Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 6, pp. 2337–2350
DOI : 10.47974/JDMSC-2283

On the friable mean-values of some arithmetic functions

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Abstract

Let $\mathbf{S}(\mathbf{w},\mathbf{z})$ denote the set of integers up to w whose prime factors are all at most z. We study the average behavior of certain multiplicative arithmetic functions on the elements of $\mathbf{S}(\mathbf{w},\mathbf{z})$ and establish estimates for these averages.

Subject Classification: *11A25, 11N37.*

Keywords: *Friable integers, Multiplicative functions.*

1. Introduction

An integer $m \geq 1$ is called z-*friable* (or z-*smooth*) if $p \leq z$ for each prime factor p of z. We use $\mathbf{S}(\mathbf{w},\mathbf{z})$ for the set of z-friable integers up to w, defined as

$$\mathbf{S}(\mathbf{w},\mathbf{z}) = \{m \leq w: P(m) \leq z\},$$

where $P(m)$ is the largest prime factor of m, with the convention $P(1) = 1$.

The cardinality of $\mathbf{S}(\mathbf{w},\mathbf{z})$, denoted $\Psi(w,z)$, has been widely investigated in the literature (see, e.g., [1, 2, 3, 5, 9, 11, 15, 18]). Among the known estimates, Tenenbaum [16] established the following asymptotic formula:

$$\Psi(w,z) = w\rho(w)\left(1 + O\left(\frac{\log w}{\log z}\right)\right),$$

uniformly for $2 \leq z \leq w$, where $c = \frac{\log w}{\log z}$. Here, $\rho(c)$ is the Dickman–de Bruijn function, defined by

$$\rho(c) = 1 \quad \text{for } 0 \leq c \leq 1,$$

and

$$\rho(c) = 1 - \int_1^c \frac{\rho(v-1)}{v} dv, \qquad c > 1.$$

A function $f: \mathbb{N} \to \mathbb{C}$ that satisfies:

(i) $f(1) = 1$, and

(ii) $f(xy) = f(x)f(y)$ for all coprime integers x and y.

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is called a *multiplicative* function. In what follows, we present some classical examples of multiplicative functions that will be central to our discussion:

- The *divisor function*, which determines the number of positive divisors of m:
$$\tau(m) = \sum_{d|m} 1.$$
- The *sum-of-divisors function* (for $s \in \mathbb{R}$), of m:
$$\sigma_s(m) = \sum_{d|m} \left(\frac{m}{d}\right)^s.$$
Note that $\sigma_0 = \tau$.
- *Euler's totient function*,
$$\varphi(m) = \sum_{\substack{1\le k\le m \\ (k,m)=1}} 1.$$
- The *Möbius function*, defined by:
$$\mu(m) = \begin{cases} (-1)^r & \text{if } m = p_1 \cdots p_r \text{ (product of distinct primes)} \\ 0 & \text{if } m \text{ is not square} - \text{free}, \\ 1 & \text{if } m = 1. \end{cases}$$
- The *Dedekind psi function*:
$$\psi(m) = \sum_{d|m} \mu_2(m/d)d,$$

where μ_2 denotes the characteristic function of square-free integers.

The functions $\sigma_s(m)$ (for $s \in \mathbb{R}^*$), $\varphi(m)$, and $\psi(m)$ share several fundamental properties: they are multiplicative, can be expressed as Dirichlet convolutions $f(m) = \sum_{d|m} g(m/d)$ for suitable g, and admit similar proof techniques. Notably, they satisfy the following identities:
$$\varphi(m) = \sum_{d|m} \mu(d)\frac{m}{d}, \quad \psi(m) = \sum_{d|m} \mu_2(d)\frac{m}{d}, \quad \text{and} \quad \mu_2(m) = \sum_{d^2|m} \mu(d).$$

For further details on these functions and their properties, we refer to Chapters 2 and 3 of Apostol's classic text [4].

