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Double power domination in graphs

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Abstract

Monitoring an electrical network using minimal measurement devices is a problem that led to the study of power domination in graphs. The power domination number of paths or cycles is one. As a result, a single-phase measuring unit (PMU) keeps track of all the vertices of a path or cycle in circuits. When the single connected PMU in a path or circle is not functioning effectively under unexpected conditions, then the observability of the electrical

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network diminishes. To overcome this problem, Double Power Domination was introduced. A subset $\mathcal{S}$ of the vertex set of a graph G is called a double power dominating set if every vertex in G is monitored at least twice by the vertices in $\mathcal{S}$. The minimal cardinality of all double power dominating sets $\mathcal{S}$ is known as the double power domination number and is symbolized by $\gamma_{DP}(G)$. In this article, we have proposed an algorithm to obtain the double power domination number for any graph. We have related the double power domination number and power domination number. We have demonstrated that the difference between the power domination number and the double power domination number is minimal.

Subject Classification: *05C69, 05C12.*

Keywords: *Double power domination, Power domination, Feasible neighborhood set, Feasible region, Propagation depth, PMU placement.*

1. Introduction

Power domination is one of the most powerful tools for analyzing the networks. Power domination is a form of domination that occurs when monitoring electric power grids. In 1993, Baldwin et al. [1] propose the idea of power domination. Models of the electrical power grid are used in factories, hospitals, apartments, etc. A power network is constructed by vertices and edges that link the vertices together and includes a number of lodes and a number of generators. The generators produce electricity, and a number of loads receive the electricity. An electrical power grid generates, distributes, and consumes electricity. Electrical power companies must continuously monitor their infrastructure. They use Phase Measuring Units (PMUs) strategically placed in certain sports to perform these measurements. Monitoring an electric power supply system with the least number of measurement devices is one of the main challenging tasks for engineers. One of the primary objectives of the power domination problems is to control an electricity distribution system with the fewest possible measurement devices.

In 2002, Haynes et al. [2] defined power domination in terms of graph theory, and the power domination mathematical formula was derived from that paper. Let $G = (V, E)$ represent an electrical grid, with a vertex representing an electrical node (ann electrical bus that connects loads,generators,s, and transmission lines) and an edge represent an energy transmission cables connecting two electrical power nodes. A PMU measures state variables such as voltage magnitude at loads and generator equipment phase measurement of the vertex. That vertex is placed in its incident edges and their end vertices (these vertices and edges are said to be observed). The additional observation guidelines are given below: [3]

1. Any vertices that are incident to an edge currently being observed are observed.
2. Any edge connecting two observed vertices is considered observed(By Ohm's law).

3. If one end vertex of an observed edge is observed, a second vertex is observed likewise (By Ohm's law).
4. According to Kirchhoff's law, if a vertex is connected to $k \geq 1$ edges and $k-1$ of these edges are monitored, then all k edges will be monitored. Haynes et al. [2] modified the concept of power domination by introducing propagation rules regarding the vertices and edges of the graphs. This paper uses a simplified version of the propagation rules equivalent to the original[4].

   For a set $S \subseteq V$ in a graph $G = (V, E)$ define $PD(S) = V$ recursively:

   (1) Domination step: $PD(S) = N[S]$.

   (2) Propagation step: While there exists $v \in PD(S)$ such that

   $$|N(v)/PD(S)| = 1: PD(S) = PD(S) \cup N(v).$$

A set $S \subseteq V$ is a power dominating set of a graph G if $PD(S) = V(G)$. The cardinality of a minimum power dominating set is known as the power domination number of G represented by $\gamma_P(G)$. Power domination numbers for paths and cycles were obtained in [2]. Power domination has been studied extensively [2, 5, 6]. In particular, the closed formulas for the power domination number of certain families of graphs, such as Grid graphs [5, 7], Block graphs [8, 9], Generalized power domination [10, 11], Corona product [12], upper eccentric domination [13] and Power domination numbers of graph products [14, 15] have been found. The complexities of power domination were discussed in [16, 17]. Power domination is NP-complete for arbitrary graphs. Moreover, different variants of power domination were discussed by different authors. For example, Connected power domination [18, 19], Global power domination [20, 21], Restricted power domination [4], and K-Fault tolerant power domination [22, 23], Failed power domination [24] and Double power domination [3].

