \{1, 2, \dots, n-1\}$ such as $f(H_i) = i$, where $i \in N_1^{n-1}$.

Based on the vertex and edge labeling mentioned above, we observe that the vertex labels of the vertices $H_1$, $H_2$, …, $H_{n-1}$ follow a strictly increasing pattern from $1$ to $n-1$. Thus, the mapping is a bijection and hence, the graph is super graceful.

**Theorem 2:** *The co-maximal subgroup graph of the group $\mathbb{Z}_{p_1^n p_2^m}$ is a super graceful graph for $n, m \in \mathbb{N}$.*

***Proof:*** Firstly, we discuss the structure of the co-maximal subgroup graph of group $\mathbb{Z}_{p_1^n p_2^m}$ (Figure 2). The group $\mathbb{Z}_{p_1^n p_2^m}$ has $mn + m + n - 1$ non-trivial proper subgroups, so the graph has $mn + m + n - 1$ vertices.

Let the vertex set be $V(\Gamma(\mathbb{Z}_{p_1^n p_2^m})) = \{H_{ij} | i \in N_0^n$ and $j \in N_0^m\}$ - $\{H_{00}, H_{nm}\}$, where $|H_{ij}| = p_1^i p_2^j$.

Now two vertices $H_{ij}$ and $H_{kl}$ are connected iff $lcm(|H_{ij}|, |H_{kl}|) = p_1^n p_2^m$. Using this condition, we observe that vertices $H_{nj}$, $j \in N_0^{m-1}$ and $H_{im}, i \in N_0^{n-1}$ are adjacent and all the remaining vertices are isolated. Thus, we have $\Gamma(\mathbb{Z}_{p_1^n p_2^m}) \cong K_{m,n} \cup (mn-1)K_1$. Clearly, the graph has $mn + m + n - 1$ vertices and $mn$ edges.

Thus, we get

$$deg(H_{ij}) = \begin{cases} n, & i = n, j \in N_0^{m-1} \\ m, & i \in N_0^{n-1}, j = m \\ 0, & i \in N_0^{n-1}, j \in N_0^{m-1} \end{cases}$$

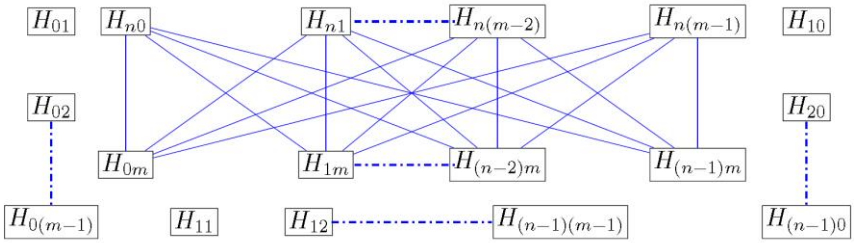


**Figure 2**
**Co-maximal Subgroup Graph of Group $\mathbb{Z}_{p_1^n p_2^m}$**

Now, we define a mapping

$f: V(\Gamma(\mathbb{Z}_{(p_1^n p_2^m)})) \cup E(\Gamma(\mathbb{Z}_{(p_1^n p_2^m)})) \to \{1,2,3,\dots,2mn+m+n-1\}$ such as

$$f(H_{ij}) = \begin{cases} j+1, & i=n, j \in N_0^{m-1} \\ 2mn+m+n-1-(m+1)i, & i \in N_0^{n-1}, j=m \\ m+j, & i=0, j \in N_1^{m-1} \\ (1+i)m+j, & i \in N_1^{n-1}, j \in N_0^{m-1} \end{cases}$$

$$f(H_{ij}H_{kl}) = \begin{cases} 2mn+m+n-2-(m+1)i-l, & i \in N_0^{n-1}, j=m, \\ & k=n, l \in N_0^{m-1} \end{cases}$$

Based on the above vertex and edge labeling, we observe that

1. Vertex labels of vertices $H_{n0}$, $H_{n1}$, ..., $H_{n(m-1)}$, $H_{01}$, $H_{02}$, ..., $H_{0(m-1)}$, $H_{10}$, $H_{11}$, ..., $H_{(n-1)(m-1)}$ follow strictly increasing pattern from $1$ to $mn+m-1$.
2. Vertex labels of vertices $H_{(n-2)m}$, $H_{(n-3)m}$, ..., $H_{0m}$ are in strictly increasing manner from $mn+3m+1$ to $2mn+m+n-1$, each vertex having label $m+1$ greater than label of its preceding vertex.
3. Vertex label of vertex $H_{(n-1)m}$ is $mn+2m$ and then labels of edges $H_{(n-1)m}H_{nj}, j \in N_0^{m-1}$ are in strictly decreasing manner from $mn+2m-1$ to $mn+m$. A similar strictly decreasing pattern of labels is followed by the vertices $H_{im}, i \in N_0^{n-2}$ and edges $H_{im}H_{nj}, i \in N_0^{n-2}, j \in N_0^{m-1}$ having label from $2mn+m+n-1$ to $mn+3m+1$.

Thus, each vertex and edge get a distinct label from $1$ to $2mn+m+n-1$. Hence, the graph is super graceful.

