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On reflexive irregularity strength of diamond network

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Abstract

Graph labeling is one of the many ideas that arise while studying graph theory and has attracted considerable attention, it yields mathematical models that are applicable to numerous high-tech applications (data security, telecommunication networks, astronomy, various problems of coding theory, cryptography, etc.). The process of assigning names to the vertices and edges of a set of numbers (natural numbers) is known as labeling "tagging" or a graph label. Each domain has its own unique collection of vertices "nodes" and edges "links", which are represented in the labeling. Consequently, for entire labeling, we consider the domain as a collection of nodes and links concurrently. The reflexive edge irregularity strength (res) is a form of entire labeling in which edge weights are different among all links and the weight of a link is calculated as the aggregate of its link tag and the tags of nodes that attached to this link. The links are tagged with positive integers, while in reflexive node irregularity strength (rvs) is entire labeling in which node weights are different among all nodes and the weight of a node is derived as the aggregate of its node tag and the tags of all links incidents at this node. In both res and rvs the links are marked with positive numbers

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while the nodes are marked with positive even numbers. Whether the tags are related with nodes or links, we must keep them to a minimum. If these taggings are present, they are referred to as the res or rvs of *H* and are expressed as res(*H*) or rvs(*H*) respectively. In this article, we computed the res and rvs for diamond network graphs.

Subject Classification: *05C78, 05C90.*

Keywords: *Diamond network graph, Vertex irregular reflexive labeling, Edge irregular reflexive labeling, Reflexive edge strength, Reflexive vertex strength, Cryptography.*

1. Introduction

Any graph H comprises vertex set $V(H)$ and a family of all subsets that consist of only two elements of $V(H)$ called edge set $E(H)$. All of the graphs used in this article are finite, without direction, nontrivial, linked, and simple. For more information about notations see [1]. A function called a "graph labeling" turns graph elements into integers, frequently positive or non-negative numbers. Labelings are classified as vertex labelings or edge labelings based on whether the domain pertains to $V(H)$ or $E(H)$. When the domain encompasses the both $V(H)$ and $E(H)$, the labeling is termed total "entire" labeling. The most recent and comprehensive study of graph labelings is [2]. For a specific edge labeling $\psi: E(H) \to \{1,2,\ldots,m\}$, where m is a nonnegative number, the comparable weight of a node $x \in V(H)$ is defined as $w(x) = \sum_{xv\in E(H)} \psi(xv)$, with the adding up taken over all nodes v adjacent to x. An edge labeling is said to be irregular if $w_\psi(x)$ is different for every node x in $V(H)$. The irregularity strength of H is expressed by $s(H)$, which is the least number m for which an irregular labeling of H occurs. In [3], Chartrand et al. made the initial argument for the irregularity strength. For more details see [4, 5, 6]. In [7], Bac′a et al. proposed two additional types: entire edge irregularity strength and entire vertex irregularity strength of a graph. An entire labeling $\beta: V(H) \cup E(H) \to \{1,2,\ldots,m\}$ is known as an edge irregular entire m-labeling if all edges have different weights that is, $wt_\beta(xv) \neq wt_\beta(x'v')$ for all different edges $xv, x'v' \in E(V)$, where $wt_\beta(xv) = \beta(x) + \beta(xv) + \beta(v)$. Also the entire labeling $\beta: V(H) \cup E(H) \to \{1,2,\ldots,m\}$ is known as a vertex irregular entire m-labeling if all nodes have different weights that is, $wt_\beta(x) \neq wt_\beta(v)$ for all different vertices x, v, where $wt_\beta(x) = \beta(x) + \sum_{xv\in E(H)} \beta(xv)$, where the aggregate is across all vertices v next to x. For further information, see [8, 9, 10, 11, 12, 13, 14, 15, 16].. Tanna et al. In [17, 18] generalised the notion of entire irregularity strength for graph will be reflexive irregularity strength. Let H be any connected graph the entire m-labeling identified the function $\beta_e: E(H) \to \{1,2,3,\ldots,m_e\}$ and $\beta_v: V(H) \to \{0,2,4,\ldots,2m_v\}$ the function β is entire m-labeling of H such that $\beta(a) = \beta_v(a)$ for all $a \in V(H)$ and $\beta(a) = \beta_e(a)$ for all $a \in E(H)$ where $m = \max\{m_e, 2m_v\}$. The entire m-labeling is edge irregular reflexive labeling of graph H (EIRL) satisfying $wt_\beta(ab) \neq wt_\beta(a'b')$

