

Invariant intersection graph of a graph

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Abstract

Studies in algebraic graph theory showcase the interplay between group theory and graph theory by defining graphs on groups, investigating their properties, and also by analysing the automorphism groups that emerge from the graphs. In this article, we introduce the idea of constructing an algebraic derived graph; that is, constructing a graph based on the algebraic properties of a graph, by introducing the invariant intersection graph of a graph, constructed based on the automorphism group of a graph. Here, we introduce the graph construction and initiate an investigation on the structure of the invariant intersection graph with respect to the graph and its automorphism group.

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1. Introduction

For concepts in group theory, we refer to [1]. For fundamentals in graph theory, see [2] and for topics in algebraic graph theory, refer to [3].

A *permutation* of a set S having r elements is an arrangement of the elements of S in a specific order, and the *symmetric group* S_r consists of all possible permutations of S . The *fix* (resp. *support*) of a permutation $\pi \in S_r$, denoted by $fix(\pi)$ (resp. $supp(\pi)$), is the set $\{x \in S : \pi(x) = x\}$ (resp. $\{x \in S : \pi(x) \neq x\}$). Note that $supp(\pi) = S - fix(\pi)$, for any $\pi \in S_r$.

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A group X is said to be *cyclic* if there exists an element $a \in X$ such that $X = \langle a \rangle$ and any cyclic group of order r is denoted by $\mathbb{Z}_r$, as it is known that any cyclic group of order r is isomorphic to $\mathbb{Z}_r$. A *dihedral group*, denoted by D_{2r} , consists of the symmetries of rotations and reflections of an r -gon, and it is given by $D_{2r} = \{a, b: a^r = b^2 = e, bab = a^{-1}\}$. Note that throughout the study, all group structures considered are the embeddings on the symmetric group S_r , for some $r > 0$, and a cyclic group $\mathbb{Z}_r$ corresponds to the one generated by the r -cycle and not the ones isomorphic to the direct product of cyclic groups.

For graphs G_1 and G_2 , the existence of a bijection $f: V(G_1) \rightarrow V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$, for any two vertices $u, v \in V(G_1)$ makes G_1 and G_2 *isomorphic*, and here, if $G_1 = G_2 = G$, then f is an *automorphism* of G . The set of all possible automorphisms defined on $V(G)$ is the *automorphism group* of G , which we denote by $Aut(G)$, and if $|V(G)| = n$, $Aut(G)$ is a subgroup of S_n . A graph G is said to be *asymmetric* if $Aut(G)$ is the trivial group.

Constructing graphs based on groups and investigating the group associated with a given graph are two prominent topics of study in the field of algebraic graph theory. Multiple graphs can be defined on a group, based on different adjacency relations that reflect the structure of the group. However, the only group that is associated with any graph G , is its *automorphism group* $Aut(G)$ and owing to this, $Aut(G)$ is known as the *algebraic invariant* of G (c.f. [4]).

The existing literature on algebraic graph theory consists of several studies on graphs that are defined on finite algebraic structures, reflecting their algebraic properties (ref. [5,6]) and investigations on the structural aspects of the automorphism group of a graph (ref. [7-14]). Motivated to combine these two prominent aspects of study in the field, an intersection graph is constructed based on the algebraic invariant of a given graph, which helps to investigate the automorphisms of a graph based on a graph, and not just a group.

This graph construction can be seen as exploring the graphical structure of the algebraic invariant, and the investigation of this graph becomes interesting as it gives the properties of a group, which is associated with a graph in the form of another graph. Owing to the structure-preserving nature of the automorphism group, the study of the newly defined graph helps us investigate the structure of the graph as well as its automorphism group.

2. Invariant Intersection Graphs of Graphs

An *intersection graph* can be viewed as a graph in which each vertex represents a set, and two distinct vertices are made adjacent based on the intersection properties of the corresponding sets represented by these vertices. For any graph G , we construct an algebraic intersection graph, called the invariant intersection graph of G , based on its algebraic invariant $Aut(G)$ as follows:

Definition 2.1: The *invariant intersection graph* of a graph G , denoted by Λ_G , is a graph with $V(\Lambda_G) \cong \{\pi \in Aut(G)\}$, and the vertices that correspond to two

distinct automorphisms $\pi, \pi' \in \text{Aut}(G)$ are adjacent in Λ_G when $\text{fix}(\pi) \cap \text{fix}(\pi') \neq \emptyset$.