The study of asymptotic evaluations for sums $\sum_{m\le x} f(m)$ of an arithmetic function f, particularly when f is additive or multiplicative, is foundational in number theory. Significant contributions to this field were made by Delange, Wirsing, and Halász in the latter half of the twentieth century. The escalating importance of friable integers has naturally led researchers to investigate mean values of arithmetic functions restricted to these integers - what we call the "friable mean-value". Specifically, we examine the asymptotic behavior of the summatory function
$$\Psi_f(w, z) := \sum_{m\in\mathbf{S(w,z)}} f(m),$$
where $\mathbf{S}(\mathbf{w}, \mathbf{z})$ denotes the set of z-friable integers up to w. These summations are instrumental in elucidating the general case via the decomposition $m = xy$ with $P(x) \le z$ and $p|y \Rightarrow p > z$ (see, for instance, [10]).

Several special cases have been thoroughly investigated. For instance:

- The Möbius function $\mu(m)$
- The characteristic function of square-free integers $\mu_2(m)$
- The divisor function $\tau(m)$
- The Piltz function $\tau_\kappa(m)$

These investigations have laid the foundation for more general theorems. For further details, we refer to [1, 3, 6, 7, 8, 13, 17].

A particularly instructive example is the case $f = \tau = \sigma_0$, where

$$\Psi_\tau(w,z) = \sum_{m\in\mathbf{S(w,z)}} \tau(m).$$

Initial studies of this function [14] were limited to the range

$$\exp\{(\log_2 w)^{5/3+\varepsilon}\} \leq z \leq w, \quad \varepsilon > 0.$$

Subsequent work, notably by Tenenbaum and Wu [17], extended these results and established the asymptotic formula

$$\Psi_\tau(w,z) = \Psi(w,z)(\log w)\frac{\rho_2(c)}{c\rho(c)}\left\{1 + O\left(\frac{\log(c+1)}{\log z}\right)\right\},$$

valid for

$$w \geq 3, \quad \exp\{(\log_2 w)^{5/3+\varepsilon}\} \leq z \leq w.$$

Here, Γ is the Euler gamma function, ρ_2 presents the unique solution of:

$$\begin{cases} \rho_2(c) = c/\Gamma(2) & \text{if } 0 < c \leq 1, \\ c\rho_2'(c) - \rho_2(c) + 2\rho_2(c-1) = 0 & \text{if } c > 1, \end{cases}$$

which is continuous on $(0, +\infty)$ and differentiable on $[1, +\infty)$.

The present paper continues the investigation of arithmetic mean values over friable integers. Building on previous work concerning $\tau(m)$, we focus on the sum of s-th powers of divisors function $\sigma_s(m)$ for $s \in \mathbb{R}^*$. Our initial principal result establishes an asymptotic formula for

$$\Psi_{\sigma_s}(w,z) := \sum_{n\in\mathbf{S(w,z)}} \sigma_s(m), \quad s \in \mathbb{R}^*.$$

Given the structural similarities between $\sigma_s(m)$ (for $s \in \mathbb{R}^*$), Euler's totient function $\varphi(m)$, and the Dedekind psi function $\psi(m)$, we also establish asymptotic formulas for

$$\Psi_\varphi(w,z) := \sum_{n\in\mathbf{S(w,z)}} \varphi(m) \quad \text{and} \quad \Psi_\psi(w,z) := \sum_{m\in\mathbf{S(w,z)}} \psi(m).$$

This paper is organized in the following manner: The preliminary results on friable integers are presented in Section 2. A key tool in our analysis is the estimation of the summatory function $\sum_{m\in\mathbf{S(w,z)}} m^s$ for $s > 0$, which we establish in Section 3 (Lemma 3.1). Section 4 contains our main results and their proofs, providing asymptotic estimates for $\Psi_{\sigma_s}(w,z)$, $\Psi_\varphi(w,z)$, and $\Psi_\psi(w,z)$ in specified ranges of w and z.

Notation: We employ standard asymptotic notation:

- $f(w) = O(g(w))$ or $f \leq g$ entails the existence of constants $C > 0$ and w_0 with $|f(w)| \leq C|g(w)|$ for all $w \geq w_0$.
- All implied constants in O-terms and $=$-symbols are absolute unless otherwise specified.

2. Preliminary Results on Friable Integers

This section presents many results that are pivotal for our subsequent proofs. We begin with fundamental estimates concerning friable integers.

