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Exploring integral solutions for ternary Diophantine equations : A multi-technique approach

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Abstract

This research paper deals with the problem of finding non-zero integral solutions for a ternary Diophantine equation expressed as $pw^2 - (2p - 1)wz + pz^2 = (k^2 + 4p - 1)q^r$. The investigation explores three distinct cases, each examining the exponent r from 1 to 3. Additionally, the paper outlines methodologies for finding solutions when r is greater than 3. The paper tailors a theorem to this ternary equation, utilizing seven unique techniques to address the problem. A crucial aspect of these methodologies involves a transformative

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approach where w is replaced with $x + y$ and z is replaced with $x - y$. Through rigorous analysis and the application of these techniques, the paper provides comprehensive insights into the solutions of the aforementioned Diophantine equation, supported by numerical examples and graphical representations.

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1. Introduction

Number theory, a key branch of mathematics, investigates the properties of integers, particularly positive ones. These numbers were some of the first mathematical concepts developed by humans. Starting around 600 BC, Greek mathematicians like Pythagoras laid the groundwork for this field. Over the years, notable figures such as Euclid, Diophantus, Fermat, Euler, Gauss, Goldbach, and Ramanujan have significantly advanced number theory. Mathematician Kronecker noted that natural numbers are divinely created, implying that all other mathematical constructs are human inventions [1]-[4].

Diophantine equations, named after the ancient mathematician Diophantus, are equations that seek integer solutions and are a key focus in number theory due to their complexity and applications in cryptography. Researchers have explored various forms of these equations, uncovering connections between solutions and special numbers, such as polygonal and pyramidal numbers. This research provides valuable insights into the characteristics of integer solutions, encouraging further exploration in the field [5]-[35].

In this paper, we focus on a specific ternary Diophantine equation given by

$$pw^2 - (2p-1)wz + pz^2 = (k^2 + 4p - 1)q^r.$$

Our objective is to find integer solutions for this equation by varying the parameter r from 1 to 3. We develop a theorem to investigate integral solutions for cases where r is a natural number, applying specific methods for values of r in this range. Through this approach, we demonstrate that the above ternary Diophantine equation admits integer solutions for any natural number r, thus broadening our understanding of solution patterns in these types of equations and highlighting potential generalizations for further study.

Notations:

$T_{m,n} - n^{th}$ m-gonal number

$P(n) - n^{th}$ pronic number

2. The Methodology: An Approach to Integer Equation Solving

This section presents a step-by-step method for solving integer equations in number theory. It simplifies finding integer solutions making it a useful tool for researchers. The following theorem illustrates this method.

Note 1: The Transformations $x+y$ and $x-y$ are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$. In this case, the transformations $x+y$ and $x-y$ satisfies conditions of linear transformations.

Theorem 1: *For any integers $p,\ k \neq 0$, and any positive integer r, the equation*

$$pw^2-(2p-1)wz+pz^2=(k^2+4p-1)q^r$$

admits a variety of non-zero distinct integer solutions. This theorem establishes a connection between integer solutions to the given equation and certain transformations and factorization techniques.

Proof: We consider the variables w, z, q, and r to show that the equation can consistently yield integral solutions by choosing suitable values. In equation

$$pw^2-(2p-1)wz+pz^2=(k^2+4p-1)q^r \tag{1}$$

By examining the equation after applying the following transformations:

$$w=x+y,\quad z=x-y \tag{2}$$

This transformation simplifies the original equation to:

$$x^2+(4p-1)y^2=(k^2+4p-1)q^r \tag{3}$$

We will conduct our analysis in several distinct cases, each corresponding to different values of the positive integer r.

Case 1: $r=1$

The equation (1) becomes

$$pw^2-(2p-1)wz+pz^2=(k^2+4p-1)q \tag{4}$$

In this instance, we have:

$$x^2+(4p-1)y^2=(k^2+4p-1)q \tag{5}$$

by substituting the transformation. We assume that q can be expressed as a sum of squares:

$$q=m^2+(4p-1)n^2 \tag{6}$$

where m and n are non-zero integers. Furthermore, we recognize that (k^2+4p-1) can be expressed as the product of two complex conjugates:

$$(k^2+4p-1)=(k+i\sqrt{4p-1})(k-i\sqrt{4p-1}) \tag{7}$$

Substituting (6) and (7) into (5) and utilizing the factorization method, we obtain:

$$(x+i\sqrt{4p-1}y)(x-i\sqrt{4p-1}y)=(k+i\sqrt{4p-1})$$
$$(k-i\sqrt{4p-1})(m+i\sqrt{4p-1}n)(m-i\sqrt{4p-1}n) \tag{8}$$

By equating like expressions and comparing the real and imaginary parts, we deduce the following equation:

$$x=km-4np+n$$
$$y=kn+m$$

Substituting these equations back into the original transformation equations, we can derive the integral solutions:

$$w = w(p, m, n, k) = (k + 1)m - (4p - k - 1)n$$
$$z = z(p, m, n, k) = (k - 1)m - (4p + k - 1)n$$
$$q = q(p, m, n) = m^2 + (4p - 1)n^2$$