An example of the graph construction concerned is given as follows. As the graph G in Figure 1a has $\text{Aut}(G) \cong D_{12}$, let $\text{Aut}(G) = \{\pi_0, \pi_1, \dots, \pi_{11}\}$, such that π_0 is the identity automorphism and

$$\begin{aligned} \pi_1 &= (v_1 v_2 v_3 v_4 v_5 v_6), & \pi_6 &= (v_1 v_4)(v_2 v_3)(v_5 v_6), \\ \pi_2 &= (v_1 v_6 v_5 v_4 v_3 v_2), & \pi_7 &= (v_1 v_2)(v_3 v_6)(v_5 v_4), \\ \pi_3 &= (v_1 v_3 v_5)(v_2 v_4 v_6), & \pi_8 &= (v_1 v_6)(v_2 v_5)(v_3 v_4), \\ \pi_4 &= (v_1 v_5 v_3)(v_2 v_6 v_4), & \pi_9 &= (v_1)(v_4)(v_2 v_6)(v_5 v_3), \\ \pi_5 &= (v_1 v_4)(v_2 v_5)(v_3 v_6), & \pi_{10} &= (v_2)(v_5)(v_1 v_3)(v_6 v_4), \\ & & \pi_{11} &= (v_3)(v_6)(v_2 v_4)(v_1 v_5). \end{aligned}$$

Based on the fix of each of the automorphisms in $\text{Aut}(G)$, the invariant intersection graph of G is constructed as shown in Figure 1b. Note that throughout the study, the vertex corresponding to an automorphism $\pi \in \text{Aut}(G)$ is denoted by $v_\pi \in V(\Lambda_G)$ and the identity automorphism is denoted by π_0 .

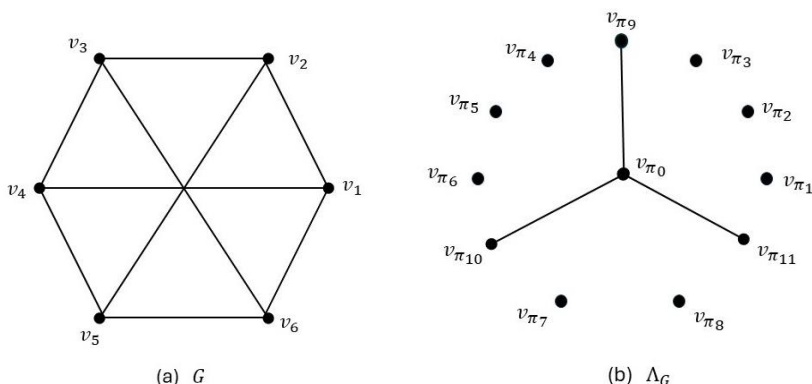


Figure 1
An example of a graph and its invariant intersection graph.

Note that for each $\pi \in \text{Aut}(G)$, as the vertex $v_\pi \in V(\Lambda_G)$ can be viewed as the set $\text{fix}(\pi)$, the graph Λ_G becomes an algebraic intersection graph derived from the graph G . For any graph G , the following observations on Λ_G are immediate from Definition 2.1.

- (i) $\Lambda_G \cong K_1$ if and only if G is asymmetric.
- (ii) $\Lambda_G \cong \Lambda_{\bar{G}}$, as the fix of every automorphism in $\text{Aut}(G)$ depends on the adjacencies in G , which corresponds to the non-adjacencies in $\bar{G}$.
- (iii) The graph Λ_G has at most one non-trivial component and its diameter is at most 2, as any vertex $v_\pi \in \Lambda_G$ corresponding to an automorphism $\pi \in \text{Aut}(G)$ with $\text{fix}(\pi) \neq \emptyset$ is adjacent to the vertex v_{π_0} .

- (iv) In Λ_G , the vertex corresponding to the identity permutation is a vertex with the maximum degree $\Delta(\Lambda_G)$.
- (v) The invariant intersection graph of a graph is either acyclic or has girth 3.

In view of the observation (i) mentioned above, the graphs considered throughout the study are non-asymmetric, unless stated otherwise, and the corresponding invariant intersection graphs are also non-trivial. Based on the initial observations made, we study the structure of the invariant intersection graphs and to aid in the study, we introduce the notion of a *primal vertex* of a graph as follows.

Definition 2.2: A *primal vertex* in a simple graph G is a vertex that cannot be mapped to any other vertex of G , by any automorphism of G ; that is, $\pi(v) = v$, for all $\pi \in \text{Aut}(G)$, and we call the number of primal vertices in G as its *primal index*, denoted by $\rho(G)$.