Hildebrand and Tenenbaum [12] developed a powerful method to estimate $\Psi(w, z)$ across a wide range of parameters w and z, utilizing a critical point they termed the "saddle point" $\alpha := \alpha(w, z) > 0$. While we do not employ their main estimate directly, an important consequence of their work will be crucial for our analysis. They established the following asymptotic formula:

$$\Psi(w, z) = \frac{w^{\alpha}\zeta(\alpha, z)}{\alpha\sqrt{2\pi(1+(\log w)/z)\log w \log z}}\left(1 + O\left(\frac{1}{\log(c+1)} + \frac{1}{\log z}\right)\right), \tag{2.1}$$

where

$$c = \frac{\log w}{\log z},$$

holds uniformly for $2 \leq z \leq w$. Here,

$$\zeta(s, z) := \sum_{P(m)\leq z} \frac{1}{m^s} = \prod_{p\leq z}\left(1 - \frac{1}{p^s}\right)^{-1} \quad (\Re(s) > 0, z \geq 2) \tag{2.2}$$

represents the z-friable part of the Riemann zeta function, and the saddle point $\alpha(w, z)$ is defined as the unique solution to the equation

$$\sum_{p\leq z} \frac{\log p}{p^{\alpha}-1} = \log w. \tag{2.3}$$

Subsequently, Hildebrand and Tenenbaum [12] derived a simplified approximation for $\alpha(w, z)$ valid throughout the range $2 \leq z \leq w$. In particular, when $\log w < z \leq w$, we have

$$\alpha(w, z) = 1 - \frac{\log(c\log(c+1))}{\log z} + O\left(\frac{1}{\log z}\right). \tag{2.4}$$

The following local result due to Bretèche and Tenenbaum [5] will play a pivotal role in our arguments, providing a comparison between $\Psi(w, z)$ and $\Psi(w/d, z)$ across a broad parameter range:

Lemma 2.1:1 *For $2 \leq z \leq w$ and $1 \leq d \leq w$, we have:*

(i) *The upper bound*

$$\Psi\left(\frac{w}{d}, z\right) \ll \frac{\Psi(w,z)}{d^{\alpha}}. \tag{2.5}$$

- The more precise estimate

$$\Psi\left(\frac{w}{d}, z\right) = \frac{\Psi(w,z)}{d^{\alpha}}\left\{1 + O\left(\frac{1}{c_z} + \frac{t}{c} + \frac{t^2\bar{c}}{c^2}\right)\right\}, \tag{2.6}$$

where

$$t := \frac{\log d}{\log z}, \quad c_z := c + \frac{\log z}{\log(c+2)}, \quad \text{and} \quad \bar{c} := \min\left\{c, \frac{z}{\log z}\right\},$$

and $\alpha = \alpha(w,z)$ is the saddle point corresponding to the z-friable integers not exceeding w.

3. An Auxiliary Estimation

As outlined in our paper's structure, a key component in proving our main theorems is establishing an estimate for the sum $\sum_{m\in\mathbf{S}(\mathbf{w},\mathbf{z})} m^s$ where $s > 0$. Indeed:

Lemma 3.1: *For any real number $s > 0$, we have the asymptotic estimate*

$$\sum_{m\in\mathbf{S}(\mathbf{w},\mathbf{z})} m^s = \frac{w^s\Psi(w,z)}{s+1}\left(1 + O\left(\frac{\log(c+1)}{\log z}\right)\right),$$

which holds uniformly for all $w \geq w_0$ and $log\ w < z \ll w$.

Proof: We begin by expressing our sum in terms of the characteristic function of y-friable integers:

$$\sum_{m\in\mathbf{S}(\mathbf{w},\mathbf{z})} m^s = \sum_{m\leq w} a_m m^s \quad \text{where} \quad a_m = \begin{cases} 1 & \text{if } P(m) \leq z, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Abel's summation formula yields:

$$\begin{aligned}\sum_{m\in\mathbf{S}(\mathbf{w},\mathbf{z})} m^s &= w^s\Psi(w,z) - s\int_1^w \Psi(t,z)t^{s-1}dt \\ &= w^s\Psi(w,z) - sw^s\int_1^w \Psi\left(\frac{w}{u},z\right)\frac{du}{u^{1+s}},\end{aligned}$$

where we made the substitution $t = w/u$.