for all $ab, a'b' \in E(H)$, where $wt_\beta(ab) = \beta_v(a) + \beta_v(b) + \beta_e(ab)$. The reflexive edge strength of graph H, represented as res(H) is least m so that graph H has an EIRL. The lower bound for reflexive graph irregularity strength is determined in [17] as follows:

$$res(H) \geq \begin{cases} \left\lceil \frac{|E(H)|}{3} \right\rceil + 1, & |E(H)| \equiv 2,3 \pmod 6, \\ \left\lceil \frac{|E(H)|}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Additional details can be found in [19, 20, 21, 22, 23, 24]. The entire m-labeling is vertex irregular reflexive labeling of graph H (VIRL) satisfying $wt_\beta(x) \neq wt_\beta(x')$ for all $x, x' \in V(H)$, where $wt_\beta(x)$ determined by aggregating of the tag of the node x addition to the tags of all links incident with the node x, the aggregate is across all v neighbouring to x. The reflexive vertex strength of graph H, represented as rvs(H) is least m so that graph H has VIRL. Suppose that a graph has a minimum degree $\delta(H)$ and a maximum degree $\Delta(H)$, the lower bound of the rvs(H) can be obtained as follows [18]:

$$rvs(H) \geq \left\lceil \frac{\ell+\delta-1}{\Delta+1} \right\rceil.$$

Where ℓ is the order of the graph H. In a sense, a better lower and upper bound of rvs can be derived from the next lemma.

Lemma 1.1: *Consider H is a graph of order ℓ and minimal degree δ then, for each vertex irregular reflexive labeling, the maximum vertex weight is at least this:*

(i) $\ell + \delta - 1$, *if $\ell \equiv 0, 1$ (mod 4) and δ is even, or $\ell \equiv 3$ (mod 4) and δ is odd,*

(ii) $\ell + \delta$, $\quad$ *otherwise.*

For clarity, throughout this paper, we denote the interval of numbers i as $i = [r, s]$, where $r \leq i \leq s$. Suppose H is r-regular graphs of order ℓ, then we get.

Corollary 1.1:

$$rvs(H) \geq \begin{cases} \left\lceil \frac{\ell+r-1}{r+1} \right\rceil & \text{if } \ell \equiv 0,1 \pmod 4, \\ \left\lceil \frac{\ell+r}{r+1} \right\rceil + 1, & \text{if } \ell \equiv 2,3 \pmod 4. \end{cases}$$

For additional details in reflexive vertex irregularity strength, see [25, 26]. Consider $L_\ell \cong P_\ell \otimes P_2$ is a ladder with vertex-set $V(L_\ell) = \{x_i, y_i : i \in [1, \ell]\}$ and edge-set $V(L_\ell) = \{x_i x_{i+1}, y_i y_{i+1} : i \in [1, \ell - 1]\} \cup \{y_i x_i : i \in [1, \ell]\}$. A triangular ladder, denoted by the symbol TL_ℓ, is obtained by attaching the links $x_i y_{i+1}, i \in [1, \ell - 1]$ to the ladder while excluding the vertex x_ℓ and the two links $x_{\ell-1} x_\ell$ and $x_\ell y_\ell$. For $\ell \geq 3$ a diamond graph D_ℓ, is gained by merging a one node v to all nodes $y_i, i \in [1, \ell]$, of a triangular ladder TL_ℓ. Hence, the vertex and edge sets of the diamond graph D_ℓ are as follows:

$$V(D_\ell) = \{v\} \cup \{x_i : i \in [1, \ell - 1]\} \cup \{y_i : i \in [1, \ell]\}$$
$$E(D_\ell) = \{x_i x_{i+1}, y_i y_{i+1} : i \in [1, \ell - 2]\} \cup \{y_i y_{i+1} : i \in [1, \ell - 1]\}$$
$$\cup \{x_i y_i : i \in [1, \ell - 1]\} \cup \{x_i y_{i+1} : i \in [1, \ell - 1]\}$$
$$\cup \{v y_i : i \in [1, \ell]\}.$$

Then, $|V(D_\ell)| = 2\ell$ and $|E(D_\ell)| = 5\ell - 5$ see [27]. Figure 1 depicts the diamond graph D_6. In this article, we analyze how diamond graphs are labeled reflexively along their edges (vertices), and then we compute the precise value of res and rvs for this category of graphs.