The concept of k-tuple domination was introduced in Harary and Haynes [25] and Haynes, Hedetniemi, and Slater [26] and it was extended to 2-tuple domination. A subset $D \subseteq V(G)$ is said to be a k-tuple dominating set if $|D \cup N[v]| \geq k$ for every vertex $v \in V(G)$ and also defined as $\gamma_{\times k}(G) = \min |D|; D$ is a k-tuple dominating set. Here, $k = 1$ is a dominating set denoted as $\gamma(G)$, and $k = 2$ is a double dominating set. A double dominating set with minimum cardinality is a double domination number, $\gamma_{\times 2}(G) = \min |D|; D$ is a 2-tuple dominating set. For more detail refer to [27, 28].

Electrical power networks have two types of observability, namely, security-based and economic-based. The work on economic-based observability focuses on the minimum number of PMUs to monitor the complete networks, namely power domination. The security-based observability was first introduced by Bashian, Assili, and Anvari-Moghaddam [29] and focuses on the minimum number of PMUs by observing the network and checking the network's efficiency. Since security-based observation is more efficient than economic-based observability, recent literature focuses on security-based observability rather than economics-based observability. The present work focuses on security-based observation explained from a mathematical direction.

An optimal PMU placement (OPP) model was developed by Lu et al. [30] that reduces the depth of Zero Injection (ZI) observations to adjust for the reliability of ZI observations. In the electrical power network represented as a simple graph, Saravanan et al.[31] identified PMU locations in electrical bus circuits through power domination and showcased the stability and efficient use of PMU placement. Momoh, Xia, and Boswell [32] provides an overview of the issue, explains a technique for tracking voltage stability, and provides examples of monitoring the weak voltage nodes. Guo, Liao, and Chu [33] proposed the idea of observability propagation depth (OPD). The propagation depth is defined as the number of steps required for a power dominating set to cover the entire graph. It is easy to see that if the observability propagation depth is two then the observability propagation depth decreases and the voltage stability increases.

Graph power domination gives a solution to the PMUs placement problem. Our previous work is, a contemporary technique to place the PMUs in an electrical power network using graph theory that paper discussed the following results[3], Double power domination is a graph theory method used to find the minimum number of PMUs required to ensure that every node in an electrical power network is monitored twice. This study [3] addressed the difference between double power domination and power domination based on the IEEE-14 bus. A comparison study of double power domination and power domination[3] in the electric bus system is included and concludes that double power domination is more efficient and stable than power domination. If one of the PMUs is not working correctly, the other PMUs monitor the PMU-placed node and its neighboring nodes. The double power domination number is the smallest set of PMUs required to ensure each node is observed two times. One can refer to the basic definitions and notations of power domination and its variants from [34-36].

2. Double Power Domination

Let $G=(V,E)$ be a graph and $S \subseteq V$, then S is said to be a **Double power dominating set (DPDS)** if all the vertices of G are monitored at least twice by the vertices of S. The minimal cardinality of S is called the double power domination number and is denoted by $\gamma_{DP}(G)$. *In double power domination, the propagation depth for the PMU is to observe any vertex to enhance the observability efficiency. The observability propagation depth of a node is the depth away from its closest PMUs. The power domination with propagation depth is strictly restricted to two, is known as the* ***Propagation depth restricted to two****, and is denoted as* $\gamma_{(p\leq 2)}(G)$.

An example for double power domination is shown in Fig. 1. In this example, the maximum observability propagation depth is two. The two vertices B and C are the vertices corresponding to PMU location, $S_{(p=0)} = \{B, C\}$. The adjacent vertices at a propagation depth of one are represented by $S_{(p=1)} = \{A, D, E, G\}$. The propagation depth is restricted to two, which means the distance between the monitoring vertex and the remaining vertices is two, $S_{(p=2)} = \{F\}$. The union of all propagation depth values equals the number of vertices of a graph. Therefore $S_{(p=0)} \cup S_{(p=1)} \cup S_{(p=2)} = V(G)$.