Here algebraic transformations and substitutions are used to simplify and solve equations for integer solutions of parameters p, m, n, and k. By applying transformations like $w = x + y$ and $z = x - y$, we reduce equations into manageable forms for each r, yielding integer solutions for w, z, and q. For instance, Cases 1 and 2 employ factorizations and substitutions that match specific r-values, while the Cross-Multiplication Method reorganizes equations into systems for further simplification. Case 3, with $r = 3$, involves higher-degree terms but remains solvable through expanded factorization, and similar methods apply as r increases, albeit with added complexity. Each distinct r yields unique solution sets, and although graphical generalization becomes difficult for large r, the iterative approach consistently derives expressions for w, z, and q across cases. These methods offer a systematic path to locate non-zero integer solutions up to any positive integer degree.

Properties of Technique 1 ($r = 1$)	**Properties of Technique 1 ($r = 2$)**	**Properties of Technique 2 ($r = 2$)**
• $2w(1, m, m, k) + (k - 1)z(1, m, m, k) = 0$	• $-4k[w(1, m, m, k) + z(1, m, m, k)]$ is a perfect square number.	• $w(p, A, B, k) + z(p, A, B, k) \equiv 0 \pmod 2$
• $w(p, m, 1,1) + q(p, m, 1) = T_{4,(m+1)}$	• $w(p, m, 1,1) - z(p, m, 1,1) + 2q(p, m, 1) = 4p(m)$	• $z(p, A, B, k) - w(p, A, B, k) - 2q(p, A, B, k) \equiv 0(\text{mod}2^2)$
• $z(p, m, 1,1) + q(p, m, 1) \equiv 0 \ (\text{mod}(m - 1))$	• $w(1, m, m, k) + 2q(1, m, n) + 6T_{4,m}$ is a nasty number.	• $z(1, A, A, 1) = 0$
• $w(1, m, m, k) - (k - 1)z(1, m, m, k) \equiv 0(\text{mod}(k - 1))$	• $w(1, m, m, k) + 2q(1, m, n) \equiv 0 \pmod 3$	• $w(1, A, A, 1) + q(1, A, A) \equiv 0 \ (\text{mod } A^2)$
	• $2w(1, m, n, 1) - z(1, m, n, 1) + 4q(1, m, n) \equiv 0 \ (\text{mod } m^2)$	• $w(1, A, B, 1) - 2z(1, A, B, 1) + 4q(1, A, B)$ is a nasty number.
		• $w(1, A, A, 1) - q(1, A, A)$ is a perfect square number.
		• $z(p, A, 1,1) - w(p, A, 1,1) + 2q(p, A, 1) \equiv 0 \ (\text{mod } 4A)$
Properties of Technique 3 ($r = 2$)	**Properties of Technique 4 ($r = 2$)**	**Properties of Technique 5 ($r = 2$)**
• $z(1, B, B, 1) = 8T_{4,b}$	• $q(p, 1, B) - B^2 \equiv 0 \ (mod(4p - 1))$	• $q(1, N, N)$ is a perfect powers number.
• $6q(1, B, B)$ is a nasty number.	• $6[q(p, 1, B) - (4p - 1)]$ is a nasty number.	• $6[q(1, N, N)]$ is a nasty number.
• $z(1, A, B, 1) - 2w(1, A, B, 1) + 4q(1, A, B) \equiv 0 \ (mod\ B^2)$	• $w(1, A, B, 1) \equiv 0 \ (mod\ B)$	• $w(1, M, M, 1)$ is a powerful number.
• $z(1, B, B, 1) + w(1, B, B, 1) + 2q(1, B, B)$ is a perfect square number.	• $q(p, 1, B) - (4p - 1)$ is a perfect square number.	• $w(1, M, N, 1) \equiv 0 \ (mod\ MN)$
	• $w(1, A, A, 1) - q(1, A, A) \equiv 0 \ (mod\ 4A)$	• $6[z(1, M, M, 1)]$ is a nasty number.

		• $q(p, M, N) \equiv 0 \ (mod\ (2^2p^2))$ • $2z(1, M, N, 1) - w(1, M, N, 1) + 4q(1, M, N) \equiv 0 \ (mod\ 13)$
Properties of Technique ($r = 3$) • $q(p, m, n) - (4p - 1)n^2$ is a perfect powers number. • $6[q(p, m, n) - (4p - 1)n^2]$ is a nasty number. • $w(1, m, m, 1) \equiv 0 \ (mod\ m^3)$ • $z(1, m, n, 1) \equiv 0 \ (mod\ 3)$ • $z(1, m, m, 1) = 0$ • $2w(1, m, n, 1) - z(1, m, n, 1) \equiv 0 \ (mod\ 4m)$ • $q(p, m, n) - (4p - 1)n^2 \equiv 0 \ (mod\ m^2)$ • $q(p, m, n) - m^2 \equiv 0 \ (mod\ n^2)$		

