Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 6, pp. 2237–2245
DOI : 10.47974/JDMSC-2188

Solving the eigenproblems of triangular and diagonal strictly double $\mathbb{R}$-astic matrices over min-plus algebra

Siswanto *
Sahmura Maula Al-Maghribi #
Study Program of Mathematics
Universitas Sebelas Maret
Surakarta
Indonesia

and

Jalan Ir. Sutami 36A Kentingan
Jebres
Surakarta 57126
Jawa Tengah
Indonesia

Abstract

The algebraic structure defined by the triple ($\mathbb{R}_{\varepsilon'}$, $\oplus'$, $\otimes$), commonly referred to as min-plus algebraic semi-field. The eigenvalue and eigenvector problem is one of the key issue in min-plus algebra. With a literature review as the research method, this paper investigates and derives explicit formulas for solving the eigenproblem of two special matrices over min-plus semi-field.

Subject Classification: *11C20, 15Axx, 15A30, 15A80.*

Keywords: *Eigenvalue, Eigenvector, Special square matrices, Min-plus algebra.*

1. Introduction

Min-plus algebra is one of many idempotent semi-field that has been studied across various fields of mathematics, another notable instance is max algebra [4]. We work wih the min-plus semi-field which is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{+\infty\}$, accompanied by minimal ($\oplus'$) operation as addition and regular addition ($\otimes$) as multiplication [9]. Formally, for every $a, b \in \mathbb{R}_{\varepsilon'}$ applies

* *E-mail:* **sis.mipa@staff.uns.ac.id (Corresponding Author)**
# *E-mail:* **sahmura_maula18@student.uns.ac.id**

© Taru Publications

$$a \oplus' b = min(a, b) \text{ and } a \otimes b = a + b.$$

Furthermore, a matrix operates under the min-plus algebra, where addition is replaced by the minimum operation, and multiplication is replaced by standard addition. Moreover, a square matrix over min-plus semi-field is an $n \times n$ matrix whose entries are elements of $\mathbb{R}_{\varepsilon'}$, where the set of these square matrices is denoted by $\mathbb{R}_{\varepsilon'}^{n\times n}$ [1]. We define various operations within $\mathbb{R}_{\varepsilon'}^{n\times n}$.

a) Let A and $B \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Then $(A \oplus' B)_{ij} = min\{a_{ij}, b_{ij}\}$.

b) Let A and $B \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Then $(A \otimes B)_{ij} = \oplus'_{k=1}^{n}(a_{ik} \otimes b_{kj})$.

c) For all set of matrix over min-plus semi-field, there exist identity matrix with respect to $\otimes$ such that $I \otimes A = A = A \otimes I$. We have

$$I = \begin{pmatrix} 0 & \cdots & \varepsilon' \\ \vdots & \ddots & \vdots \\ \varepsilon' & \cdots & 0 \end{pmatrix},$$

d) Let $\alpha \in \mathbb{R}_{\varepsilon'}$ and $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Then $(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij}$.

e) Let $B \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $p \in \mathbb{N}$. Then the power operation of B is expressed as

$$\underbrace{B^{\otimes p} = B \otimes \ldots \otimes B}_{p \text{ times}}.$$

In conventional algebra, we can define the determinant for any square matrices. The determinant is calculated using a specific formula involving the matrix element. Since the concept of a determinant in min-plus algebra has no straightforward equivalent in classical algebra due to the lacks of an additive inverse, the determinant of matrices over min-plus algebra can be expressed using the permanent [6]. It is described as the minimum sum of the products of elements forming specific paths within the matrix. A specific path often refers to selecting elements in a particular way, such as choosing one element from each row and column without repeating any row or column, which can also be described as a permutation of elements. This influences how characteristic polynomials are determined.

Formally, we expressed for a matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ as

$$perm(A) = \bigoplus_{\sigma\in P_n}{}' \bigotimes_{i=1}^{n} (a_{i\sigma(i)})$$

where σ denote the permutation and P_n is a set of all permutations from $\{1,2,\ldots,n\}$ [6]. The concept of the permanent replaces the determinant structure and provides a way to define the characeristic polynomial structure in the context of the minimal and addition operations, such that the definition of the characeristic min-polynomial can be adapted to the definition of the characteristic polynomial in classical algebra. Therefore, by [5], the characteristic min-polynomial of A is defined as

$$\chi_A(x) = perm(I \otimes x \oplus' A).$$

Moreover, for each square matrix $\mathbb{R}_{\varepsilon'}^{n\times n}$ can also get its characteristic polynomial and satisfies its own characteristic equation by using Cayley-Hamilton theorem in min-plus algebra [7].