Let $V = \{v_1, v_2, v_3 \ldots\ldots v_n\}$ be the vertex set of the graph $G = (V, E)$ and $\mathcal{S} \subseteq V$. Let F_{v_i} be the feasible neighboured set for $v_i \in \mathcal{S}$. Then the set $X_{(i,j)} = F_{v_i} \cap F_{v_j}$ is called the double feasible neighboured set and will have the G vertices observed by the PMUs placed at v_i and v_j. Thus every vertices in $X_{(i,j)}$ is observed twice. For every $v_i, v_j \in \mathcal{S}$, if $\cup\ X_{(i,j)} = V$, then $\mathcal{S}$ is called the double power dominating set of G. The two vertices B and C are the PMU-placed vertices and the feasible regions are $F_B = \{B, A, C, D, G, E, F\}$ and $F_C = \{B, C, G, E, F, D, A\}$. Therefore, $F_B \cap F_C = X_{(B,C)} = V(G)$. The ultimate analysis finds out that the union of restricted propagation depth of two equals the union of all feasible neighborhood intersections.

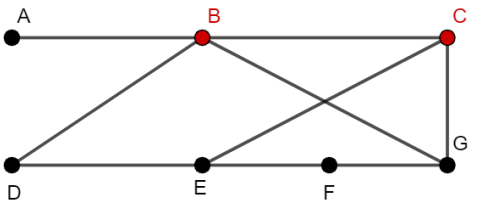


Figure 1
Example of double power domination

Example 2.1: Consider the set $\{B, C\}$. Here, $F_B = \{B, A, D, C, G, F, E\}$ and $F_C = \{C, B, G, F, E, D, A\}$ these corresponding regions are known as the feasible neighboured regions.

It is clear that $X = F_B \cap F_C = \{A, B, C, D, E, F, G\} = V(G)$, that is, every vertex in a graph G is monitored by at least twice the vertices in a set $\{B, C\}$. Therefore $\{B, C\}$ is a double power dominating set and hence $\gamma_{DP}(G) = 2$.

Example 2.2: Double power domination in IEEE-14 Bus

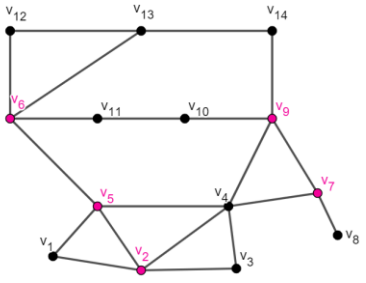


Figure 2
Double power domination with in the 14-IEEE Bus

The graph model for14-IEEE Bus and its double power domination is showin in Fig. 2. Consider the set $B_1 = \{v_2, v_5, v_6, v_9, v_7\}$. The vertex v_2

monitor the vertices in $F_{v_2} = \{v_5, v_2, v_1, v_3, v_6, v_4\}$, $S_{(p=0)} \Rightarrow \{v_2\}$, $S_{(p=1)} \Rightarrow \{v_1, v_3, v_4, v_5\}$ and the propagation depth one condition gives $S_{(p=2)} \Rightarrow \{v_6\}$.

Similarly, the vertices v_5, v_6, v_7 and v_9 monitor the sets respectively, $F_{(v_5)} = \{v_6, v_2, v_4, v_1, v_5, v_3\}, F_{(v_6)} = \{v_6, v_5, v_{12}, v_{11}, v_{14}, v_{10}, v_{13}\}$, $F_{(v_7)} = \{v_4, v_7, v_8, v_9\}$ and $F_{(v_9)} = \{v_4, v_{12}, v_{13}, v_{14}, v_{10}, v_9, v_{11}, v_7, v_8\}$.

$F_{(v_2)} \cap F_{(v_5)} = \{v_1, v_2, v_4, v_5, v_6, v_3\}$ $\quad$ $F_{(v_2)} \cap F_{(v_6)} = \{v_5, v_6\}$
$F_{(v_2)} \cap F_{(v_9)} = \{v_4\}$ $\quad$ $F_{(v_2)} \cap F_{(v_7)} = \{v_4\}$
$F_{(v_5)} \cap F_{(v_6)} = \{v_5, v_6\}$ $\quad$ $F_{(v_5)} \cap F_{(v_9)} = \{v_4\}$
$F_{(v_5)} \cap F_{(v_7)} = \{v_4\}$ $\quad$ $F_{(v_6)} \cap F_{(v_9)} = \{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$
$F_{(v_6)} \cap F_{(v_7)} = \phi$ $\quad$ $F_{(v_9)} \cap F_{(v_7)} = \{v_4, v_7, v_8, v_9\}$

Therefore $\cup X_{i,j} = V(G)$. Hence the given set is a double power dominating set that is $B_1 = \{v_2, v_5, v_6, v_9, v_7\}$ and $\gamma_{DP}(G) = 5$.