**Illustration:** Figure 3 given below for Super Graceful Labeling of co-maximal subgroup graph of group $\mathbb{Z}_{p_1^2 p_2^3}$

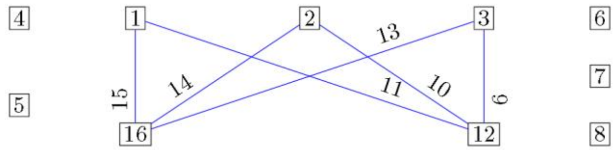


**Figure 3**
**Super Graceful Labeling of $\Gamma(\mathbb{Z}_{p_1^2 p_2^3})$**

**Theorem 3:** *The co-maximal subgroup graph of group $\mathbb{Z}_{p_1^n p_2 p_3}$ is a super graceful graph for $n \in \mathbb{N}$.*

***Proof:*** Firstly, we discuss the structure of the co-maximal subgroup graph of group $\mathbb{Z}_{p_1^n p_2 p_3}$ (Figure 4). The number of vertices in this graph is $4n+2$, as the group $\mathbb{Z}_{p_1^n p_2 p_3}$ has $4n+2$ non-trivial proper subgroups. Let the vertex set be
$V(\Gamma(\mathbb{Z}_{p_1^n p_2 p_3})) = \{H_{ijk} | i \in N_0^n \ , \ j = 0,1, \quad k = 0,1\} - \{H_{000}, H_{n11}\}$ , where $|H_{ijk}| = p_1^i p_2^j p_3^k$, $i \in N_0^n$, $j = 0,1$ and $k = 0,1$. Now two vertices $H_{ijk}$ and $H_{lrs}$ are adjacent in $\Gamma(\mathbb{Z}_{p_1^n p_2 p_3})$ iff $lcm(|H_{ijk}|, |H_{lrs}|) = p_1^n p_2 p_3$. Using this condition, we observe that

1. The vertices $H_{n00}, H_{n10}, H_{n01}$ are adjacent to the vertices $H_{i11}$, $i \in N_0^{n-1}$ and vertices $H_{i00}$ are isolated, where $i \in N_1^{n-1}$.
2. The vertex $H_{n10}$ is adjacent with vertices $H_{i01}$ and vertex $H_{n01}$ is adjacent with vertices $H_{i10}$ where $i \in N_0^{n-1}$.

   Thus, we get

$$deg(H_{ijk}) = \begin{cases} n, & i = n, j = 0, k = 0 \\ 3, & i \in N_0^{n-1}, j = 1, k = 1 \\ 2n+1, & i = n, j = 0,1, k = 0,1, j \neq k \\ 1, & i \in N_0^{n-1}, j = 0,1, k = 0,1, j \neq k \\ 0, & i \in N_1^{n-1}, j = 0, k = 0 \end{cases}$$

   Thus the edge set of this graph contains $5n+1$ edges.

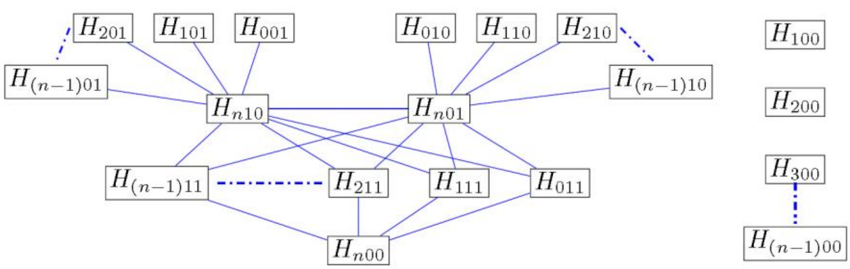


**Figure 4**
**Co-maximal Subgroup Graph of Group $\mathbb{Z}_{p_1^n p_2 p_3}$**

In this graph, there are $4n+2$ vertices and $5n+1$ edges. Let us consider a mapping $f: V(\Gamma(\mathbb{Z}_{p_1^n p_2 p_3})) \cup E(\Gamma(\mathbb{Z}_{p_1^n p_2 p_3})) \rightarrow \{1,2,3,\ldots,9n+3\}$ such as

$$f(H_{ijk}) = \begin{cases} i+1, & i \in N_0^{n-1}, j=1, k=1 \\ 3i+1, & i=n, j=0, k=1 \\ 6i+1, & i=n, j=0, k=0 \\ 9i+3, & i=n, j=1, k=0 \\ n+i+1, & i \in N_0^{n-1}, j=0, k=1 \\ 6n+i+3, & i \in N_0^{n-1}, j=1, k=0 \\ 4n+i+1, & i \in N_1^{n-1}, j=0, k=0 \end{cases}$$

$$f(e) = \begin{cases} 6n-i, & e = H_{n00}H_{i11}, i \in N_0^{n-1} \\ 9n+2-i, & e = H_{n10}H_{i11}, i \in N_0^{n-1} \\ 3n-i, & e = H_{n01}H_{i11}, i \in N_0^{n-1} \\ 8n+2-i, & e = H_{n10}H_{i01}, i \in N_0^{n-1} \\ 3n+2+i, & e = H_{n01}H_{i10}, i \in N_0^{n-1} \end{cases}$$

where $e$ denotes the edge of the graph and $f$ is a bijective function with $f(H_{ijk}H_{lrs}) = |f(H_{ijk}) - f(H_{lrs})|$ for every edge $H_{ijk}H_{lrs} \in E(\Gamma(\mathbb{Z}_{p_1^n p_2 p_3}))$.