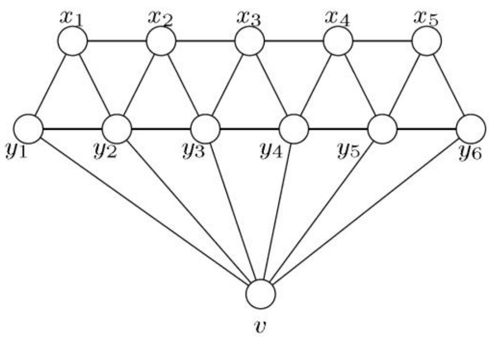


Figure 1
The diamond graph D_6.

2. Reflexive edge strength of D_ℓ.

The precise value of res(D_ℓ) is estimated by the following theorem.

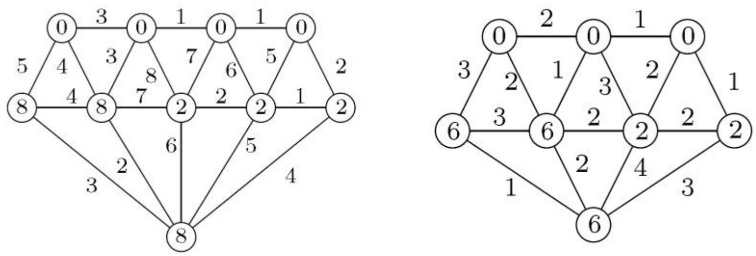
Theorem 1: Consider $D_\ell, \ell \geq 3$ is diamond graph. Then

$$res(D_\ell) = \begin{cases} \left\lceil \frac{5\ell - 5}{3} \right\rceil, & \ell \equiv 0,1,2,3 \pmod 6, \\ \left\lceil \frac{5\ell - 5}{3} \right\rceil + 1, & \ell \equiv 4,5 \pmod 6. \end{cases}$$

Proof: As D_ℓ has $5\ell - 5$ edges, hence, from Lemma 1.1. We get

$$res(D_\ell) \geq \begin{cases} \left\lceil \frac{5\ell - 5}{3} \right\rceil, & \ell \equiv 0,1,2,3 \pmod 6, \\ \left\lceil \frac{5\ell - 5}{3} \right\rceil + 1, & \ell \equiv 4,5 \pmod 6, \end{cases}$$

For $\ell = 3$, the diamond graph D_3 isomorphic to the wheel graph W_5, Tanna showed that res$(W_5) = 4$, and hence res$(D_3) = 4$. The optimal reflexive edge irregular res(D_ℓ)-labeling for $\ell = 4,5$ are given in Figures 2(a), 2(b). For $\ell \geq 6$, put $m = \left\lceil \frac{5\ell - 5}{3} \right\rceil$ for $\ell \equiv 0,1,2,3 \pmod 6$ and $m = \left\lceil \frac{5\ell - 5}{3} \right\rceil + 1$ for $\ell \equiv 4,5$ (mod 6). Next, we suggest m-labeling to demonstrate that m is the upper bound for the strength of reflexive edge irregularities in the diamond graph



(a) A reflexive edge 8-labeling of D_5. (b) A reflexive edge 6-labeling of D_4.