In its application, min-plus semi-field proves invaluable for selecting the shortest and most effective route for a journey, which is highly useful for supply chain efficiency [3]. The problem is closely related to an algebraic concept known as the eigenproblem solving. The eigenproblem is done by determining the eigenvalue (λ) and the corresponding eigenvector (v) such in a way that it satisfies the equation $A \otimes v = \lambda \otimes v$ [1]. In min-plus semi-field, we can determine the eigenvalues and eigenvectors corresponding to eigenvalues by the characteristic min-polynomial [10].

Beyond the general discussion of square matrices, some researchers delve into special types of matrices. For instance, Jafari and Hosseinyazdi [2] explored triangular matrices and diagonal (or certain) strictly double $\mathbb{R}$-*astic* matrices in their paper. Both matrices are examples of square matrices that possess special properties. Furthermore, research by Siswanto and Maghribi [8] extends these two matrices to min-plus algebra and establishes properties for determining the eigenvalue of these matrices through their characteristic min-polynomial.

Let $T \in \mathbb{R}_{\varepsilon'}^{n\times n}$. A square matrix T is triangular if every element either above or below the main diagonal is equal to ε' [8]. And let $D \in \mathbb{R}_{\varepsilon'}^{n\times n}$. A square matrix D is called diagonal strictly double $\mathbb{R}$-*astic* if $d_{ij} \leq +\infty$, there is only one finite element on each row and on each column of D, and its finite element is the diagonal element on the matrix D [8]. Based on the definitions of these two matrices, in general, matrix T and matrix D can be written as follows:

$$T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \varepsilon' & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & a_{nn} \end{bmatrix}, \qquad D = \begin{bmatrix} a_{11} & \varepsilon' & \cdots & \varepsilon' \\ \varepsilon' & a_{22} & \cdots & \varepsilon' \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & a_{nn} \end{bmatrix}$$

Building on the min-plus algebraic structure, Jafari and Hosseinyazdi's paper, and also Siswanto and Maghribi's research, we will dive into the specific formula for solving the eigenvalue and eigenvector problem specifically of these two special matrices.

2. Methods

A literature review serves as the research method for this paper, with references consisting of books, journals, and other relevant writings related to this research. Additionally, the researcher also uses references that discuss characteristic polynomials and eigenvalue problems of triangular matrices and diagonal strictly double $\mathbb{R}$-*astic* matrices over max-plus semi-field. From this

method, the solution to the eigenproblem of these two matrices over min-plus semi-field can be determined.

The steps taken in this research are as follows:

1) Reviewing fundamental concepts
2) Exploring the relationship with the permanent
3) Recalling the form of the characteristic polynomial by the previous research
4) Formulating and proving the special formula for solving eigenvalues and corresponding eigenvectors of these two matrices over min-plus semi-field
5) Implementing in specific cases

3. Solving the eigenvalue and eigenvector problem of the triangular matrix

In conventional algebra, there is a specific method to determine the eigenvalues of upper or lower triangular matrices. The eigenvalues of a triangular matrix correspond to the elements along its diagonal. Because of the utilization of the minimum operation in the min-plus algebra, eigenvalues can be interpreted as unique values that optimize a particular function. The formula for determining the eigenvalue is contained in the following theorem.

Theorem 3.1: *The principal eigenvalue λ of a triangular matrix $T \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is given by*

$$\lambda(T) = \bigoplus_{i=1}^{n}{}' t_{ii},$$

which is equal to the smallest corner of the matrix T.

Proof: The proof start with the theorem of smallest corner. Based on the theorem in the research by Maghribi *et al.* [5], the smallest corner of matrix T is

$$\bigoplus_{r=0}^{n-1}{}' \frac{\delta_{n-r}}{n-r} \equiv \bigoplus_{k=1}^{n}{}' \frac{\delta_k}{k}.$$

where δ_{n-r} represent the sum of the $n-r$ diagonal elements of T. According to the theorem by Siswanto and Maghribi [8], δ_{n-r} corresponds to the cumulative sum of the smallest $n-r$ diagonal elements. As $n-r$ decreases, the denominator $n-r$ also decreases, which cause the value of $\frac{\delta_{n-r}}{n-r}$ to increases. Hence, the smalles value of $\frac{\delta_{n-r}}{n-r}$ occurs when $n-r=1$, leading to

$$\frac{\delta_{n-r}}{n-r} = \frac{\delta_1}{1} = \delta_1.$$

Thus, the smallest corner is determined to be $\oplus'_{r=0,\ldots,n-1} \frac{\delta_{n-r}}{n-r} = \delta_1$, which is the smallest diagonal element of T.