Double power dominating sets in a network are not necessarily unique. For example, It is easy to see that the minimal double power dominating set for the IEEE-14 Bus architecture are $B_1 = \{v_2, v_5, v_6, v_9, v_7\}$, $B_2 = \{v_7, v_4, v_{11}, v_6, v_2\}$, $B_3 = \{v_5, v_4, v_7, v_{11}, v_9\}$.

It is easy to see that (i). Every double power dominating set is always a power dominating set. (ii). For any graph G, where $G \in \{K_n, W_n, K_{(n,m)}, B_{(n,m)}, F_n\}$,$\gamma_{DP}(G) = 2$.

Theorem 2.3: *The minimum double power domination number in any graph is two.*

Proof: Let $G = (V, E)$ be a connected graph and $S \subset V$. Consider $V = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices of a graph G and its feasible neighbourhood regions are $\{F_{b_1}, F_{b_2}, F_{b_3}, \ldots, F_{b_n}\}$. Let b_k be any of the vertex in G. From the definition of double power domination $b_k \in F_{b_i} \cap F_{b_j}$ Here F_{b_i} and F_{b_j} are the feasible regions included in b_k. Clearly, b_k is a double feasible neighborhood set member and will have the vertices in G observed by the PMUs at b_i and b_j. Therefore $X_{i,j} = F_{b_i} \cap F_{b_j}$ and this implies that $b_k \in \bigcup_{i,j=1}^{k} X_{i,j} = V(G)$. Therefore, b_k serves as a double power dominating set with a propagation depth limited to two, and the minimum cardinality of a double power dominating set in any graph is two.

Theorem 2.4: *For any cycle* C_n, $\gamma_{DP}(C_n) = \left\lceil \frac{2n}{5} \right\rceil$.

Proof: Let $V = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices of cycle C_n. Consider $S = \{b_3, b_8, b_{13}, \ldots, b_{5k+3}\}$ where $k = 0,1,2,\ldots.$ be the PMU placed vertices in the cycle C_n. It is easy to see that $S_{(p=0)} = \{b_i : i \equiv 3(mod5)\}$, $S_{(p=1)} = \{b_i : i \equiv 2(mod5)$ and $i \equiv 4(mod5)\}$ The propagation depth value is one, and $S_{(p=2)} = \{b_i : i \equiv 1(mod5)$ and $i \equiv 0(mod5)\}$. Therefore $S_{(p=0)} \cup S_{(p=1)} \cup$

$S_{(p=2)} = V(G)$. Here the maximum propagation depth of every vertex in graph G is two. That is, the entire graph is monitored by S.

Let $S = \{b_3, b_8, b_{13}, \ldots, b_{5k+3}\}$, where $k = 0,1,2,\ldots$ be the set of PMUs placed vertices of G, where $\{F_{b_3}, F_{b_8}, F_{b_13}, \ldots, F_{(5k+3)}\}$ is the feasible neighborhood set for the set of vertices of S. Hence F_{bi} is a cluster with cluster head bi, which means the vertex bi monitored set of vertices is F_{b_i}. Consider $X_{i,j} = F_{b_i} \cap F_{b_j}$ for all $i, j = 1$ to k, then

$\bigcup_{i,j=1}^{k} X_{i,j} = V(G)$. Hence, S is a double power dominating set with propagation depth restricted to two. It is clear that the union of propagation depth restricted to two equals the union of all feasible neighborhood intersections.

Claim: $|S| = \left\lceil \frac{2n}{5} \right\rceil, n = 3,4,5,\ldots$

Case (i) Suppose $|S| \leq \left\lceil \frac{2n}{5} \right\rceil$.

Consider $\{b_1, b_2, \ldots, b_n\}$ be the vertices of a cycle C_n and $S \subset V$. Let b_i be one of the vertex in S, $deg(b_i) = 2$ for all i. Then assume that $S - b_i$ is a double power dominating set. $b_i \notin N(S) \cup S$. Again consider two feasible regions F_{u_i}, F_{u_j}. The vertex b_i also lies in these two feasible regions and b_i is monitored twice. Otherwise, b_i monitoring twice is impossible. Therefore $|S| \leq \left\lceil \frac{n}{2} \right\rceil$

Case (ii) Suppose $|S| \geq \left\lceil \frac{2n}{5} \right\rceil$.