An illustration of the notion of primal vertices is given in Figure 2. A vertex v in $V(G)$ can be identified as a primal vertex if it satisfies either or all of the following properties.

- (i) If $d(v) \neq d(v')$, for any vertex $v' \in V(G)$.
- (ii) If there exists no vertex $v' \in V(G)$ other than $v \in V(G)$, such that the degree sequence of the vertices in $N(v')$ and $N(v)$ are the same.
- (iii) If there exists a $v' \in V(G)$ such that $d(v', v) \neq d(v^*, v)$, for some $v^* \in V(G)$.

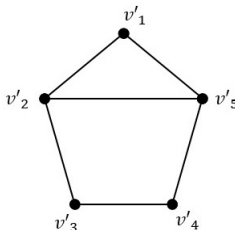


Figure 2
An example of a graph G' with $\rho(G') > 0$.

It can be observed that $d(v'_1) = d(v'_3) = d(v'_4) = 3$, and $d(v'_2) = d(v'_5) = 2$, in the graph G' given in Figure 2. However, it can be seen that the vertices v'_3 and v'_4 are adjacent to two vertices, among which one is of degree 2 and the other one is of degree 3; whereas, both the vertices adjacent to v'_1 have degree 3. Therefore, this makes v'_1 a primal vertex of G even if the graph has other vertices of the same degree.

In order to examine the structure of the invariant intersection graphs of graphs, we first address the uniqueness of the vertex $v_{\pi_0} \in V(\Lambda_G)$ in the following theorem, as $\pi_0 \in \text{Aut}(G)$ is unique.

Theorem 2.3: *The vertex v_{π_0} is a primal vertex in Λ_G if and only if Λ_G is non-regular.*

Proof: In the invariant intersection graph Λ_G of a graph G , the vertex v_{π_0} is adjacent to all vertices that correspond to automorphisms that have a non-empty fix. To prove the primality of the vertex v_{π_0} , we will prove that $d(v_{\pi_0}) \neq d(v_\pi)$, for any $\pi \in \text{Aut}(G)$. For this, assume that there exists another vertex, say v_{π_1} , where $\pi_1 \in \text{Aut}(G)$ is a non-trivial automorphism, such that the degrees of v_{π_0} and v_{π_1} are equal. Therefore, $|\text{fix}(\pi_1)| = k$; $1 \leq k \leq n - 2$, and since all vertices of Λ_G corresponding to automorphisms that have non-empty fixes are adjacent to v_{π_1} , it implies that there exists no non-trivial mapping in $\text{Aut}(G)$, that fixes the remaining $n - k$ vertices of G . Therefore, $\mathbb{Z}_{n-k} \subseteq \text{Aut}(G)$. If $\mathbb{Z}_{n-k} \cong \text{Aut}(G)$, then either the remaining k vertices of G are its primal vertices or the fix of all non-identity permutations are empty. In both the cases, Λ_G is regular, as $\rho(G) > 0$ implies $\Lambda_G \cong K_{|\text{Aut}(G)|}$ and in the latter case, $\Lambda_G \cong |\text{Aut}(G)|K_1$. Hence, $\mathbb{Z}_{n-k} \not\cong \text{Aut}(G)$. If $\mathbb{Z}_{n-k} \subset \text{Aut}(G)$, then there exists at least one non-trivial automorphism of G in which the remaining k vertices of G mapped to other vertices of G , and in this case, we assume there are no primal vertices, if otherwise, we encounter the previous case.

Here, either all non-identity permutations in $\text{Aut}(G)$ should have empty fix, or if a non-identity permutation has a non-empty fix, it must have a non-empty support. As Λ_G is empty in the former case, we consider only the latter case. Since G has no primal vertices, every vertex of G must be in the support of at least one of its automorphisms and also there exists at least two automorphisms $\pi, \pi' \in \text{Aut}(G)$ such that $\text{fix}(\pi) \neq \text{fix}(\pi')$. Assume that they differ by $k_1 \geq 2$ elements. As $\text{Aut}(G)$ is a group, its closure property leads to the existence of an automorphism π'' that is not adjacent to v_{π_1} ; yielding a contradiction.

As mentioned above in the observation (v), the girth of the invariant intersection graph Λ_G of a graph G is either 3 or ∞ . In this regard, we proceed to characterise the graphs for which the invariant intersection graph is acyclic, in the following results.