Further, the eigenvectors can be determined after finding the eigenvalue of a triangular matrix over min-plus semi-field. However, in triangular matrices under certain conditions, the eigenvectors can be found using special steps outlined in the following theorem.

Theorem 3.2: *Let the triangular matrix $T \in \mathbb{R}_{\varepsilon'}^{n\times n}$ satisfying the following conditions:*

(1) *it has multiple diagonal elements satisfying $t_{ii} \geq \lambda(T)$;*

(2) *every element positioned either below or above the diagonal is greater than or equal to $\lambda(T)$,*

then the columns of T_λ with zero diagonal entries serve as the eigenvectors of T corresponding to the eigenvalue of T.

Proof: The proof of will be presented for upper triangular matrices. According to [5],

$$T_\lambda = \begin{pmatrix} t_{11} - \lambda(T) & \cdots & t_{1n} - \lambda(T) \\ \vdots & \ddots & \vdots \\ \varepsilon' & \cdots & t_{nn} - \lambda(T) \end{pmatrix}.$$

Now, matrix T_λ contains at least one column whose diagonal element is zero. This is because there is a column in T that contains the smallest diagonal element. Due to the fact that $\lambda(T)$ is the smallest diagonal element of T, the elements $t_{ij} - \lambda(T)$, where $t_{ij} > 0$, and $t_{ii} - \lambda(T)$, where $t_{ii} \neq \lambda(T)$ are non-negative, while the elements $t_{ii} - \lambda(T)$ are equal to 0 for $t_{ii} = \lambda(T)$. On the other hand, the matrix $T_\lambda^+ = T_\lambda \oplus' T_\lambda^{\otimes 2} \oplus' \ldots \oplus' T_\lambda^{\otimes n}$ [5]. The elements of $T_\lambda^{\otimes 2}$ are

$$T_\lambda^{\otimes 2} = \bigoplus_{k=1}^{2}{}' \left(t_{ik} \otimes t_{kj}\right).$$

The elements of $T_\lambda^{\otimes n}$ can be defined in a similar pattern to $T_\lambda^{\otimes 2}$. Furthermore, elements from T_λ that have a non-negative value will always have a non-negative value for the elements from $T_\lambda^{\otimes n}$ with the same location, and elements from T_λ will have a value retated to $T_\lambda^{\otimes n}$ such that $t_{ii} \leq \left(t_{ii} - \lambda(T)\right)^n$. On the opposite hand, the diagonal element of T_λ that has a value equal to zero also always has the value zero for the diagonal element of $T_\lambda^{\otimes n}$ with the same location. However, the results obtained from matrix T_λ^+ are the same as those obtained from matrix T_λ. Hence, based on the theorem by Maghribi et al. [5], the

corresponding eigenvectors of T are obtained by the columns of T_λ with zero diagonal entries.

Example: Consider the triangular matrix

$$T = \begin{bmatrix} 3 & 9 & 1 & 1 \\ \varepsilon' & 4 & 5 & 5 \\ \varepsilon' & \varepsilon' & 6 & 7 \\ \varepsilon' & \varepsilon' & \varepsilon' & 1 \end{bmatrix}.$$

Then the eigenvalue of T is

$$\lambda(T) = \min(3,4,6,1) = 1.$$

Now, we will determine the corresponding eigenvector of $\lambda(T)$ based on the theorem. Since $\lambda(T) = 1$, then

$$T_\lambda = -1 \otimes \begin{bmatrix} 3 & 9 & 1 & 1 \\ \varepsilon' & 4 & 5 & 5 \\ \varepsilon' & \varepsilon' & 6 & 7 \\ \varepsilon' & \varepsilon' & \varepsilon' & 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 0 & 0 \\ \varepsilon' & 3 & 4 & 4 \\ \varepsilon' & \varepsilon' & 5 & 6 \\ \varepsilon' & \varepsilon' & \varepsilon' & 0 \end{bmatrix}.$$

It can be seen that $a_{44} = 0$, so the corresponding eigenvector is the fourth column of T_λ. Thus, we obtain

$$v = [0 \quad 4 \quad 6 \quad 0]^T.$$

4. Solving the eigenvalue and eigenvector problem of diagonal strictly double ℝ-*astic* matrix

As explained in conventional algebra, eigenvalues of a diagonal matrix are the elements on the diagonal itself. Since a diagonal strictly double ℝ-*astic* matrix is operated with the minimal operation, then eigenvalues can also be interpreted as unique values that optimize a particular function. The formula for determining the eigenvalue is contained in the following theorem.