Figure 2

$$\beta(x_i) = \begin{cases} 0, & i \in \left[1, \left\lfloor \frac{\ell}{2} \right\rfloor\right], \\ 2\left\lfloor \frac{\ell}{8} \right\rfloor, & i \in \left[\left\lfloor \frac{\ell}{2} \right\rfloor + 1, \ell - 1\right]. \end{cases}$$

$$\beta(y_i) = \begin{cases} 2\left\lfloor \frac{m}{4} \right\rfloor, & i \in \left[1, \left\lfloor \frac{\ell}{2} \right\rfloor\right], \\ 2\left\lfloor \frac{m}{2} \right\rfloor, & i \in \left[\left\lfloor \frac{\ell}{2} \right\rfloor + 1, \ell\right]. \end{cases}$$

$$\beta(v) = 2\left\lfloor \frac{m}{2} \right\rfloor - 2.$$

$$\beta(x_i x_{i+1}) = \begin{cases} i, & i \in \left[1, \left\lfloor \frac{\ell}{2} \right\rfloor - 1\right], \\ \left\lfloor \frac{\ell}{2} \right\rfloor - 2\left\lfloor \frac{\ell}{8} \right\rfloor, & i = \left\lfloor \frac{\ell}{2} \right\rfloor, \\ i - 4\left\lfloor \frac{\ell}{8} \right\rfloor, & i \in \left[\left\lfloor \frac{\ell}{2} \right\rfloor + 1, \ell - 2\right]. \end{cases}$$

$$\beta(y_i x_i) = \begin{cases} \ell - 2\left\lfloor \frac{m}{4} \right\rfloor - 3 + 2i, & i \in \left[1, \left\lfloor \frac{\ell}{2} \right\rfloor\right], \\ \ell + \left\lfloor \frac{\ell}{2} \right\rfloor - 2\left\lfloor \frac{\ell}{8} \right\rfloor - 2\left\lfloor \frac{m}{2} \right\rfloor - 4 + 2i, & i \in \left[\left\lfloor \frac{\ell}{2} \right\rfloor + 1, \ell - 1\right]. \end{cases}$$

$$\beta(y_{i+1} x_i) = \begin{cases} \ell - 2\left\lfloor \frac{m}{4} \right\rfloor - 2 + 2i, & i \in \left[1, \left\lfloor \frac{\ell}{2} \right\rfloor - 1\right], \\ \ell + 3\left\lfloor \frac{\ell}{2} \right\rfloor - 2\left\lfloor \frac{m}{2} \right\rfloor - 3, & i = \left\lfloor \frac{\ell}{2} \right\rfloor, \\ \ell + \left\lfloor \frac{\ell}{2} \right\rfloor - 2\left\lfloor \frac{\ell}{8} \right\rfloor - 2\left\lfloor \frac{m}{2} \right\rfloor - 3 + 2i, & i \in \left[\left\lfloor \frac{\ell}{2} \right\rfloor + 1, \ell - 1\right]. \end{cases}$$

$$\beta(y_i v) = \begin{cases} 3\ell + \left\lfloor \frac{\ell}{2} \right\rfloor - 2\left\lfloor \frac{m}{4} \right\rfloor - 2\left\lfloor \frac{m}{2} \right\rfloor - 3 + i, & i \in \left[1, \left\lfloor \frac{\ell}{2} \right\rfloor\right], \\ 3\ell + \left\lfloor \frac{\ell}{2} \right\rfloor - 4\left\lfloor \frac{m}{2} \right\rfloor - 2 + i, & i \in \left[\left\lfloor \frac{\ell}{2} \right\rfloor + 1, \ell\right]. \end{cases}$$

$$\beta(y_iy_{i+1}) = \begin{cases} \ell + 2\left\lfloor\frac{\ell}{2}\right\rfloor - 4\left\lfloor\frac{m}{4}\right\rfloor - 3 + i, & i \in \left[1, \left\lfloor\frac{\ell}{2}\right\rfloor - 1\right], \\ 3\ell + 2\left\lfloor\frac{\ell}{2}\right\rfloor - 2\left\lfloor\frac{m}{2}\right\rfloor - 2\left\lfloor\frac{m}{4}\right\rfloor - 4, & i = \left\lfloor\frac{\ell}{2}\right\rfloor, \\ 4\ell - 4\left\lfloor\frac{m}{2}\right\rfloor - 4 + i, & i \in \left[\left\lfloor\frac{\ell}{2}\right\rfloor + 1, \ell - 1\right]. \end{cases}$$