Recall that an *involution* on a set S is a permutation of S whose cyclic decomposition contains only cycles of length 1 or 2. We define two involutions π and π' on a set S to be *disjoint involutions* if for any distinct $a, b, c \in S$, there exist no transposition $(a \ c)$ in π' when π has the transposition $(a \ b)$.

Theorem 2.4: *The invariant intersection graph of a graph is a non-empty forest if and only if every pair of its non-identity automorphisms are disjoint involutions having disjoint fixes.*

Proof: Let the invariant intersection graph Λ_G of a graph G be acyclic. Therefore, Λ_G can either be a star $K_{1, |\text{Aut}(G)|-1}$ or $K_{1, r-1} \cup (|\text{Aut}(G)| - r)K_1$, where $1 < r < |\text{Aut}(G)|$. In both the cases, for any two distinct non-identity

automorphisms $\pi_i, \pi_j \in \text{Aut}(G)$, $\text{fix}(\pi_i) \cap \text{fix}(\pi_j) = \emptyset$. If $\Lambda_G \cong K_{1, |\text{Aut}(G)|-1}$, the fix of each non-identity automorphism of G must be non-empty; whereas, if $\Lambda_G \cong K_{1, r-1} \cup (|\text{Aut}(G)| - r)K_1$, for some $1 < r < |\text{Aut}(G)|$, $r - 1$ non-identity automorphisms of G have non-empty, non-intersecting fixes and $|\text{Aut}(G)| - r$ automorphisms of G have empty fixes as the vertices corresponding to these $|\text{Aut}(G)| - r$ automorphisms of G having empty fixes are isolated in Λ_G .

As Λ_G is acyclic, every $\pi \in \text{Aut}(G)$ can only be a transposition, as the existence of cycles of length more than 2 in the cyclic decomposition of the permutations gives rise to at least two automorphisms with the same fix, inducing at least a C_3 in Λ_G . Furthermore, these transpositions present in each of the permutations must be disjoint, because if these transpositions are not disjoint, Λ_G contains cliques of order at least 3, owing to the closure property of $\text{Aut}(G)$; thereby, establishing the necessary condition.

To prove the converse of the theorem, consider a non-identity automorphism $\pi \in \text{Aut}(G)$ such that $\text{fix}(\pi) \neq \emptyset$, as the vertices corresponding to the automorphisms for which $\text{fix}(\pi) = \emptyset$ are isolated in Λ_G and assume that π is not an involution. Therefore, there exists a permutation cycle of length at least three, say $(u' v' w')$, in the cyclic decomposition of π , where $u', v', w' \in V(G)$. According to the closure property of the group, the automorphism π' with the cyclic decomposition $(u' w' v')$ also exists in the group, which fixes the same elements as π and thereby, inducing a cycle of order 3 in the graph Λ_G . Similarly, if there exists two automorphisms $\pi, \pi' \in \text{Aut}(G)$ with the involutions $(u' v')$ and $(v' w')$, then there exists an automorphism with the 3-cycle $(u' v' w')$, fixing the vertices that both π and π' fixes, thereby inducing a cycle in Λ_G . This argument can be extended for any cycle of length $t \geq 4$ as any t -cycle generates a $\mathbb{Z}_t$; a subgroup of $\text{Aut}(G)$; thereby completing the proof.

Theorem 2.5: *The invariant intersection graph Λ_G of a graph G is a non-trivial tree if and only if $\Lambda_G \cong K_2$.*

Proof: Let the invariant intersection graph of a simple graph G be a tree. By Theorem 2.4, it can be seen that if Λ_G is a tree, every automorphism of G has a unique non-empty fix and every pair of non-identity automorphisms of G are disjoint involutions of $V(G)$.

According to the structure of the automorphisms of G that have been deduced, there will exist at least one vertex $w \in V(G)$ that cannot be in the support of any automorphism of G , making $\rho(G) \geq 1$, when $|V(G)|$ is odd. As the invariant intersection graph is a complete graph, in this case, it can only be K_2 .

When $|V(G)|$ is even, each automorphism of G has to fix even number of (at least two) unique vertices, and the automorphism group of the graph must not contain any cyclic subgroup apart from $\mathbb{Z}_2$, as the cyclic decomposition of every

automorphism contains only transpositions. Therefore, $\text{Aut}(G) \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{k\text{-times}}$, $k \geq 1$. If $k = 1$, $\Lambda_G \cong 2K_1$ or $\Lambda_G \cong K_2$, depending on the existence of primal vertices in G . If $k \geq 2$, then $\Lambda_G \cong K_{2^k}$, when G has a primal vertex, and Λ_G has at least one isolated vertex that corresponds to the automorphism of the form $(v_1 v_2)(v_3 v_4) \dots (v_{n-1} v_n)$, otherwise, where $v_i \in V(G)$, $1 \leq i \leq n$. Hence, Λ_G cannot be a tree. Therefore, in this case Λ_G is a tree only when it is a K_2 . This completes the proof, as the converse immediate.