To estimate the integral, we introduce the parameter

$$A := \exp\left(a\sqrt{\log(c+1)\log w}\right) \quad (a > 0), \tag{3.1}$$

and decompose the integral as:

$$I := \int_1^w \Psi\left(\frac{w}{u},z\right)\frac{du}{u^{1+s}} = \Delta_1 + \Delta_2,$$

where Δ_1 and Δ_2 correspond to the ranges $[1,A]$ and $[A,w]$ respectively.

Estimation of $\mathbf{\Delta_1}$: For $u \leq A$ and $z > \log w$, Lemma 2.1(ii) gives

$$\Psi\left(\frac{w}{u},z\right) = \frac{\Psi(w,z)}{u^\alpha}\left(1 + O\left(\frac{\log(c+1)}{\log z} + \frac{\log u}{\log w}\right)\right).$$

Thus,

$$\Delta_1 = \Psi(w,z)\left(1 + O\left(\frac{\log(c+1)}{\log z}\right)\right)\int_1^A \frac{du}{u^{1+s+\alpha}} + O\left(\frac{\Psi(w,z)}{\log w}\int_1^A \frac{\log u}{u^{1+s+\alpha}}du\right).$$

Evaluating these integrals:

$$\int_1^A \frac{du}{u^{1+s+\alpha}} = \frac{1-A^{-s-\alpha}}{s+\alpha},$$

$$\int_1^A \frac{\log u}{u^{1+s+\alpha}}du = O\left(\frac{\log A}{(s+\alpha)A^{s+\alpha}}\right).$$

Estimation of Δ_2: Using Lemma 2.1(i), we obtain:

$$\Delta_2 \ll \Psi(w,z)\int_A^w \frac{du}{u^{1+s+\alpha}} \ll \frac{\Psi(w,z)}{(s+\alpha)A^{s+\alpha}}.$$

Combining these estimates yields:

$$I = \frac{\Psi(w,z)}{s+\alpha}\left(1+O\left(\frac{\log(c+1)}{\log z}\right)\right).$$

From the approximation of α in (2.4), we have:

$$\frac{1}{s+\alpha} = \frac{1}{s+1} + O\left(\frac{\log(c+1)}{\log z}\right).$$

Substituting back into our original expression completes the proof.

4. Main Results

Having established the necessary groundwork in previous sections, we now present the principal results of this paper.

Theorem 4.1: *For any real number $s \geq 1$, we have the asymptotic formula*

$$\Psi_{\sigma_s}(w,z) = \frac{w^s}{s+1}\Psi(w,z)\zeta(s+\alpha,z)\left(1+O\left(\frac{\log(c+1)}{\log z}\right)\right),$$

which holds uniformly for all $w \geq w_0$ and $log w < z \leq w$. Here, $\alpha = \alpha(w,z)$ is the saddle point associated with z-friable integers not exceeding w, and $\zeta(\cdot,z)$ is the z-friable zeta function defined in (2.2).

Remark 4.2: The asymptotic of $\Psi_{\sigma_s}(w,z)$ still valid in the narrower range $log^{1/s}w < z \leq w$ where $0 < s < 1$. This restriction arises because the asymptotic for $\sum\limits_{m \in S(w,z)} 1/m^{s+\alpha}$ requires $s+\alpha > 1$. Indeed, for $w \geq w_0$ and $z > log^{1/s}w$, we deduce from (2.4) that

$$1-\alpha \leq \frac{\log\log w}{\log z} + O\left(\frac{1}{\log z}\right) < s + O\left(\frac{1}{\log z}\right),$$

ensuring $s+\alpha > 1$ for sufficiently large w. The proof then follows similarly to Theorem 4.1.

Theorem 4.3: *For any real number $s \leq -1$, we have the asymptotic formula*

$$\Psi_{\sigma_s}(w,z) = \Psi(w,z)\zeta(-s+\alpha,z)\left(1+O\left(\frac{\log(c+1)}{\log z}\right)\right),$$

uniformly valid for $w \geq w_0$ and $\log w < z \leq w$, with α and $\zeta(\cdot,z)$ as defined previously.