Intuitively, it is obvious that all vertices are labeled by even integers, and the marks of the vertices and edges cannot exceed m. Under reflexive labeling, the weights of the D_ℓ edges are presented in the following way:

$$wt_\beta(x_ix_{i+1}) = i, \quad i \in [1, \ell - 2].$$

$$wt_\beta(y_ix_i) = \begin{cases} \ell - 3 + 2i, & i \in \left[1, \left\lfloor\frac{\ell}{2}\right\rfloor\right], \\ \ell + \left\lfloor\frac{\ell}{2}\right\rfloor - 4 + 2i, & i \in \left[\left\lfloor\frac{\ell}{2}\right\rfloor + 1, \ell - 1\right]. \end{cases}$$

$$wt_\beta(y_{i+1}x_i) = \begin{cases} \ell - 2 + 2i, & i \in \left[1, \left\lfloor\frac{\ell}{2}\right\rfloor - 1\right], \\ \ell + 3\left\lfloor\frac{\ell}{2}\right\rfloor - 3, & i = \left\lfloor\frac{\ell}{2}\right\rfloor, \\ \ell + \left\lfloor\frac{\ell}{2}\right\rfloor - 3 + 2i, & i \in \left[\left\lfloor\frac{\ell}{2}\right\rfloor + 1, \ell - 1\right]. \end{cases}$$

$$wt_\beta(y_iv) = \begin{cases} 3\ell + \left\lfloor\frac{\ell}{2}\right\rfloor - 5 + i, & i \in \left[1, \left\lfloor\frac{\ell}{2}\right\rfloor\right], \\ 3\ell + \left\lfloor\frac{\ell}{2}\right\rfloor - 4 + i, & i \in \left[\left\lfloor\frac{\ell}{2}\right\rfloor + 1, \ell\right]. \end{cases}$$

$$wt_\beta(y_iy_{i+1}) = \begin{cases} \ell - 2 + i, & i \in \left[1, \left\lfloor\frac{\ell}{2}\right\rfloor - 1\right], \\ 3\ell + 2\left\lfloor\frac{\ell}{2}\right\rfloor - 4, & i = \left\lfloor\frac{\ell}{2}\right\rfloor, \\ 4\ell - 4 + i, & i \in \left[\left\lfloor\frac{\ell}{2}\right\rfloor + 1, \ell - 1\right]. \end{cases}$$

clearly, the edge weights gradually accumulate the values $1, 4, \ldots, 5\ell - 5$. As a result, the tagging is an appropriate EIRL, finally we have achieved our mission. Figure 3 illustrates an example of reflexive edge labeling.

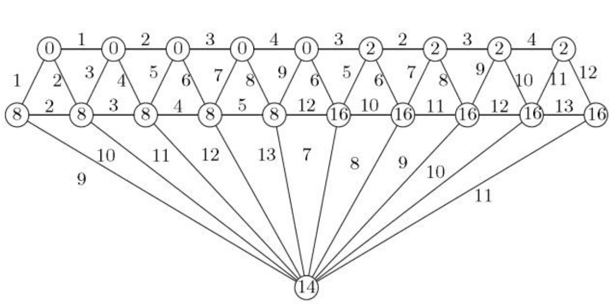


Figure 3

A reflexive edge 16-labeling of D_{10}.

3. Reflexive vertex strength of D_ℓ.

This section utilizes a precise calculation to estimate the rvs of diamond graphs.

Theorem 2: *Suppose* $D_\ell, \ell \geq 3$ *is diamond graph. Thus*

$$rvs(D_\ell) = \begin{cases} 3, & \ell = 5, \\ \left\lceil \frac{2\ell+1}{6} \right\rceil, & otherwise. \end{cases}$$

Proof: The least degree of the vertices of diamond graph is 3. Consequently, the lowest vertex weight must be at least 3 . For $\ell \geq 6$ the diamond graph has a single vertex of degree ℓ, 4 vertices of degree three, $\ell - 3$ vertices of degree four and $\ell - 2$ vertices of degree five. Given that the degree of the single vertex is greater than other vertices, so we can exclude this vertex to calculate the lower bound of rev(D_ℓ). As a result, we use Corollary 1.1 to estimate the lower bound of rvs(D_ℓ) in the following way.

$$rvs(D_\ell) \geq \left\lceil \frac{2\ell+1}{6} \right\rceil \tag{1}$$

The best reflexive vertex irregular rvs(D_ℓ)-labelings for $\ell = 3,4,5,6,7,8,9$ are shown in Figures 4, 5 and 6 respectively.