Based on the above vertex and edge labeling, we observe that

1. Vertex labels of vertices $H_{011}$, $H_{111}$,…, $H_{(n-1)11}$, $H_{001}$, $H_{101}$, …, $H_{(n-1)01}$ follow strictly increasing pattern from $1$ to $2n$.
2. Edge labels of edges $H_{n01}H_{i11}$, $i \in N_0^{n-1}$ are in strictly decreasing manner from $3n$ to $2n+1$.
3. Vertex $H_{n01}$ is labeled as $3n+1$.
4. Edge labels of edges $H_{i10}H_{n01}$, $i \in N_0^{n-1}$ follow strictly increasing pattern from $3n+2$ to $4n+1$.
5. Vertex labels of vertices $H_{i00}$, $i \in N_1^{n-1}$ follow strictly increasing pattern from $4n+2$ to $5n$.
6. Edge labels of edges $H_{n00}H_{i11}$, $i \in N_0^{n-1}$ follow strictly decreasing pattern from $6n$ to $5n+1$.
7. Vertex $H_{n00}$ is labeled as $6n+1$ and edge $H_{n10}H_{n01}$ is labeled as $6n+2$.
8. Vertex labels of vertices $H_{010}$, $H_{110}$, …,$H_{(n-1)10}$ are in strictly increasing order from $6n+3$ to $7n+2$.
9. Edge labels of edges $H_{n10}H_{i11}$, $H_{n10}H_{i01}$, $i \in N_0^{n-1}$ are in strictly decreasing order from $9n+2$ to $7n+3$.
10. Vertex $H_{n10}$ is labeled as $9n+3$.

Thus, each vertex and edge get a distinct label from $1$ to $9n+3$. Hence, the graph is super graceful.

**Illustration:** Figure 5 given below for Super Graceful Labeling of co-maximal subgroup graph of group $\mathbb{Z}_{p_1^3 p_2 p_3}$

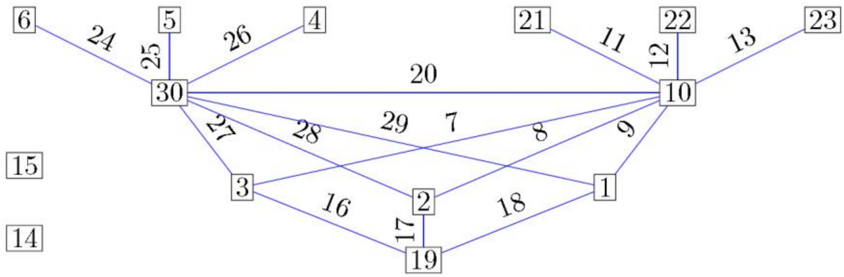


**Figure 5**
**Super Graceful Labeling of $\Gamma(\mathbb{Z}_{p_1^3 p_2 p_3})$**

**Theorem 4:** *The co-maximal subgroup graph of group $\mathbb{Z}_{p_1^n p_2^m p_3}$ is a super graceful graph for $n, m \in \mathbb{N}$ and $m \leq n$.*

***Proof:*** Firstly, we discuss the structure of the co-maximal subgroup graph of group $\mathbb{Z}_{p_1^n p_2^m p_3}$ (Figure 6). As the group $\mathbb{Z}_{p_1^n p_2^m p_3}$ has $2mn + 2m + 2n$ non-trivial proper subgroups, we have $|V(\Gamma(\mathbb{Z}_{(p_1^n p_2^m p_3)}))| = 2mn + 2m + 2n$. Let the vertex set be $V(\Gamma(\mathbb{Z}_{p_1^n p_2^m p_3})) = \{H_{ijk} | i \in N_0^n, \ j \in N_0^m, \ k = 0,1\} - \{H_{000}, H_{nm1}\}$, where $|H_{ijk}| = p_1^i p_2^j p_3^k$, $i \in N_0^n$, $j \in N_0^m$ and $k = 0,1$. Now, two vertices $H_{ijk}$ and $H_{lrs}$ are adjacent in $\Gamma(\mathbb{Z}_{p_1^n p_2^m p_3})$ iff $lcm(|H_{ijk}|, |H_{lrs}|) = p_1^n p_2^m p_3$. Using this condition, we examine the structure of $\Gamma(\mathbb{Z}_{p_1^n p_2^m p_3})$, where $n, m \in \mathbb{N}$ and $m \leq n$,

1. Vertices $H_{im1}$, $i \in N_0^{n-1}$ are adjacent with vertices $H_{nj0}$ and $H_{nj1}$, $j \in N_0^{m-1}$.
2. Vertices $H_{im0}$, $i \in N_0^n$ are adjacent with vertices $H_{nj1}, j \in N_0^{m-1}$.
3. Vertex $H_{nm0}$ is adjacent with vertices $H_{ij1}, i \in N_0^{n-1}, j \in N_0^m$.
4. Vertices $H_{ij0}$, $i \in N_0^{n-1}$, $j \in N_0^{m-1}$, $i = j \neq 0$ are not adjacent with any vertex.