Remark 4.4: The same asymptotic holds for $-1 < s < 0$ under the condition $z > log^{1/|s|}w$, since (2.4) guarantees $-s+\alpha > 1$ in this range.

We note that the ranges specified for $0 < |s| < 1$ may not be optimal. While it might be possible to extend these to smaller values of z (e.g., powers of log$w$), this would require establishing asymptotics for $\sum_{m\in\mathbf{S(w,z)}} 1/m^{|s|+\alpha}$ in the critical range $0<|s|+\alpha\le 1$, which presents substantial additional challenges. We therefore restrict our attention to the ranges stated above.

Using methods analogous to those employed in proving Theorem 4.1, we establish the following results for Euler's totient and Dedekind's psi functions:

Theorem 4.5: *We have the asymptotic formula*

$$\Psi_{\varphi}(w,z)=\frac{w}{2\zeta(1+\alpha)}\Psi(w,z)\left(1+O\left(\frac{\log(c+1)}{\log z}\right)\right),$$

valid uniformly for $w\ge w_0$ *and* $\log w<z\le w$, *where* α *is as before.*

Theorem 4.6: *We have the asymptotic formula*

$$\Psi_{\psi}(w,z)=\frac{\zeta(1+\alpha,z)}{2\zeta(2+2\alpha)}w\Psi(w,z)\left(1+O\left(\frac{\log(c+1)}{\log z}\right)\right),$$

holding uniformly for $w\ge w_0$ *and* $\log w<z\le w$, *with* α *and* $\zeta(\cdot,z)$ *as previously defined.*

5. Proof of Theorem 4.1

For sufficiently large w and z verifying $\log w<z\le w$, We begin by expressing $\Psi_{\sigma_s}(w,z)$ as a double sum:

$$\begin{aligned}\Psi_{\sigma_s}(w,z) &= \sum_{m\in S(w,z)}\sum_{d\mid m}\left(\frac{m}{d}\right)^s\\ &= \sum_{d\in\mathbf{S(w,z)}}\sum_{\substack{ld\le w\\ P(ld)\le z}} l^s\\ &= \sum_{d\in\mathbf{S(w,z)}}{}'\sum_{\substack{l\le w/d\\ P(l)\le y}} l^s\end{aligned}$$

where the last equality follows since $P(d)\le z$ implies $P(ld)\le z$ if and only if $P(l)\le z$.

We decompose this sum into two parts:

$$\Psi_{\sigma_s}(w,z)=\Delta_1+\Delta_2, \tag{5.1}$$

where

$$\begin{aligned}\Delta_1 &= \sum_{\substack{d\le A\\ P(d)\le z}}\sum_{\substack{l\le w/d\\ P(l)\le z}} l^s,\\ \Delta_2 &= \sum_{\substack{d>A\\ P(d)\le z}}\sum_{\substack{l\le w/d\\ P(l)\le z}} l^s,\end{aligned}$$

with A as defined in (3.1).

5.1 *Estimation of* $\boldsymbol{\Delta_1}$

Applying Lemma 3.1 to the inner sum yields:

$$\sum_{l\in S(w/d,z)} l^s=\frac{1}{s+1}\left(\frac{w}{d}\right)^s\Psi\left(\frac{w}{d},z\right)\left(1+O\left(\frac{\log(c(d)+1)}{\log z}\right)\right),$$

where $c(d) = \frac{\log(w/d)}{\log z}$ for $d \leq A$.

Using Lemma 2.1(ii), we estimate $\Psi(w/d, z)$:

$$\Psi\left(\frac{w}{d}, z\right) = \frac{\Psi(w,z)}{d^{\alpha}}\left(1 + O\left(\frac{\log(c+1)}{\log z} + \frac{\log d}{\log w}\right)\right).$$

Combining these estimates gives:

$$\Delta_1 = \frac{w^s \Psi(w,z)}{s+1} \sum_{d \in S(A,z)} \frac{1}{d^{s+\alpha}}\left(1 + O\left(\frac{\log(c+1)}{\log z} + \frac{\log d}{\log w}\right)\right). \tag{5.2}$$