Corollary 2.6: *For a graph G , Λ_G is a tree if and only if $\rho(G) > 1$, and $\text{Aut}(G) \cong \mathbb{Z}_i$; $i = 1, 2$.*

In view of Theorem 2.3, it can be seen that Λ_G is regular if and only if it is a complete graph or an empty graph. Therefore, we characterise the graphs whose invariant intersection graphs are complete or empty in the following results.

Theorem 2.7: *The invariant intersection graph of a graph G is complete if and only if there exists a primal vertex in G .*

Proof: The necessary condition is immediate, as any primal vertex of a graph G is fixed by all automorphisms of G . To prove the converse, assume $\Lambda_G \cong K_{|\text{Aut}(G)|}$ and $\rho(G) = 0$. Therefore, every vertex in G must be mapped to at least one other vertex by at least one automorphism of G . Also, based on our assumption, every automorphism of G has non-empty intersecting fix and there exists at least two automorphisms $\pi, \pi' \in \text{Aut}(G)$ such that $\text{fix}(\pi) \neq \text{fix}(\pi')$. The fixes intersect only when $|\text{fix}(\pi)| > \frac{n}{2}$, for all $\pi \in \text{Aut}(G)$. Owing to the closure property of $\text{Aut}(G)$, the existence of 2 or more such permutations will give rise to permutations with empty fixes, leading to the existence of isolated vertices in Λ_G , and contradicting our assumption. Therefore, this completes the proof.

In [15], it was proved that almost all graphs are asymmetric. Based on this fact and Theorem 2.7, the following proposition is deduced.

Proposition 2.8: Invariant intersection graphs of almost all graphs are complete.

Proof: As a trivial case, it can be observed that Λ_G is complete when G is asymmetric. As any asymmetric graph G has $\rho(G) = |V(G)|$, it can be deduced that the invariant intersection graphs of almost all graphs are complete, by Theorem 2.7.

Next, we characterise the graphs whose invariant intersection graph is empty in Proposition 2.10 using the following definition.

Definition 2.9: [16] A non-asymmetric graph G is said to be *strongly fixed-point-free* (SFP-free), if $\pi(v) \neq v$, for any $v \in V(G)$ and every $\pi \in \text{Aut}(G) - \{\pi_0\}$.

Proposition 2.10: For any graph G , $\Lambda_G \cong |Aut(G)|K_1$, where $|Aut(G)| > 1$, if and only if G is SFP-free.

Proof: By Definition 2.9, if G is a SFP-free graph, no vertex is fixed by any non-identity automorphisms in $Aut(G)$. Hence, $\Lambda_G \cong |Aut(G)|K_1$; $|Aut(G)| > 1$, for a strongly fixed-point-free graph G . Conversely, if $\Lambda_G \cong |Aut(G)|K_1$; $|Aut(G)| > 1$, it can be deduced that $fix(\pi) = \emptyset$, for all $\pi \in Aut(G)$, except the identity map. Therefore, the graph G is strongly fixed-point-free.

Based on the results obtained thus far, the following corollaries on the regularity of the invariant intersection graphs are immediate.

Corollary 2.11: *The invariant intersection graph of a graph G is regular if and only if either G is strongly fixed-point-free or G has a primal vertex.*

Corollary 2.12: *The invariant intersection graph of a graph is either regular or has a primal vertex.*

3. Structure of the Invariant Intersection Graphs

In this section, we first begin by obtaining all the possible graph structures that can be realised as an invariant intersection graph Λ_G of a graph G , and following which certain properties of these graphs are investigated.

Theorem 3.1: *Given any graph G , its invariant intersection graph is either $K_{|Aut(G)|}$, $|Aut(G)|K_1$ or a disconnected graph with exactly one non-trivial component of diameter 2 with a unique universal vertex.*

Proof: Let Λ_G be the invariant intersection graph of a graph with $|V(G)| = n$ and $|Aut(G)| = k$. Then, based on the observation (iii) and Theorem 2.3, Λ_G is one of the following structures.