Let S be a set for C_n and $\Delta(C_n) = 2$. Let $deg(b_i) = 2$ for all $b_i \in S$ because every vertex of b_i lies in a cycle C_n. Therefore $b_i \notin S$, [Now suppose that $\gamma_{DP}(C_n) \leq \left\lceil \frac{2n}{5} \right\rceil$.] There is a vertex $b_i \in C_n$ such that $d(b_i, N[S]) = 2$. Here $b_i \notin N(S) \cup S$, $b_j \in N[S]$ there is a path from (b_i, b_j), this is shown in Fig. 3. Similarly, $b_k \in N[S]$ and has a path from (b_i, b_k). These conditions are satisfied $u_i, u_j \varepsilon S$, F_{u_i} and F_{u_j} are any of the two clusters. $F_{u_i} \cap F_{u_j} = b_i$ that means b_i monitored at least twice by the vertices of S. Otherwise, b_i monitoring is not possible. (because $S - max\{S - (v)\} \prec 2$.) Therefore S is not a DPDS.

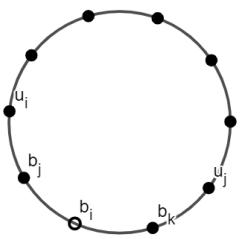


Figure 3
Case (ii) example

Obviously, it is evident that $|S| = \left\lceil \frac{2n}{5} \right\rceil$ for all n= 3,4,5.....

Hence Proved.

Theorem 2.5: $\gamma_{DP}(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & if\ n \not\equiv 2\ (mod\ 4) \\ \left\lceil \frac{n}{2} \right\rceil + 1 & if\ n \equiv 2\ (mod\ 4). \end{cases}$

Proof: Let $G = (V, E)$ be a graph with $V = \{b_1, b_2, \ldots . b_n\}$ be the vertices of P_n. Define a set S where $S \subset V$ is the PMU locating vertices in the path P_n.

Consider $U = \{F_{b_1}, F_{b_2}, F_{b_3} \ldots\ldots\ldots F_{b_k}\}$ be the feasible neighboured set of vertices monitored by $S = \{b_1, b_2, b_3, \ldots . . b_k\}$.

If

$$F_{b_1} \cap F_{b_2} = X_{(1,2)}$$
$$F_{b_2} \cap F_{b_3} = X_{(2,3)}$$
$$F_{b_3} \cap F_{b_4} = X_{(3,4)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots..$$
$$F_{b_{(k-1)}} \cap F_{b_k} = X_{((k-1),k)}$$

For every vertex $b_i \varepsilon V$; the given vertex is monitored at least twice in the vertices of S. Thus every vertex in $X_{i,j}$ is observed twice and it is a finite set.

$$\cup X_{i,j} = V(G)$$

Each vertex is a member of at least two clusters in a path, and then the cluster heads are said to be double power dominating sets.

Claim: $|S| = \left\lceil \frac{n}{2} \right\rceil ; n \not\equiv 2 \pmod 4$

Case(i)

$|S| \leq \left\lceil \frac{n}{2} \right\rceil$ where $n \not\equiv 2 \pmod 4$ Let $\{b_1, b_2, \ldots . b_n\}$ be the vertices of a path P_n and $S \subset V$. Again b_i be one of the vertex in S, $deg(b_i) = 2$. Then assume that $S - b_i$ is a double power dominating set. $b_i \notin N(S) \cup S$. Again consider two feasible regions $F_{b_{(i-2)}}, F_{b_{(i+2)}}$. Also, b_i is a member of two of the feasible regions. These two vertices are added to the set S, then b_i is monitored twice, this is shown in Fig. 4. Otherwise b_i monitored twice is not possible. Therefore $|S| \leq \left\lceil \frac{n}{2} \right\rceil$.

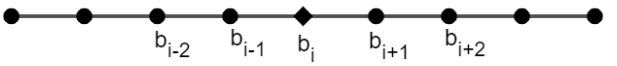


Figure 4
Case (i) example

If $deg(b_i) = 1$. Then assume that $S - b_i$ is a double power dominating set. $b_i \notin S$. Again consider two feasible region $F_{b_{(i+1)}}, F_{b_{(i+2)}}$. b_i also lies these

two feasible regions. Hence b_i is a DPDS. An example is shown in Fig. 5. Otherwise, monitoring is not possible because the propagation depth is two in double power domination. Therefore $|S| \leq \left\lceil \frac{n}{2} \right\rceil$.