Expanding the error terms and noting that $c(d)c$, we obtain:

$$\Delta_1 = \frac{w^s \Psi(w,z)}{s+1}\left\{\Gamma_1 + O\left(\frac{\log(c+1)}{\log z}\Gamma_1\right) + O\left(\frac{1}{\log w}\Gamma_2\right)\right\},$$

where

$$\Gamma_1 = \sum_{d \in S(A,z)} \frac{1}{d^{s+\alpha}} \quad \text{and} \quad \Gamma_2 = \sum_{d \in S(A,z)} \frac{\log d}{d^{s+\alpha}}.$$

5.2 *Evaluation of $\boldsymbol{\Gamma_1}$ and $\boldsymbol{\Gamma_2}$*

For Γ_1, we write:

$$\Gamma_1 = \zeta(s + \alpha, z) - R(A), \tag{5.3}$$

where the last term satisfies:

$$|R(A)| \leq \sum_{m > A} \frac{1}{m^{s+\alpha}} \leq \int_A^{\infty} \frac{dt}{t^{s+\alpha}} = \frac{1}{(s+\alpha-1)A^{s+\alpha-1}}.$$

Since $s \geq 1$ and $\alpha > 0$, we have $s + \alpha > 1$, yielding:

$$\Gamma_1 = \zeta(s + \alpha, z) + O\left(\frac{1}{A^{s+\alpha-1}}\right). \tag{5.4}$$

For Γ_2, we estimate:

$$\Gamma_2 \leq \sum_{d \leq A} \frac{\log d}{d^{s+\alpha}} \leq \int_1^A \frac{\log t}{t^{s+\alpha}} dt \ll 1,$$

where the last bound follows from $s + \alpha > 1$.

Combining these results gives the asymptotic for Δ_1:

$$\Delta_1 = \frac{w^s \Psi(w,z)}{s+1} \zeta(s + \alpha, z)\left(1 + O\left(\frac{\log(c+1)}{\log z}\right)\right). \tag{5.5}$$

5.3 *Estimation of $\boldsymbol{\Delta_2}$*

Using Lemma 2.1(i), we bound Δ_2 as:

$$\Delta_2 \ll w^s \Psi(w, z) \sum_{d > A} \frac{1}{d^{s+\alpha}} \ll \frac{w^s \Psi(w,z)}{A^{s+\alpha-1}}. \tag{5.6}$$

5.4 *Conclusion*

The theorem follows by combining (5.1), (5.5), and (5.6), noting that the error term from Δ_2 is absorbed into the existing error term since A grows sufficiently rapidly with w.

6. Proof of Theorem 4.3

Let w a large enough real and $\log w < z \le w$. Let $\beta = -s \ge 1$. We begin by expressing $\Psi_{\sigma_s}(w,z)$ as:

$$\begin{aligned}\Psi_{\sigma_s}(w,z) &= \sum_{m\in \mathbf{S}(\mathbf{w},\mathbf{z})} \sum_{d|m} \left(\frac{d}{m}\right)^{\beta} \\ &= \sum_{d\in \mathbf{S}(\mathbf{w},\mathbf{z})} \sum_{\substack{l\le w/d \\ P(l)\le z}} \frac{1}{l^{\beta}} \\ &= \sum_{l\in \mathbf{S}(\mathbf{w},\mathbf{z})} \frac{1}{l^{\beta}} \sum_{\substack{d\le w/l \\ P(d)\le z}} 1 \\ &= \sum_{l\in \mathbf{S}(\mathbf{w},\mathbf{z})} \frac{\Psi(w/l,z)}{l^{\beta}}.\end{aligned}$$

We decompose this sum into two parts:

$$\Psi_{\sigma_s}(w,z) = \Delta_1 + \Delta_2, \tag{6.1}$$

where

$$\Delta_1 = \sum_{\substack{l\le A \\ P(l)\le z}} \frac{\Psi(w/l,z)}{l^{\beta}},$$
$$\Delta_2 = \sum_{\substack{l> A \\ P(l)\le z}} \frac{\Psi(w/l,z)}{l^{\beta}}.$$