- (i) The trivial graph K_1 ,
- (ii) A complete graph K_k ; $k \geq 2$,
- (iii) An empty graph kK_1 ; $k \geq 2$,
- (iv) A star graph $K_{1,k-1}$, $k \geq 2$,
- (v) $rK_1 \cup K_{k-r}$; $k \geq 3$, $1 \leq r \leq k-2$,
- (vi) $rK_1 \cup K_{1,k-r-1}$, $k \geq 3$, $1 \leq r \leq k-2$,
- (vii) $rK_1 \cup H$, where H is a graph with $|V(H)| = k-r$, having a spanning subgraph $K_{1,k-r-1}$,
- (viii) A connected graph H' with $|V(H')| = k$, with a spanning subgraph $K_{1,k-1}$.

Case 1: If Λ_G is connected, Theorem 2.3 restricts the structure of the invariant intersection graphs to be a complete graph K_k , a star $K_{1,k-1}$ or a graph with

diameter 2 having a spanning tree $K_{1,k-1}$, and by Theorem 2.5, $\Lambda_G \cong K_{1,k-1}$ implies $\Lambda_G \cong K_2$; a complete graph.

Let Λ_G be a non-acyclic graph of diameter 2, with a spanning subgraph $K_{1,a-1}$. Owing to Theorem 2.7, $\rho(G) = 0$, and G is not SFP-free. As there exists a cyclic subgroup $\mathbb{Z}_j$ of any group X , for some $2 \leq j \leq |X|$, and since G is not SFP-free with $\rho(G) = 0$, $\mathbb{Z}_n \subset \text{Aut}(G)$, by Theorem 2.3. As $\text{fix}(\pi) = \emptyset$, for any $\pi \in \mathbb{Z}_n$, the vertices corresponding to these automorphisms of G in Λ_G are isolated; establishing that Λ_G is connected if and only if it is complete.

Case 2: If Λ_G is not connected, it can have at most one non-trivial component as any vertex corresponding to an automorphism of G , with non-empty fix must be adjacent to v_{π_0} . If there exists no non-trivial components in Λ_G , then $\Lambda_G \cong kK_1$, $k \geq 1$, and by Theorem 2.10, the existence of graphs with empty invariant intersection graphs is guaranteed.

If there exists exactly one non-trivial component, then the structure of this non-trivial component must again be either a complete graph K_{k-r} , a star $K_{1,k-r-1}$ or a graph with diameter two having a spanning tree $K_{1,k-r-1}$, where r is the number of trivial components and $1 \leq r \leq k-2$.

By Theorem 2.3, the existence of one unique universal vertex in Λ_G , when it is not-regular makes the diameter 2 graphs mentioned in (vi) and (vii) have exactly one such vertex. Since star graph $K_{1,k-r-1}$ is also a diameter 2 graph with spanning subgraph as star, $rK_1 \cup K_{1,k-r-1}$, $k > 2$, $1 \leq r \leq k-2$, just with girth ∞ , it can be considered as a subset of a set of graphs $rK_1 \cup H$, where the graph H is a graph with spanning subgraph $K_{1,r-k-1}$.

When $\text{Aut}(G) \cong S_n$, there exists at least two automorphisms of G whose fixes are disjoint, making Λ_G^* a graph with a spanning tree $K_{1,n!-r-1}$, and as $\text{Aut}(K_n) \cong S_n$, the existence of such Λ_G is guaranteed. Also, as the existence of exactly one non-identity automorphism of G with a non-empty fix is not possible unless $\text{Aut}(G) \cong \mathbb{Z}_2$, it is enough to establish that $\Lambda_G \not\cong K_{k-r} \cup rK_1$, $k > 2$, $1 \leq r \leq k-2$, to prove the result.

For this, assume the contrary. If $\Lambda_G \cong K_{k-r} \cup rK_1$; $k > 2$, $1 \leq r \leq k-2$, then $\rho(G) = 0$ and all $\pi \in \text{Aut}(G)$ with $\text{fix}(\pi) \neq \emptyset$ must have $|\text{fix}(\pi)| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. As Λ_G is not acyclic, there exists an automorphism $\pi \in \text{Aut}(G)$ such that $|\text{supp}(\pi)| \geq 3$ and as $\text{Aut}(G)$ is a group, there exists at least another automorphism $\pi' \in \text{Aut}(G)$ such that $|\text{fix}(\pi)| = |\text{fix}(\pi')| = s'$, for all $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq s' \leq n-3$. Therefore, the automorphism $\pi \circ \pi' \in \text{Aut}(G)$ has $|\text{fix}(\pi \circ \pi')| = 2s' - n$, if $\text{supp}(\pi) \cap \text{supp}(\pi') = \emptyset$; or $|\text{fix}(\pi \circ \pi')| = n - (s' - s_1)$; otherwise, where $s_1 = |\text{supp}(\pi) \cap \text{supp}(\pi')|$.