Theorem 4.1: *The eigenvalue of a diagonal strictly double* ℝ*-astic matrix* $D \in \mathbb{R}_{\varepsilon'}^{n\times n}$ *is given by*

$$\lambda(D) = \bigoplus_{i=1}^{n}{}' d_{ii},$$

which is equal to the smallest corner of the matrix D.

Proof: The proof start with the theorem of the smallest corner. Based on the theorem in the research by Maghribi *et al.* [5], the smallest corner of matrix D is

$$\bigoplus_{r=0}^{n-1}{}' \frac{\delta_{n-r}}{n-r} \equiv \bigoplus_{k=1}^{n}{}' \frac{\delta_k}{k},$$

where δ_{n-r} represent the sum of the $n-r$ diagonal elements of D. According to the theorem by Siswanto and Maghribi [8], δ_{n-r} corresponds to the cumulative

sum of the smallest $n-r$ diagonal elements. As $n-r$ decreases, the denominator $n-r$ also decreases, which cause the value of $\frac{\delta_{n-r}}{n-r}$ to increases. Hence, the smalles value of $\frac{\delta_{n-r}}{n-r}$ occurs when $n-r=1$, leading to

$$\frac{\delta_{n-r}}{n-r}=\frac{\delta_1}{1}=\delta_1.$$

Thus, the smallest corner is determined to be $\oplus'_{r=0,\dots,n-1}\frac{\delta_{n-r}}{n-r}=\delta_1$, which is the smallest diagonal element of D.

Theorem 4.2: *Let a diagonal strictly double $\mathbb{R}$-astic matrix $D\in\mathbb{R}_{\varepsilon'}^{n\times n}$, where the diagonal entries of D satisfy $d_{ii}\geq\lambda(D)$ for all i. Then the corrresponding eigenvectors of D are given by the columns of D_λ with zero diagonal entries.*

Proof: According to [5],

$$D_\lambda=\begin{pmatrix} d_{11}-\lambda(D) & \cdots & \varepsilon' \\ \vdots & \ddots & \vdots \\ \varepsilon' & \cdots & d_{nn}-\lambda(D)\end{pmatrix}.$$

Now, matrix D_λ contains at least one column whose diagonal element is zero. This is because there is a column in D that contains the smallest diagonal element. Considering that $\lambda(D)$ is the smallest diagonal element of D, the elements $d_{ij}-\lambda(D)$, where $d_{ij}>0$, and $d_{ii}-\lambda(D)$, where $d_{ii}\neq\lambda(D)$ are non-negative, while the elements $d_{ii}-\lambda(D)$ are equal to 0 for $d_{ii}=\lambda(D)$. On the other hand, the matrix $D_\lambda^+=D_\lambda\oplus' D_\lambda^{\otimes 2}\oplus'\dots\oplus' D_\lambda^{\otimes n}$ [5]. The elements of $D_\lambda^{\otimes n}$ are

$$D_\lambda^{\otimes n}=\begin{pmatrix} \big(d_{11}-\lambda(D)\big)^n & \cdots & \varepsilon' \\ \vdots & \ddots & \vdots \\ \varepsilon' & \cdots & \big(d_{nn}-\lambda(D)\big)^n\end{pmatrix}.$$

Furthermore, elements from D_λ that have a non-negative value will always have a non-negative value for the elements from $D_\lambda^{\otimes n}$ with the same location, and elements from D_λ will have a value retated to $D_\lambda^{\otimes n}$ such that $d_{ii}\leq\big(d_{ii}-\lambda(D)\big)^n$. On the opposite hand, the diagonal element of D_λ that has a value equal to zero also has the value zero for the diagonal element of $D_\lambda^{\otimes n}$ with the same location. However, the results obtained from matrix D_λ^+ are the same as those obtained from matrix D_λ. Hence, based on the theorem by Maghribi et al. [5], the corresponding eigenvectors of D are obtained by the columns of D_λ with zero diagonal entries.