6.1 *Estimation of* $\boldsymbol{\Delta_1}$

Applying Lemma 2.1(ii), we have:

$$\Delta_1 = \Psi(w,z) \sum_{l\in S(A,z)} \frac{1}{l^{\beta+\alpha}} \left(1 + O\left(\frac{\log(c+1)}{\log z} + \frac{\log l}{\log w}\right)\right). \tag{6.2}$$

We can rewrite Δ_1 as:

$$\Delta_1 = \Psi(w,z)\left\{\Gamma_1 + O\left(\frac{\log(c+1)}{\log z}\Gamma_1\right) + O\left(\frac{1}{\log w}\Gamma_2\right)\right\},$$

where

$$\Gamma_1 = \sum_{l\in S(A,z)} \frac{1}{l^{\beta+\alpha}} \quad \text{and} \quad \Gamma_2 = \sum_{l\in S(A,z)} \frac{\log l}{l^{\beta+\alpha}}.$$

Since $\beta + \alpha > 1$, we have:

$$\begin{aligned}\Gamma_1 &= \zeta(\beta+\alpha, z) + O\left(\frac{1}{A^{\beta+\alpha-1}}\right), \\ \Gamma_2 &\ll 1.\end{aligned}$$

This yields the asymptotic:

$$\Delta_1 = \Psi(w,z)\zeta(\beta+\alpha,z)\left(1 + O\left(\frac{\log(c+1)}{\log z}\right)\right). \tag{6.3}$$

6.2 *Estimation of* $\boldsymbol{\Delta_2}$

Using Lemma 2.1(i), we bound Δ_2 as:

$$\Delta_2 = \frac{\Psi(w,z)}{A^{\beta+\alpha-1}}. \tag{6.4}$$

6.3 *Conclusion*

The theorem follows by combining (6.1), (6.3), and (6.4), noting that the secondary term from Δ_2 is absorbed into the existing error term.

7. Proof of Theorem 4.5

We begin with the identity for Euler's totient function:

$$\varphi(m) = \sum_{h|m} \mu(h)\frac{m}{h}. \tag{7.1}$$

Expressing $\Psi_\varphi(w,z)$ using this relation yields:

$$\begin{aligned}\Psi_\varphi(x,y) &= \sum_{m\in\mathbf{S(w,z)}} \sum_{h|m} \mu(h)\frac{m}{h}\\ &= \sum_{h\in\mathbf{S(w,z)}} \mu(h) \sum_{\substack{l\le w/h\\ P(l)\le z}} l\\ &= \sum_{h\in\mathbf{S(w,z)}} \mu(h) \sum_{l\in S(w/h,z)} l.\end{aligned}$$

Following the approach used in Theorem 4.1 with $s = 1$, we obtain:

$$\Psi_\varphi(w,z) = \frac{w\Psi(w,z)}{2}\left\{\Gamma_1 + O\left(\frac{\log(c+1)}{\log z}\Gamma_2\right) + O\left(\frac{1}{\log w}\Gamma_3\right)\right\} + O(\Delta),$$

where

$$\begin{aligned}\Gamma_1 &= \sum_{h\in S(A,z)} \frac{\mu(h)}{h^{1+\alpha}},\\ \Gamma_2 &= \sum_{h\in S(A,z)} \frac{|\mu(h)|}{h^{1+\alpha}},\\ \Gamma_3 &= \sum_{h\in S(A,z)} |\mu(h)|\frac{\log h}{h^{1+\alpha}},\\ \Delta &= \sum_{\substack{h>A\\ P(h)\le z}} \mu(h)\sum_{\substack{l\le w/h\\ P(l)\le z}} l^s.\end{aligned}$$

7.1 *Evaluation of* $\mathbf{\Gamma_1}$

We decompose Γ_1 as:

$$\begin{aligned}\Gamma_1 &= \sum_{P(h)\le z} \frac{\mu(h)}{h^{1+\alpha}} - \sum_{\substack{h>A\\ P(h)\le z}} \frac{\mu(h)}{h^{1+\alpha}}\\ &= \sum_{P(h)\le z} \frac{\mu(h)}{h^{1+\alpha}} + O\left(\sum_{h>A} \frac{1}{h^{1+\alpha}}\right)\\ &= \sum_{P(h)\le z} \frac{\mu(h)}{h^{1+\alpha}} + O\left(\frac{1}{A^\alpha}\right).\end{aligned}$$