If $\text{supp}(\pi) \cap \text{supp}(\pi') = \emptyset$, $\mathbb{Z}_{n-s'} \times \mathbb{Z}_{n-s'} \subseteq \text{Aut}(G)$. If $\mathbb{Z}_{n-s'} \times \mathbb{Z}_{n-s'} \cong \text{Aut}(G)$, then $\rho(G) = 2s' - n$, which is a contradiction, as $2s' - n > 0$, for any $s' \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Hence, $\mathbb{Z}_{n-s'} \times \mathbb{Z}_{n-s'} \subset \text{Aut}(G)$. As $\rho(G) = 0$, there must exist at least one automorphism of G that maps the remaining $2s' - n$

vertices of G to its other vertices. If these $2s' - n$ vertices are mapped among themselves, then $\mathbb{Z}_{n-s'} \times \mathbb{Z}_{n-s'} \times \mathbb{Z}_{2s'-n} \subseteq \text{Aut}(G)$, in which there exists one automorphism fixing exactly $2s' - n$ vertices and another one automorphism fixing only the remaining $2(n - s')$ vertices of G . As the fixes of these two automorphisms do not intersect, the corresponding vertices in Λ_G are non-adjacent; yielding $\Lambda_G \not\cong K_{k-r} \cup rK_1$. If the remaining $2s' - n$ vertices of G are mapped to the other $2n - 2s'$ vertices of G , then $S_t \subseteq \text{Aut}(G)$; for some $t > s'$, in which there exists at least one automorphism of G which fixes at most $\left\lfloor \frac{n}{2} \right\rfloor$ vertices of G ; a contradiction. Therefore, $\Lambda_G \not\cong K_{k-r} \cup rK_1$, $k > 2$, $1 \leq r \leq k - 2$.

If $|\text{fix}(\pi \circ \pi')| = n - (s' - s_1)$, for two distinct automorphisms $\pi, \pi' \in \text{Aut}(G)$ with $|\text{fix}(\pi)| = |\text{fix}(\pi')| = s'$, where $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq s' \leq n - 3$, and $|\text{supp}(\pi) \cap \text{supp}(\pi')| = s_1$, then $\mathbb{Z}_{s'-s_1} \subseteq \text{Aut}(G)$. As this leads to the same situation as encountered in the previous case, $\mathbb{Z}_{s'-s_1} \subset \text{Aut}(G)$. Using the similar arguments based on the non-existence of primal vertices in G , as in the previous case, it follows that $\Lambda_G \not\cong K_{k-r} \cup rK_1$, $k > 2$, $1 \leq r \leq k - 2$.

In view of Theorem 3.1, the following corollaries are deduced.

Corollary 3.2: *The invariant intersection graph Λ_G of a graph G is connected if and only if Λ_G is complete.*

Corollary 3.3: *The diameter of the invariant intersection graph Λ_G of a graph G is either 1 or ∞ .*

On obtaining the diameter of the graph Λ_G , we compute the degree of the vertices in Λ_G and domination number of Λ_G in the following results. Note that the set of all permutations of a set S that fixes an element $a \in S$ is the stabilizer of a , denoted by X_a , where X represents the permutation group acting on S .

Theorem 3.4: *For any vertex $v_\pi \in V(\Lambda_G)$ that corresponds to the automorphism $\pi \in \text{Aut}(G)$,*

$$d(v_\pi) = \left| \bigcup_{v \in \text{fix}(\pi)} X_v \cap \text{Aut}(G) \right|,$$

where $v \in V(G)$.

Proof: In the invariant intersection graph Λ_G of a graph G , the vertex $v_\pi \in V(\Lambda_G)$, for any $\pi \in \text{Aut}(G)$ is adjacent to all the other vertices that correspond to the automorphisms that fix at least one of the elements that π fixes. Therefore, it can be deduced that the vertex v_π will be adjacent to all the vertices that corresponds to the automorphisms that belong to the stabilizer of at least one of the elements that π fixes. Since $\text{Aut}(G) \subseteq S_n$, for any graph G having n

vertices, the degree of any vertex $v_\pi \in V(\Lambda_G)$ that corresponds to the automorphism $\pi \in \text{Aut}(G)$ is given as $d(v_\pi) = |\bigcup_{v \in \text{fix}(\pi)} X_v \cap \text{Aut}(G)|$.