Thus, we get

$$deg(H_{ijk}) = \begin{cases} 2m+1, & i \in N_0^{n-1}, j = m, k = 1 \\ n, & i = n, j \in N_0^{m-1}, k = 0 \\ 2n+1, & i = n, j \in N_0^{m-1}, k = 1 \\ m, & i \in N_0^{n-1}, j = m, k = 0 \\ mn+m+n, & i = n, j = m, k = 0 \\ 0, & i \in N_0^{n-1}, j \in N_0^{m-1}, i = j \neq 0 \end{cases}$$

Here the edge set of the graph contains $4mn + m + n$ edges.

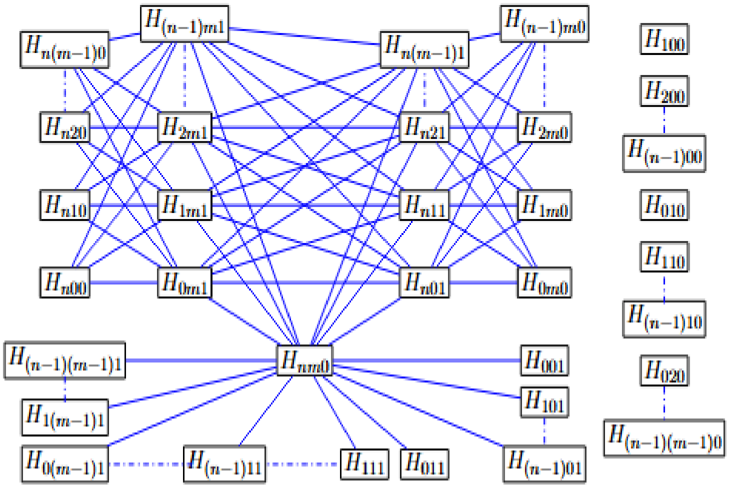


**Figure 6**
**Co-maximal Subgroup Graph of Group $\mathbb{Z}_{p_1^n p_2^m p_3}$**

In this graph, there are $2mn + 2m + 2n$ vertices and $4mn + m + n$ edges. Let $f: V(\Gamma(\mathbb{Z}_{(p_1^n p_2^m p_3)})) \cup E(\Gamma(\mathbb{Z}_{(p_1^n p_2^m p_3)})) \to \{1,2,3,\ldots,6mn + 3m + 3n\}$ be a mapping such that

$$f(H_{ijk}) = \begin{cases} j+1, & i = n, j \in N_0^{m-1}, k = 1 \\ m + i + nj + 1, & i \in N_0^{n-1}, j \in N_0^{m-1}, k = 1 \\ 3mn + m - (m+1)i + 2n, & i \in N_1^{n-1}, j = m, k = 0 \\ 3mn + 3m + 2n - 1, & i = 0, j = m, k = 0 \\ 6mn + 3m + 3n, & i = n, j = m, k = 0 \\ 3mn + m + 2n + j - 1, & i = n, j \in N_0^{m-1}, k = 0 \\ 5mn + 2m + 3n - 1 - (m+1)i, & i \in N_0^{n-1}, j = m, k = 1 \\ 3mn + 3m + n + i + nj, & i \in N_0^{n-1}, j \in N_1^{m-1}, k = 0 \\ 3mn + 2n + i - 1, & i \in N_1^{m-1}, j = 0, k = 0 \\ 4mn + 2m + n + i, & i \in N_m^{n-1}, j = 0, k = 0 \end{cases}$$

$$f(e) = \begin{cases} 2mn + m + n - i(m+1) - j, & e = H_{im1}H_{nj0}, i \in N_0^{n-1}, j \in N_0^{m-1} \\ 5mn + 2m + 3n - i(m+1) - j - 2, & e = H_{im1}H_{nj1}, i \in N_0^{n-1}, j \in N_0^{m-1} \\ 3mn + 2n + m - (m+1)i - j - 1, & e = H_{im0}H_{nj1}, i \in N_1^{n-1}, j \in N_0^{m-1} \\ 3mn + 3m + 2n - j - 2, & e = H_{0m0}H_{nj1}, j \in N_0^{m-1} \\ 6mn + 3m + 3n - j - 1, & e = H_{nm0}H_{nj1}, j \in N_0^{m-1} \\ mn + m + i(m+1) + 1, & e = H_{nm0}H_{im1}, i \in N_0^{n-1} \\ 6mn + 2m + 3n - i - nj - 1, & e = H_{nm0}H_{ij1}, i \in N_0^{n-1}, j \in N_0^{m-1} \end{cases}$$

where $e$ denotes the edge of the graph. On the basis of the above vertex and edge labeling, we observe that