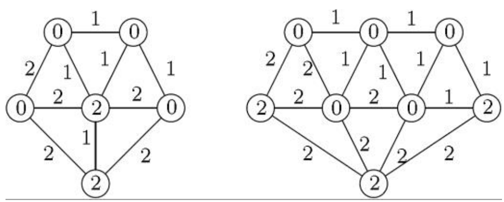


Figure 4
A reflexive vertex 2-labeling of D_3 and D_4.

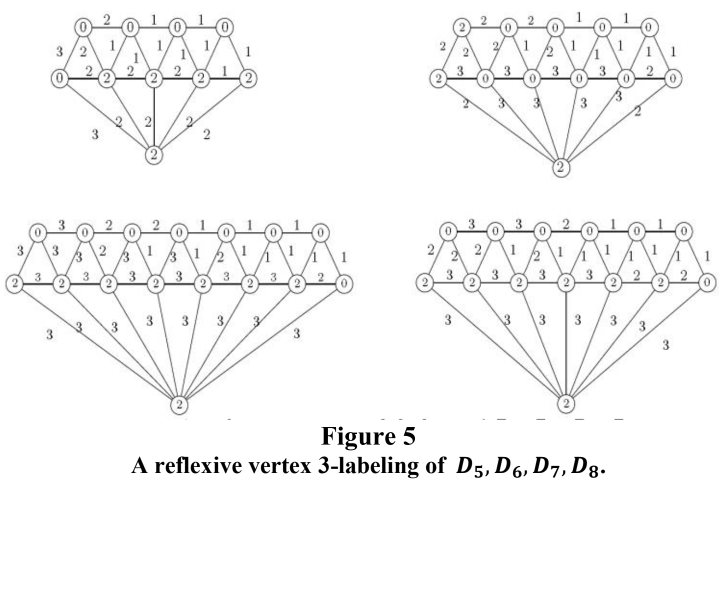


Figure 5
A reflexive vertex 3-labeling of D_5, D_6, D_7, D_8.

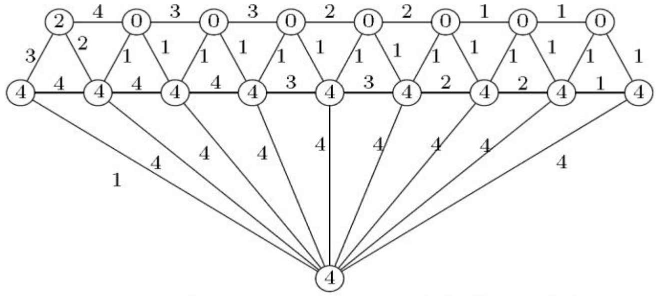


Figure 6
A reflexive vertex 4-labeling of D_9.

For $\ell \geq 10$, say $m = \left\lceil \frac{2\ell+1}{6} \right\rceil$. To get the converse of the inequality (1), it is adequate to show the presence of a VIRL for the diamond graph. We propose vertex irregular reflexive labeling as follows:

$$\beta(x_i) = 0, \quad i \in [1, \ell - 2]$$

$$\beta(x_{\ell-1}) = \begin{cases} 2\left\lfloor \frac{\left\lceil \frac{\ell-m}{2} \right\rceil + \left\lceil \frac{m-1}{2} \right\rceil + 2}{4} \right\rfloor, & \text{If } \ell \equiv\ 0, 2 \pmod 4, \\ 2\left\lfloor \frac{\left\lceil \frac{m-1}{2} \right\rceil + \left\lceil \frac{\ell-m}{2} \right\rceil + 1}{4} \right\rfloor & \text{If } \ell \equiv 1, 3 \pmod 4. \end{cases}$$