Example: Consider the matrix

$$D = \begin{bmatrix} 5 & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & 8 & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & 7 \end{bmatrix}.$$

Then the eigenvalue of D is

$$\lambda(D) = \min(5,3,8,7) = 3.$$

Now, we will determine the corresponding eigenvector of $\lambda(D)$ based on the theorem. Since $\lambda(D) = 3$, then

$$D_\lambda = -3 \otimes \begin{bmatrix} 5 & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & 8 & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & 7 \end{bmatrix} = \begin{bmatrix} 2 & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & 0 & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & 5 & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & 4 \end{bmatrix}.$$

It can be seen that $a_{22} = 0$, so the corresponding eigenvector is the second column of D_λ. Thus, we obtain

$$v = [\varepsilon' \quad 0 \quad \varepsilon' \quad \varepsilon']^T.$$

What should be emphasized is that eigenvectors of these two special matrices over min-plus semi-field are not necessarilly unique. However, any linear combination of these eigenvectors will also be an eigenvector for the eigenvalue λ.

5. Conclusions

It can be concluded that the formulas for finding the eigenvalue and eigenvector of triangular matrices and diagonal strictly double $\mathbb{R}$-*astic* matrices are same. Those formulas are easier to solve than the formula for solving the eigenvalue and eigenvector problem of conventional square matrices.

Acknowledgements

This research was made possible through the support of Universitas Sebelas Maret, for which the author is profoundly grateful during the 2025 academic year.

References

[1] A. W. Nowak, "The Tropical Eigenvalue-Vector Problem from Algebraic, Graphical, and Computational Perspectives," Honors Theses, Bates College (2014). [Online]. Available: http://scarab.bates.edu/honorstheses/97.

[2] F. Jafari and M. Hosseinyazdi, "Characteristic Max-Polynomial of a Triangular and Certain Strictly Double R-astic Matrices," *Int J Pure Appl Sci Technol*, vol. 8, no. 1, pp. 47–56 (2012), [Online]. Available: www.ijopaasat.in.

[3] H. Suprayitno, "Correctness Proof of Min-plus Algebra for Network Shortest-Paths Simultaneous Calculation," *Journal of Technology and Social Science* (JTSS)-61-J. Tech. Soc. Sci, vol. 1, no. 1 (2017), Accessed: Sep. 07, 2023. [Online]. Available: http://jtss.e-jikei.org/issue/archives/vol01_no01/9_A010/Camera%20ready%20manuscript_JTSS_A010.pdf.

[4] P. Butkovic, Max-linear Systems: Theory and Algorithms. London: Springer, 2010. Accessed: Sep. 07 (2023). [Online]. Available: https://link.springer.com/book/10.1007/978-1-84996-299-5.

[5] S. Maula Al Maghribi, Siswanto, and Sutrima, "JTAM (Jurnal Teori dan Aplikasi Matematika) Characteristic Min-Polynomial and Eigen Problem of a Matrix over Min-Plus Algebra," vol. 7, no. 4, pp. 1108–1117 (2023), doi: 10.31764/jtam.v7i4.16498.

[6] Siswanto, A. Gusmizain, and S. Wibowo, "Determinant of a matrix over min-plus algebra," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 24, no. 6, pp. 1829–1835 (2021), doi: 10.1080/09720529.2021.1948663.

[7] Siswanto, N. Nurhayati, and Pangadi, "Cayley-Hamilton theorem in the min-plus algebra," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 24, no. 6, pp. 1821–1828 (2021), doi: 10.1080/09720529.2021.1948662.

[8] Siswanto and S. M. Al Maghribi, "Characteristic min-polynomial of a triangular and diagonal strictly double R-astic matrices," F1000Res, vol. 13, p. 771, Jul. 2024, doi: 10.12688/f1000research.147931.1.

[9] S. Watanabe and Y. Watanabe, "Min-Plus Algebra and Networks," Research Institute for Mathematical Sciences Kyoto University, vol. 47, pp. 41–54 (2014), [Online]. Available: http://hdl.handle.net/2433/226217

[10] S. Watanabe, Y. Tozuka, Y. Watanabe, A. Yasuda, and M. Iwasaki, "Two characteristic polynomials corresponding to graphical networks over min-plus algebra," ArXiv, pp. 1–16 (2017), doi: 10.48550/arXiv.1705.09513.

Received March, 2024