1. Vertex labels of vertices $H_{n01}$, $H_{n11}$, …,$H_{n(m-1)1}$,$H_{001}$, $H_{101}$, …,$H_{(n-1)01}$, $H_{011}$, $H_{111}$, …,$H_{(n-1)11}$, $H_{021}$, …,$H_{(n-1)(m-1)1}$ follow strictly increasing pattern from $1$ to $m+mn$.
2. Vertex labels of vertex $H_{0m0}$ is $3mn+3m+2n-1$ and then vertex label of vertices $H_{1m0}$, …,$H_{(n-1)m0}$ are in strictly decreasing manner from $3mn+2n-1$ to $2mn+2m+n+1$ decreasing $m+1$ at each step.
3. Vertex labels of vertices $H_{n00}$, $H_{n10}$, …, $H_{n(m-1)0}$ follow strictly increasing pattern from $3mn+m+2n-1$ to $3mn+2m+2n-2$.
4. Vertex labels of vertices $H_{0m1}$, $H_{1m1}$, …, $H_{(n-1)m1}$ are in strictly decreasing manner from $5mn+2m+3n-1$ to $4mn+3m+2n$ decreasing $m+1$ at each step.
5. Vertex labels of vertices $H_{010}$, $H_{110}$, …,$H_{(n-1)10}$, $H_{020}$, …,$H_{(n-1)(m-1)0}$ follow strictly increasing pattern from $3mn+3m+2n$ to $4mn+3m+n-1$ and vertex label of vertices $H_{m00}$, $H_{(m+1)00}$, …,$H_{(n-1)00}$ are also in strictly increasing manner from $4mn+3m+n$ to $4mn+2m+2n-1$.
6. Vertex labels of vertices $H_{100}$, $H_{200}$, …,$H_{(m-1)00}$ are in strictly increasing order from $3mn+2n$ to $3mn+m+2n-2$.
7. The vertex $H_{nm0}$ gets the highest label, $6mn+3m+2n$.
8. Edge labels of edges $H_{0m1}H_{nj0}$, $j \in N_0^{m-1}$ are in strictly decreasing manner from $2mn+m+n$ to $2mn+n+1$ followed by the edge $H_{nm0}H_{(n-1)m1}$ having label $2mn+n$. Further strictly decreasing pattern of labels is followed for edges $H_{im1}H_{nj0}$, $i \in N_1^{n-1}$, $j \in N_0^{m-1}$ followed by edge labels of $H_{nm0}H_{im1}$, $i = n-2, n-3, \dots, 0$ in similar $n-1$ more steps considering for each value of $i$ as one step. In these steps, edges get label from $2mn+n-1$ to $mn+m+1$.
9. Edge labels of edges $H_{0m0}H_{nj1}$, $j \in N_0^{m-1}$ are in strictly decreasing manner from $3mn+3m+2n-2$ to $3mn+2m+2n-1$.
10. Edge labels of edges $H_{imo}H_{nj1}$, $i \in N_1^{n-1}$, $j \in N_0^{m-1}$ are in strictly decreasing manner from $3mn+2n-2$ to $2mn+m+n+1$ skipping one vertex label that is already given to vertex $H_{im0}$, $i \in N_2^{n-1}$ after each set of $m$ edge labels.
11. Edge labels of edges $H_{im1}H_{nj1}$, $i \in N_0^{n-1}$, $j \in N_0^{m-1}$ are in strictly decreasing manner from $5mn+2m+3n-2$ to $4mn+2m+2n$ skipping one vertex label that is already assigned to vertex $H_{im1}$, $i \in N_1^{n-1}$ after each set of $m$ edge labels.
12. Edge labels of edges $H_{nm0}H_{nj1}$, $j \in N_0^{m-1}$ are in strictly decreasing manner from $6mn+3m+3n-1$ to $6mn+2m+3n$.

13. Edge labels of edges $H_{nm0}H_{ij1}$, $i \in N_0^{n-1}$ are in strictly decreasing manner for each value of $j$, where $j \in N_0^{m-1}$ from $6mn + 2m + 3n - 1$ to $5mn + 2m + 3n$.

Thus, each vertex and edge get a distinct label from $1$ to $6mn + 3m + 3n$. Hence, the graph is super graceful.

**Illustration:** Figure 7 given below for Super Graceful Labeling of co-maximal subgroup graph of group $\mathbb{Z}_{p_1^3 p_2^3 p_3}$

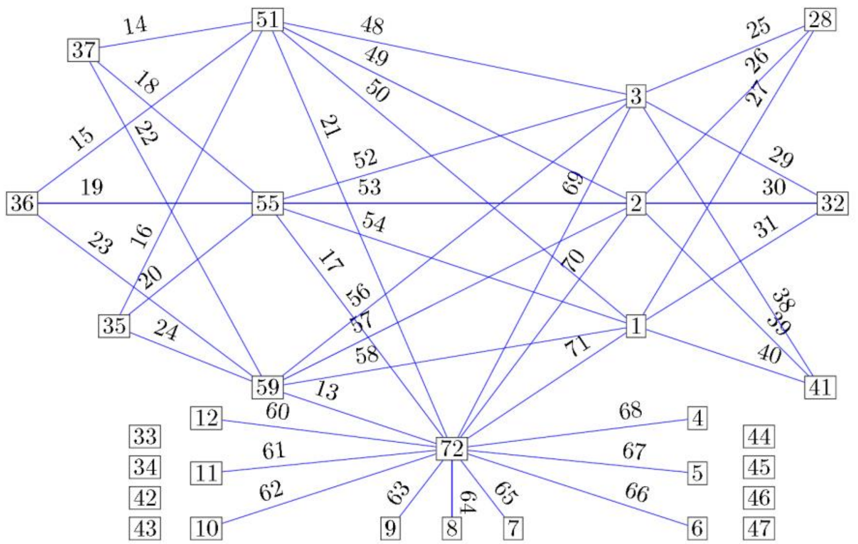

**Figure 7**
**Super Graceful Labeling of $\Gamma(\mathbb{Z}_{p_1^3 p_2^3 p_3})$**

**Theorem 5:** *The co-maximal subgroup graph of group $\mathbb{Z}_{p_1^n p_2^m p_3^l}$ is a super graceful graph for $n, m, l \in \mathbb{N}$, where $l \leq m \leq n$ and $l \geq 2$.*