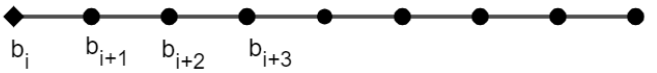


Figure 5
Case (i) example

Case (ii)

$|S| \geq \left\lceil \frac{n}{2} \right\rceil$ where $n \not\equiv 2 \pmod 4$

Let $\{b_1, b_2, \ldots . b_n\}$ be the vertices of a path P_n and $S = \{b_1, b_2, \ldots . b_k\}$ is a finite set, and its feasible region is $\{F_{b_1}, F_{b_2}, \ldots . . F_{b_k}\}$. Consider another vertex $b_{(k+1)}$ and its feasible region is $F_{b(k+1)}$. $\therefore$ $S = \{b_1, b_2, \ldots . b_k, b_{k+1}\}$ is not a minimal DPDS. $F_{b_i} \cap F_{b_j} = X_{(i,j)}$; for all $i, j = 1,2,3, \ldots k+1$. Therefore $\bigcap X_{(i,j)} = V(G)$ which means the set S is monitored all the vertices at least twice. So S is a double power dominating set. $|S| \geq \left\lceil \frac{n}{2} \right\rceil$.

From case (i) and case (ii) $|S| = \left\lceil \frac{n}{2} \right\rceil ; n \not\equiv 2 \pmod 4$.

Hence proved.

Theorem 2.6: *For any graph* $1 \leq \gamma_P(G) \leq \gamma_{DP}(G) \leq \gamma(G) \leq n$.

Proof: Consider a connected graph $G = (V, E)$ its vertex set is $V = \{b_1, b_2, b_3, \ldots, b_n\}$ and edge sets $E = \{e_1, e_2, e_3, \ldots, e_n\}$. Let D be a dominating set of G, and is represented as $\gamma(G)$. Haynes et al. [26] gave the lower bound and upper bound of $\gamma(G)$. In any graph of order n, the domination number $\gamma(G)$ is between 1 and n, with both limits being attainable. For example, the complete graph K_n; $\gamma(K_n) = 1$ and $\gamma(\overline{K}_n) = n$. Again Liao and Lee [37] demonstrated that the lower bound found and the upper bound of the power dominating set $1 \leq \gamma_P(G) \leq \gamma(G)$. Based on the above findings, we conclude that

$$1 \leq \gamma_p(G) \leq \gamma(G) \leq n \tag{1}$$

The double power dominating set in a graph G is denoted by $\gamma_{DP}(G)$. It is easy to see that the minimum double power domination number in any graph is two. Also, every double power dominating set is always a power domination se. From (1) it is easy to see that $2 \leq \gamma_{DP}(G)$ and $\gamma_P(G) \leq \gamma_{DP}(G)$. Therefore, $2 \leq \gamma_P(G) \leq \gamma_{DP}(G)$.

In the restricted propagation depth concept used to represent the domination graph (Dominating set and its neighborhood) $S_{(p=0)} \cup S_{(p=1)} = V(G)$ Here the maximum propagation depth is one. However, in double power domination, the propagation depth is limited to two. Compared to domination, more of the number

of vertices monitored in a single vertex is possible. $S_{(p=0)} \cup S_{(p=1)} \cup S_{(p=2)} = V(G)$. We see that

$$\gamma(G) \geq \gamma_{DP}(G).$$

Therefore

$$2 \leq \gamma_P(G) \leq \gamma_{DP}(G) \leq \gamma(G). \tag{2}$$

Combining equations (1) and (2)

$$1 \leq \gamma_P(G) \leq \gamma_{DP}(G) \leq \gamma(G) \leq n.$$

Hence Proved.

Theorem 2.7: *If G is a double dominating graph, then G is also a double power dominating graph.*

Proof: Let $G = (V, E)$ for all $U \in V$ is a double dominating set. To prove that U is a double power dominating set.