$$\beta(y_i) = \begin{cases} 2\left\lfloor \frac{3m-\ell+3\left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{m-\ell}{2} \right\rceil}{4} \right\rfloor, & i = 1, \\ 2\left\lfloor \frac{m}{2} \right\rfloor, & i \in [2, \ell]. \end{cases}$$

$$\beta(v) = 2\left\lfloor \frac{m}{2} \right\rfloor$$

$$\beta(y_i y_{i+1}) = \begin{cases} \frac{i-1}{2} - \left\lceil \frac{m-\ell}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor + 1, & i \in \left[1, 3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell\right] (i \text{ odd}), \\ \frac{i}{2} - \left\lceil \frac{m-\ell}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor, & i \in \left[2, 3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell\right] (i \text{ even}), \\ m, & i \in \left[3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell + 1, \ell - 1\right]. \end{cases}$$

$$\beta(x_i x_{i+1}) = \begin{cases} 1 + \frac{i-1}{2}, & i \in \left[1, 3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell - 1\right] \ (i \text{ odd}), \\ 1 + \frac{i-2}{2}, & i \in \left[1, 3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell - 1\right] \ (i \text{ even}), \\ m + \left\lfloor \frac{m-\ell}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor, & i \in \left[3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell, \ell - 2\right] \ (i \text{ even}), \\ m + \left\lfloor \frac{m-\ell}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor, & i \in \left[3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell, 2m - 2\right] \ (i \text{ odd}), \\ m + \left\lfloor \frac{m-\ell}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + 1, & i \in [2m - 1, \ell - 2] \ (i \text{ odd}), \end{cases}$$

$$\beta(y_i x_i) = \begin{cases} 1, & i \in [1, 2m-1], \\ i - 2m + 2, & i \in [2m, \ell - 1]. \end{cases}$$

$$\beta(y_{i+1} x_i) = \begin{cases} 1, & i \in \left[1, 3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell - 1\right], \\ i - 3m - 2\left\lfloor \frac{m}{2} \right\rfloor + \ell + 2, & i \in \left[3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell, 2m - 3\right], \\ \ell - m - 2\left\lfloor \frac{m}{2} \right\rfloor & i \in [2m - 2, \ell - 1]. \end{cases}$$

$$\beta(y_{\ell+1} x_\ell) = \begin{cases} \left\lceil \frac{\ell - m}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor - \\ -2\left\lfloor \frac{\left\lceil \frac{\ell-m}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor + 2}{4} \right\rfloor + 2 & \text{If } \ell \equiv 0, 2 \pmod 4, \\ \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{\ell - m}{2} \right\rfloor - \\ -2\left\lfloor \frac{\left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{\ell-m}{2} \right\rfloor + 1}{4} \right\rfloor + 1 & \text{If } \ell \equiv 1, 3 \pmod 4. \end{cases}$$

$$\beta(v y_i) = \begin{cases} 3m - \ell + 3\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m-\ell}{2} \right\rceil - \\ \quad -2\left\lfloor \frac{3m - \ell + 3\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m-\ell}{2} \right\rceil}{4} \right\rfloor, & i = 1, \\ m, & i \in [2, \ell - 1]. \end{cases}$$

$$\beta(v y_\ell) = \begin{cases} \ell - 2m - \left\lfloor \frac{m}{2} \right\rfloor - \left\lceil \frac{\ell - m}{2} \right\rceil + \\ \quad +2\left\lfloor \frac{\left\lceil \frac{\ell-m}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor + 2}{4} \right\rfloor + 2, & \text{If } \ell \equiv 0, 2 \pmod 4, \\ \ell - 2m - \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{\ell - m}{2} \right\rfloor + \\ \quad +2\left\lfloor \frac{\left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{\ell-m}{2} \right\rfloor + 1}{4} \right\rfloor + 3, & \text{If } \ell \equiv 1, 3 \pmod 4. \end{cases}$$