***Proof:*** Firstly, we discuss the structure of the co-maximal subgroup graph of group $\mathbb{Z}_{p_1^n p_2^m p_3^l}$ (Figure 8). As the group $\mathbb{Z}_{p_1^n p_2^m p_3^l}$ has $(l + 1)(mn + m + n) + l - 1$ non-trivial proper subgroups, we have $|V(\Gamma(\mathbb{Z}_{p_1^n p_2^m p_3^l}))| = (l + 1)(mn + m + n) + l - 1$. Let the vertex set be $V(\Gamma(\mathbb{Z}_{p_1^n p_2^m p_3^l})) = \{H_{ijk} | i \in N_0^n$, $j \in N_0^m$, $k \in N_0^l\} - \{H_{000}, H_{nml}\}$, where $|H_{ijk}| = p_1^i p_2^j p_3^k$. Now two vertices $H_{ijk}$ and $H_{rst}$ are adjacent in $\Gamma(\mathbb{Z}_{p_1^n p_2^m p_3^l})$ iff $lcm\ (|H_{ijk}|, |H_{rst}|) = p_1^n p_2^m p_3^l$.

Using this condition, we examine the structure of $\Gamma(\mathbb{Z}_{p_1^n p_2^m p_3^l})$, where $n, m, l \in \mathbb{N}$, $l \leq m \leq n$ and $l \geq 2$.

1. Vertices $H_{njk}, j \in N_0^{m-1}, k \in N_0^{l-1}$ are adjacent with vertices $H_{iml}, i \in N_0^{n-1}$.
2. Vertices $H_{iml}, i \in N_0^{n-1}$ are adjacent with vertices $H_{njl}, j \in N_0^{m-1}$ and $H_{nmk}, k \in N_0^{l-1}$.
3. Vertices $H_{njl}, j \in N_0^{m-1}$ are adjacent with vertices $H_{nmk}, k \in N_0^{l-1}$ and $H_{imk}, i \in N_0^{n-1}, k \in N_0^{l-1}$.
4. Vertices $H_{ijl}, i \in N_0^{n-1}, j \in N_0^{m-1}$ are adjacent with vertices $H_{nmk}, k \in N_0^{l-1}$.
5. Vertices $H_{ijk}, i \in N_0^{n-1}, j \in N_0^{m-1}, k \in N_0^{l-1}$ are isolated.

Thus, we get

$$deg(H_{ijk}) = \begin{cases} n, & i = n, j \in N_0^{m-1}, k \in N_0^{l-1} \\ m, & i \in N_0^{n-1}, j = m, k \in N_0^{l-1}, \\ l, & i \in N_0^{n-1}, j \in N_0^{m-1}, k = l, \\ lm + m + l, & i \in N_0^{n-1}, j = m, k = l, \\ ln + n + l, & i = n, j \in N_0^{m-1}, k = l, \\ mn + m + n, & i = n, j = m, k \in N_0^{l-1}, \\ 0, & otherwise \end{cases}$$

Thus, the graph has $3lmn + mn + l(m + n)$ edges.

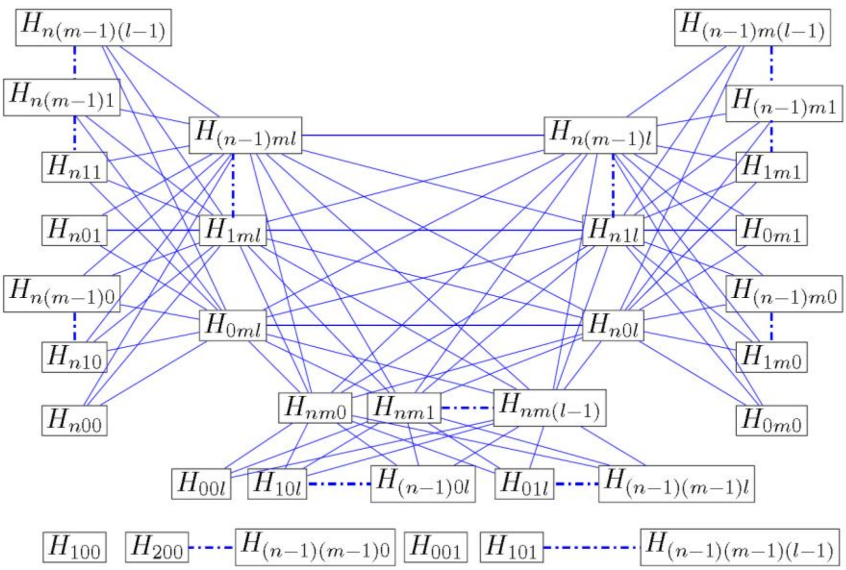


**Figure 8**
**Co-maximal Subgroup Graph of Group $\mathbb{Z}_{p_1^n p_2^m p_3^l}$**

In this graph, we have $(l+1)(mn+m+n)+l-1$ vertices and $3lmn+mn+l(m+n)$ edges.