The main term evaluates to:

$$\begin{aligned}\sum_{P(h)\le z} \frac{\mu(h)}{h^{1+\alpha}} &= \frac{1}{\zeta(1+\alpha)} + O\left(\sum_{\substack{h>z\\ P(h)>z}} \frac{1}{h^{1+\alpha}}\right)\\ &= \frac{1}{\zeta(1+\alpha)} + O\left(\frac{1}{z^\alpha}\right).\end{aligned}$$

7.2 *Bounds for Remaining Terms*

We have the estimates:

$$\begin{aligned}\Gamma_2, \Gamma_3 &\ll 1,\\ \Delta &\ll \frac{w\Psi(w,z)}{A^\alpha}.\end{aligned}$$

Combining these results yields the theorem.

8. Proof of Theorem 4.6

Recall the identity for Dedekind's psi function:

$$\psi(m) = \sum_{h|m} \mu_2(h)\frac{m}{h}. \tag{8.1}$$

We express $\Psi_\psi(w,z)$ as:

$$\begin{aligned}\Psi_\psi(w,z) &= \sum_{m\in\mathbf{S(w,z)}} \sum_{h|m} \mu_2(h)\frac{m}{h} \\ &= \sum_{h\in\mathbf{S(w,z)}} \mu_2(h) \sum_{l\in S(w/h,z)} l.\end{aligned}$$

Following the same approach as before, we obtain:

$$\Psi_\psi(w,z) = \frac{w\Psi(w,z)}{2}\left\{\Gamma_1 + O\left(\frac{\log(c+1)}{\log z}\Gamma_2\right) + O\left(\frac{1}{\log w}\Gamma_3\right)\right\} + O(\Delta),$$

where

$$\begin{aligned}\Gamma_1 &= \sum_{h\in S(A,z)} \frac{\mu_2(h)}{h^{1+\alpha}}, \\ \Gamma_2 &= \sum_{h\in S(A,z)} \frac{|\mu_2(h)|}{h^{1+\alpha}}, \\ \Gamma_3 &= \sum_{h\in S(A,z)} |\mu_2(h)|\frac{\log h}{h^{1+\alpha}}, \\ \Delta &= \sum_{\substack{h>A \\ P(h)\le z}} \mu_2(m) \sum_{\substack{l\le w/h \\ P(l)\le z}} l^s.\end{aligned}$$

8.1 *Evaluation of* $\mathbf{\Gamma_1}$

Using the Möbius function representation of μ_2, we have:

$$\begin{aligned}\Gamma_1 &= \sum_{h\in S(A,z)} \frac{1}{h^{1+\alpha}} \sum_{t^2|d} \mu(t) \\ &= \sum_{t\in\mathbf{S(A^{1/2},z)}} \frac{\mu(t)}{t^{2+2\alpha}} \sum_{h\in\mathbf{S(A/t^2,z)}} \frac{1}{h^{1+\alpha}} \\ &= \sum_{t\in\mathbf{S(A^{1/2},z)}} \frac{\mu(t)}{t^{2+2\alpha}}\left(\zeta(1+\alpha,z) + O\left(\frac{t^{2\alpha}}{A^\alpha}\right)\right) \\ &= \frac{\zeta(1+\alpha,z)}{\zeta(2+2\alpha)} + O\left(\frac{1}{A^{\alpha+1/2}}\right).\end{aligned}$$

8.2 *Bounds for Remaining Terms*

We estimate the remaining terms:

$$\begin{aligned}&\Gamma_2 \ll 1, \\ &\Gamma_3 \ll \sum_{t\le A^{\frac{1}{2}}} \frac{\log t^2}{t^{2+2\alpha}} + \sum_{h\le A} \frac{\log h}{h^{1+\alpha}} \ll 1, \\ &\Delta \ll \frac{w\Psi(w,z)}{A^\alpha} \sum_{t\ge 1} \frac{1}{t^2} \ll \frac{w\Psi(w,z)}{A^\alpha}.\end{aligned}$$

Combining these results completes the proof.

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Received October, 2024