It is routinely checked to ensure that all node and link tags are not exceed m and that the weights of D_ℓ nodes are the following:

$$wt_\beta(x_i) = \begin{cases} 3, & i = 1, \\ 2 + i, & i \in \left[2, 3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell - 1\right], \\ 3 + i, & i \in \left[3m + 2\left\lfloor \frac{m}{2} \right\rfloor - \ell, \ell - 1\right]. \end{cases}$$

$$wt_\beta(y_i) = \begin{cases} 3m - \ell + 2\left\lceil \frac{m}{2} \right\rceil + 2, & i = 1, \\ \ell + 2 + i, & i \in [2, \ell - 1], \\ \ell + 3, & i = \ell. \end{cases}$$

$$wt_\beta(v) = \begin{cases} m\ell - \ell + 4\left\lfloor \frac{m}{2} \right\rfloor + 2\left\lfloor \frac{\left\lceil \frac{\ell - m}{2} \right\rceil + \left\lceil \frac{m-1}{2} \right\rceil + 2}{4} \right\rfloor - \\ \quad - \left\lfloor \frac{3m - \ell + 3\left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{m-\ell}{2} \right\rceil}{4} \right\rfloor + 2, & \text{If } \ell \equiv 0, 2 \pmod 4. \\ \quad m\ell - m + 4\left\lfloor \frac{m}{2} \right\rfloor - 2\left\lfloor \frac{\ell - m}{2} \right\rfloor + \\ +2\left\lfloor \frac{\left\lceil \frac{\ell - m}{2} \right\rceil + \left\lceil \frac{m-1}{2} \right\rceil + 2}{4} \right\rfloor - \left\lfloor \frac{3m - \ell + 3\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m-\ell}{2} \right\rfloor}{4} \right\rfloor + 2, & \text{If } \ell \equiv 1, 3 \pmod 4. \end{cases}$$

Figure 7 illustrates an example of reflexive vertex labeling. Precisely, the weights of nodes $x_i, i \in [1\ \ell - 1]$, and $y_i, i \in [1, \ell]$, construct a succession of sequential numbers from 3 to $2\ell + 1$, and node v has a weight more than $2\ell + 4$. It signifies that the weights of D_ℓ nodes vary for all pairs of different nodes. As a result, the needed VIRL β is used. In fact,

$$rvs(D_\ell) \leq \left\lceil \frac{2\ell+1}{6} \right\rceil \tag{2}$$

Hence, from (1) and (2) we can deduce that: $rvs(D_\ell) = \left\lceil \frac{2\ell+1}{6} \right\rceil$ for $\ell \neq 5$.

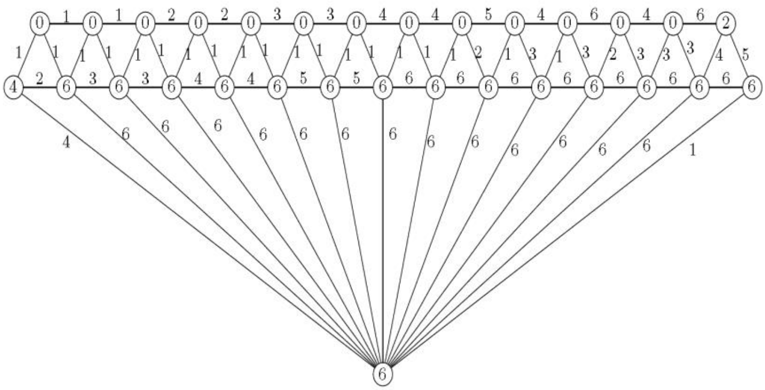


Figure 7
A reflexive vertex 6-labeling of D_{15}.

4. Conclusion

From the preceding sections, we compute the precise values of res and rvs of diamond network graphs D_ℓ. We prove that $\text{res}(D_\ell) = \left\lceil \frac{5\ell - 5}{3} \right\rceil$, for $\ell \equiv 0,1,2,3$

(mod 6) and $\mathrm{res}(D_\ell) = \left\lceil \frac{5\ell-5}{3} \right\rceil + 1$, for $\ell \equiv 4{,}5 \pmod 6$. Furthermore we prove that $\mathrm{rvs}(D_\ell) = 3$, for $\ell = 3$ and $\mathrm{res}(D_\ell) = \left\lceil \frac{2\ell+1}{6} \right\rceil + 1$, otherwise.

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