Let $f: V(\Gamma(\mathbb{Z}_{(p_1^n p_2^m p_3^l)})) \cup E(\Gamma(\mathbb{Z}_{(p_1^n p_2^m p_3^l)})) \to \{1,2,\dots,4lmn+2mn+2lm+2ln+m+n+l-1\}$ be a mapping such that

$$f(H_{ijk}) = \begin{cases} j+1, & i=n, j \in N_0^{m-1}, k=l \\ m+nj+i+1, & i \in N_0^{n-1}, j \in N_0^{m-1}, k=l, \\ lmn+mn+lm+m+n+l+(m+1)((l-k)n-(i+1)), & \\ \qquad i \in N_0^{n-1}, j=m, k \in N_1^{l-1} & \\ 2lmn+mn+2ln+m-l+j+k(mn+m+1), & \\ \qquad i=n, j \in N_0^{m-1}, k \in N_0^{l-1} & \\ 3lmn+2mn+lm+2ln+m+n-1-i(m+1), & \\ \qquad i \in N_0^{n-1}, j=m, k=l & \\ 2lmn+2mn+2ln+m+n-l-2-(m+1)(i-1), & \\ \qquad i \in N_1^{n-1}, j=m, k=0 & \\ 3lmn+lm+2ln+2m-1, & i=0, j=m, k=0 \\ 3lmn+3mn+lm+2ln+2m+n+k(mn+m+1), & \\ \qquad i=n, j=m, k \in N_0^{l-1} & \end{cases}$$

$$f(e) = \begin{cases} lmn+mn+lm+n+l-j-1-i(m+1)-k(mn+m+1), \\ \qquad e=H_{iml}H_{njk}, i \in N_0^{n-1}, j \in N_0^{m-1}, k \in N_0^{l-1}, \\ 3lmn+2mn+lm+2ln+m+n-j-2-i(m+1), \\ \qquad e=H_{iml}H_{njl}, i \in N_0^{n-1}, j \in N_0^{m-1}, \\ mn+m+1+i(m+1)+k(mn+m+1), \\ \qquad e=H_{nmk}H_{iml}, i \in N_0^{n-1}, k \in N_0^{l-1} \\ 3lmn+3mn+lm+2ln+2m+n+k(mn+m+1)-j-1, \\ \qquad e=H_{nmk}H_{njl}, j \in N_0^{m-1}, k \in N_0^{l-1}, \\ lmn+mn+lm+m+n+l+(m+1)((l-k)n-(i+1))-j-1, \\ \qquad e=H_{imk}H_{njl}, i \in N_0^{n-1}, k \in N_1^{l-1}, j \in N_0^{m-1}, \\ 2lmn+2mn+2ln+m+n-l-3-j-(m+1)(i-1), \\ \qquad e=H_{im0}H_{njl}, i \in N_1^{n-1}, j \in N_0^{m-1} \\ 3lmn+lm+2ln+2m-2-j, \\ \qquad e=H_{0m0}H_{njl}, j \in N_0^{m-1}, \\ 3lmn+3mn+lm+2ln+m+n+k(mn+m+1)-jn-i-1, \\ \qquad e=H_{nmk}H_{ijl}, k \in N_0^{l-1}, i \in N_0^{n-1}, j \in N_0^{m-1} \end{cases}$$

where, $e$ denotes edge of the graph. Remaining $lmn-1$ labels make one-one correspondence with the set of isolated vertices, that contains $lmn-1$ vertices.