2.1 *Numerical Illustrations with Graphical Insights*

In this section, we examine the relationship between numerical examples and visual representations, utilizing various values for parameters k, p, A, B, m, and M in terms of n and N for each technique. For different values of the exponent r, distinct parameter sets are applied. When $r = 1$, p is set to 25, k to 15, and n to $-500m$, with N and B unspecified. For $r = 2$, Technique 1 assigns $p = 20$, $k = 10$, and $n = -50m$; Technique 2 specifies $p = 40$, $k = 30$, and $B = 25A$; Technique 3 defines $p = 50$, $k = 40$, and $B = 15A$; Technique 4 sets $p = 60$, $k = 70$, and $B = 10A$; and Technique 5 provides $p = 2$ and $k = 10$ with $N = 5M$. Lastly, for $r = 3$, p is set to 5, k to 10, and n to m, while N and B are left unspecified. This systematic selection of values establishes a structured basis for identifying trends and illustrating data patterns effectively.

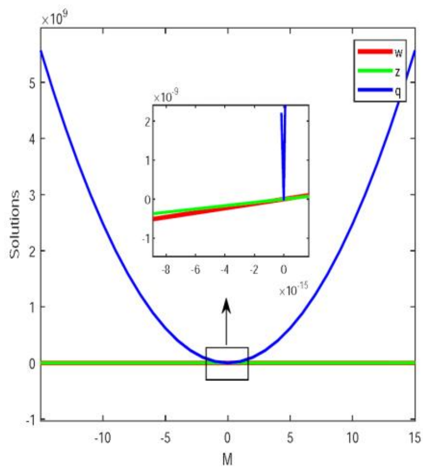

Figure 1
Graphical Presentation of technique 1: case 1.

Table 1
Numerical Solutions of the Ternary Diophantine Equation
for $-15 \leq a \leq 15$ when $r = 1$

m	w	z	q
–10	–420160	–570140	2475000100
–9	–378144	–513126	2004750081
–8	–336128	–456112	1584000064
–7	–294112	–399098	1212750049
–6	–252096	–342084	891000036
–5	–210080	–285070	618750025
–4	–168064	–228056	396000016
–3	–126048	–171042	222750009
–2	–84032	–114028	99000004
-1	–42016	–57014	24750001
0	0	0	0
1	42016	57014	24750001
2	84032	114028	99000004
3	126048	171042	222750009
4	168064	228056	396000016
5	210080	285070	618750025
6	252096	342084	891000036
7	294112	399098	1212750049
8	336128	456112	1584000064
9	378144	513126	2004750081
10	420160	570140	2475000100

Table 1 and **Figure 1** provided for Technique 1 illustrate the numerical solutions of the ternary Diophantine equation for a range of values when $r = 1$. In a similar manner, one can generate the graphs and tables for other techniques by varying the parameters as specified for each case. By adjusting the values of p, m, n, k, and other parameters according to the different techniques, similar graphical representations and numerical solutions can be obtained for each technique, allowing for a consistent comparison across various scenarios. This approach showcases the flexibility of the method in handling different cases.

3. Comparison to Recent Works

To contextualize our findings, we provide a comparative analysis with recent studies, as summarized in **Table 2**. This section highlights key similarities and differences, underscoring the contributions and advantages of our approach.

Table 2
Comparison of this work with other recent studies

Aspect	This Article	Other Recent Works
Scope of Equations Studied	Examines multiple types of Diophantine equations, including linear, quadratic, and cubic equations.	Focuses on specific classes of equations (e.g., quadratic or higher-degree), such as [6] (quadratic) and [12] (cubic).

Solution Techniques	Uses substitution-based methods to derive integer solutions for w, z, and q in terms of m.	Methods include modular arithmetic (e.g., [5]) and lattice-based or geometric approaches (e.g., [21]).
Graphical Representation	Provides graphical representations of the solutions, offering structural insights.	Often omitted, with many works focusing on purely algebraic solutions, such as [32].
Future Research Directions	Suggests further exploration of different manifestations and properties of Diophantine equations with a focus on theoretical insights.	Papers like [24] propose extensions of Diophantine equations specifically in cryptographic or algorithmic contexts.

4. Conclusions

This article covers several types of equations, including linear, quadratic, and cubic equations. It explores various methods and techniques for identifying multiple patterns of non-zero, distinct integer solutions. By substituting the value of n in terms of cm, where c is a fixed integer and k and p are fixed integers, we derive expressions for w, z and q in terms of m. By varying the value of m over integers, we obtain the corresponding solutions. Additionally, we provide a graphical representation of the solutions for a given numerical example. The study gives important information about Diophantine equations. It could be used in cryptography and number theory. Finally, it proposes potential avenues for future research to investigate different manifestations of Diophantine equations and their inherent properties.

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