Proposition 3.5: For any graph G , the domination number γ of Λ_G ,

$$\gamma(\Lambda_G) = \begin{cases} 1, & \rho(G) > 0; \\ |\text{Aut}(G)|, & G \text{ is strongly fixed-point-free}; \\ 1 + \zeta(\Lambda_G), & \text{otherwise,} \end{cases}$$

where $\zeta(\Lambda_G)$ is the number of isolated vertices in Λ_G .

Upon the construction of a derived graph from a graph G , the necessary and sufficient conditions for G to be isomorphic to its derived graph is one of the main problem of interest. This intriguing question of when $G \cong \Lambda_G$ is addressed in the following theorem.

Theorem 3.6: For any graph G , $G \cong \Lambda_G$ if and only if $G \cong iK_1, i = 1, 2$.

Proof: When $G \cong K_1$, it is evident that $\text{Aut}(G) = \{\pi_0\}$ and hence $\Lambda_G \cong K_1 \cong G$. If the graph $G \cong 2K_1$ with the vertex set $\{v_1, v_2\}$, the automorphism group $\text{Aut}(G) = \{\pi_0 = (v_1)(v_2), \pi_1 = (v_1, v_2)\}$ and hence $\Lambda_G \cong 2K_1$.

Conversely, assume that $G \cong \Lambda_G$ and $|G| > 1$. Then, G must be one of the following structures with $|G| = |\text{Aut}(G)| = n$.

- (i) A complete graph K_n ; $n > 1$,
- (ii) An empty graph nK_1 ; $n > 1$,
- (iii) $rK_1 \cup K_{1, n-r-1}$; $n > 2, 1 \leq r \leq n-2$,
- (iv) $rK_1 \cup H$, where H is a graph of order $n-r$ and diameter 2, with a spanning subgraph $K_{1, n-r-1}$.

If $G \cong K_n$ or $G \cong nK_1$, for $n > 1$, $\text{Aut}(G) \cong S_n$. Then, $|V(G)| = |\text{Aut}(G)| = n$ implies $n = 2$. Therefore, $G \cong K_2$ or $G \cong 2K_1$ and the invariant intersection graph of both K_2 and $2K_1$ is $2K_1$. Therefore, in this case, $G \cong \Lambda_G$ only when $G \cong 2K_1$.

When $G \cong rK_1 \cup K_{1, n-r-1}$, $1 \leq r \leq n-2$, $\text{Aut}(G) \cong S_r \times S_{n-r-1}$. Therefore, $|\text{Aut}(G)| = n$ if and only if $(n, r) = (2, 1)$, and hence, $G \cong 2K_1$ in this case.

If $G \cong rK_1 \cup H$, where H is a graph with $n-r$ vertices and a spanning subgraph $K_{1, n-r-1}$, then $\text{Aut}(G) \cong S_r \times \text{Aut}(H)$. Based on the symmetry of H , the graphs can be investigated in two cases. Assume H is asymmetric. Then, $\text{Aut}(H) = \{\pi_0\}$ and hence, $\text{Aut}(G) \cong S_r$. Here, if $r = 1$, then G is asymmetric graph of order at least 2, a contradiction.

When H is not asymmetric, $|\text{Aut}(H)| \geq 2$. If $r = 1$, then Λ_G has a primal vertex and $\Lambda_G \cong K_{|\text{Aut}(H)|}$, a contradiction. If $r > 2$, $\text{Aut}(G) \cong S_r \times |\text{Aut}(H)|$.

In this case, as the order of the graph can never be equal the order of its automorphism group, we get that $G \not\cong \Lambda_G$, when $G \cong rK_1 \cup H$, for any graph H of order $n - r$, containing a spanning subgraph $K_{1,n-r-1}$; completing the proof.

4. Conclusion

The notion of algebraic derived graph is introduced in this article by defining the invariant intersection graph based on the automorphism group of a graph. The structure of this new graph has been examined based on the graph with respect to which it is constructed. As this study depends on the structure of the graph as well as its automorphism group, the problem becomes interesting and challenging. We have obtained the graph structures that can be realised as invariant intersection graphs and characterised the structure of the graphs based on it. As the article focuses on the structural aspects of the invariant intersection graph, the future study can explore various graph properties and parameters of these graphs.

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