Using the vertex and edge labeling described above, we observe that

1. Vertex labels of vertices $H_{n0l}, H_{n1l}, \dots, H_{n(m-1)l}$, $H_{00l}, H_{10l}, \dots, H_{(n-1)(m-1)l}$ are in strictly increasing order from $1$ to $m + mn$.
2. Vertex $H_{nm(l-1)}$ gets label $4lmn + 2mn + 2lm + 2ln + m + n + l - 1$, then edges $H_{nm(l-1)}H_{njl}, j \in N_0^{m-1}$ get labels in decreasing manner. They are decreasing by $1$ for each succeeding edge upto label $4lmn + 2mn + 2lm + 2ln + n + l - 1$.
3. Strictly decreasing pattern of labels is observed for edges $H_{nm(l-1)}H_{ijl}, i \in N_0^{n-1}, j \in N_0^{m-1}$ from $4lmn + 2mn + 2lm + 2ln + n + l - 2$ to $4lmn + mn + 2lm + 2ln + n + l - 1$.
4. Vertex $H_{nm(l-2)}$ gets label $4lmn + mn + 2lm + 2ln + n + l - 2$ and then the same type of decreasing pattern is followed for edges $H_{nm(l-2)}H_{ijl}, i \in N_0^{n-1}, j \in N_0^{m-1}$ as followed in (4). Similar pattern of labels is followed by vertices $H_{nmk}, k = l-3, l-4, \dots, 0$ and edges $H_{nmk}H_{ijl}, k = l-3, l-4, \dots, 0, i \in N_0^{n-1}, j \in N_0^{m-1}$. Thus we come across the label upto $3lmn + 2mn + lm + 2ln + m + n$.
5. Vertices $H_{iml}, i \in N_0^{n-1}$ get labels decreasing $(m+1)$ at each vertex starting from $3lmn + 2mn + lm + 2ln + m + n - 1$. At each step the in-between $m$ labels are given to edges $H_{iml}H_{njl}, i \in N_0^{n-1}, j \in N_0^{m-1}$. Last label in this sequence is $3lmn + mn + lm + 2ln + 2m$.
6. Vertex $H_{0m0}$ gets label $3lmn + lm + 2ln + 2m - 1$, then labels are given to edges $H_{0m0}H_{njl}, j \in N_0^{m-1}$ in strictly decreasing manner upto label $3lmn + lm + 2ln + m - 1$.
7. For each $k$, labels of vertices $H_{njk}$ $j = m-1, m-2, \dots, 0, k \in N_1^{l-1}$ are decreasing by $1$ and for each $j$, labels of vertices $H_{njk}$, $j = m-1, m-2, \dots, 0, k = l-1, l-2, \dots, 1$ are decreasing by $mn + m + 1$. Thus we get labels from $3lmn + lm + 2ln + m - 2$ to $2lmn + 2ln + 2mn + 2m - l + 1$. The in-between remaining labels are given to isolated vertices.
8. Vertex $H_{1m0}$ gets label $2lmn + 2mn + 2ln + m + n - l - 2$, then labels decreasing by 1 are obtained for edges $H_{1m0}H_{njl}, j \in N_0^{m-1}$. The next label is given to vertex $H_{2m0}$ and labels decreasing by 1 are obtained for edges $H_{2m0}H_{njl}, j \in N_0^{m-1}$. Same pattern is followed for vertices $H_{im0}, i \in N_3^{n-1}$ and edges $H_{im0}H_{njl}, i \in N_3^{n-1}, j \in N_0^{m-1}$ upto label $2lmn + mn + 2ln + 2m - l$.
9. The next label $2lmn + mn + 2ln + 2m - l - 1$ is given to $H_{n(m-1)0}$, then labels are decreasing by 1 for vertices $H_{nj0}, j = m-2, m-3, \dots, 0$ upto label $2lmn + mn + 2ln + m - l$.
10. Vertex $H_{0m1}$ gets label $2lmn + lm + ln + l - 1$, then labels decreasing by 1 are obtained for edges $H_{0m1}H_{njl}, j \in N_0^{m-1}$. The next decreasing label is given to $H_{1m1}$ followed by edges $H_{1m1}H_{njl}, j \in N_0^{m-1}$. Similar pattern follows for vertices $H_{imk}, i \in N_2^{n-1}, k \in N_1^{l-1}$ and edges $H_{imk}H_{njl}$, $i \in N_2^{n-1}, k \in N_1^{l-1}, j \in N_0^{m-1}$ upto label $lmn + mn + lm + n + l$.

11. Edges $H_{nj0}H_{0ml}, j \in N_0^{m-1}$ get decreasing pattern of labels from $lmn + mn + lm + n + l - 1$ to $lmn + mn + lm - m + n + l$ followed by edge label $lmn + mn + lm - m + n + l - 1$ of edge $H_{(n-1)ml}H_{nm(l-1)}$. Similar decreasing pattern of labels is obtained for edges $H_{nj0}H_{iml}, j \in N_0^{m-1}, i \in N_1^{n-1}$ followed by edge $H_{0ml}H_{nm(l-1)}$ having label $lmn + lm + l$. For each $k$ the same pattern of decreasing labels is obtained for edges $H_{njk}H_{iml}, j \in N_0^{m-1}, i \in N_0^{n-1}$ followed by edge labels of edges $H_{iml}H_{nmk}, i \in N_0^{n-1}, k = l-2, l-3, \ldots, 1$ upto label $mn + m + 1$.
12. The remaining labels are given to the isolated vertices.

Thus, we obtain distinct labels from 1 to $4lmn + 2mn + 2lm + 2ln + m + n + l - 1$ for the vertices and edges of this graph. So, the function $f$ is bijective. Hence, we obtain super graceful graph.

**Illustration:** Figure 9 given below for Super Graceful Labeling of co-maximal subgroup graph of group $\mathbb{Z}_{p_1^2 p_2^2 p_3^2}$

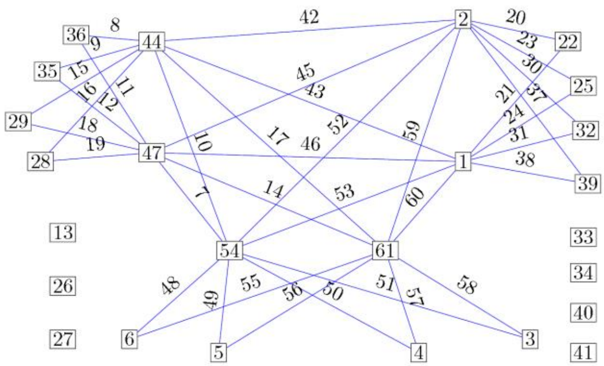

**Figure 9**
**Super Graceful Labeling of $\Gamma(\mathbb{Z}_{p_1^2 p_2^2 p_3^2})$**

## Conclusion

The results proved in the present paper establish a new class of super graceful graphs with the help of co-maximal subgroup graphs of finite cyclic groups $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p_1^n p_2^m}$, $\mathbb{Z}_{p_1^n p_2 p_3}$, $\mathbb{Z}_{p_1^n p_2^m p_3}$ and $\mathbb{Z}_{p_1^n p_2^m p_3^l}$ for $n, m, l \in \mathbb{N}$.

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## Conflict of Interest

There is no conflict of interest regarding this paper.

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