Consider $U = \{v_1, v_2, v_3, \ldots, v_n\}$ be the double dominating set in graph G. Here $\{F_{v_1}, F_{v_2}, F_{v_3}, \ldots, F_{v_n}\}$ be the feasible region monitored by vertices $v_1, v_2, \ldots, v_n$.

$$F_{v_1} \cap F_{v_2} = X_{(1,2)}$$
$$F_{v_2} \cap F_{v_3} = X_{(2,3)}$$
$$F_{v_3} \cap F_{v_4} = X_{(3,4)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots..$$
$$F_{v_n} \cap F_{v_{n_1}} = X_{(n,n-1)}.$$

That is every vertices of $X_{i,j}$ is monitored twice by the feasible region of the double dominating set. Therefore $X_{i,j} = V(G)$ is a finite value. By the definition of $\gamma_{DP}(G)$, every vertex of V dominates at least twice in the vertex of U. So U is a double power dominating set.

The converse of the above theorem is not true.

Theorem 2.8: *A graph G of order $n \geq 2$ is connected if and only if it possesses at least one double power dominating set.*

Proof: Let $G = (V, E)$ be a connected graph of order $n \geq 2$. G contains at least two vertices. Using the statement of theorem 2.1 then G is a double power dominating set.

Conversely assume that G is a double power dominating set. A set S is monitored every vertex at least twice is the definition of a double power dominating set, and $S \subseteq G$. Therefore G is a connected graph.

Theorem 2.9: *Let G be a graph with a cut vertex b and if G_1 and G_2 two components of $G - b$ then $\gamma_{DP}(G) \leq \gamma_{DP}(G_1) + \gamma_{DP}(G_2)$.*

Proof: Let $G = (V, E)$ be a graph with vertex set $V = \{b_1, b_2, b_3, \ldots, b_n\}$ and $b_i \in V$ act as a cut vertex.

If $G - b_i$ is a disconnected graph, then b_i act as a cut vertex for all i. Two connected components $G_1, G_2 \subset G$. Here $b_j \varepsilon G_1$ and $b_k \varepsilon G_2$. In double power domination, the maximum restricted propagation depth is 2. Here b_j and b_k are the neighboring vertices of v_i, and its restricted propagation depth is one.

Case (i)

If $b_i \in \gamma_{DP}(G)$ (PMU placed on b_i)

If $G - b_i$ is a disconnected graph, b_j and b_k becomes disconnected. Consider b_q and b_p are another PMU placed vertices in G_1 and G_2. Therefore b_j is monitored by b_p and b_k is monitored by b_q, but the propagation depth is increased. Adding one or more PMUs in G_1 and G_2 to monitor properly the vertex b_j and b_k.

$$\gamma_{DP}(G) \leq \gamma_{DP}(G_1) + \gamma_{DP}(G_2).$$

Case (ii) If $b_i \notin \gamma_{DP}(G)$ (PMU not placed on b_i).

If $G - v_i$ is a vertex, remove the case that time the connected graph becomes disconnected. Two parts are here, (i) PMU placed b_j or b_k. Suppose PMU is placed in b_j, we are not able to monitor b_k. so add one more PMU similar to the other case.

$$\gamma_{DP}(G) \leq \gamma_{DP}(G_1) + \gamma_{DP}(G_2).$$

(ii) PMU monitoring other vertices as b_p and b_k monitoring the vertices b_j and b_k.increasing the number of PMUs in that situation.

Our observation is if G is a graph and b is a cut vertex. Then $G - b$ is a disconnected graph with n components $G_1, G_2, G_3, \ldots, G_n$ then $\gamma_{DP}(G) \leq \gamma_{DP}(G_1) + \gamma_{DP}(G_2) + \gamma_{DP}(G_3) + \ldots + \gamma_{DP}(G_n)$.

3. Algorithm of Double Power Domination

Here, we discuss the complexity and an algorithm to obtain the double power dominating set for a graph. [h] Double Power Domination Graph $G = (V, E)$ where $V = \{b_1, b_2, b_3, \ldots, b_n\}$ and $E = \{e_1, e_2, e_3, \ldots, e_m\}$ with $e = b_i b_j$ for all $i, j = 1,2,3, \ldots, m$; $S = \{b_1, b_2, \ldots, b_i\} \subseteq V(G)$. Here S is a maximum degree vertex. A double power dominating set

Step 1: Set $i = 1$ to n where n is the number of vertices in $V(G)$. ($i = 1$; $i \leq n$; $i++$) Condition is satisfied

Step 2: Place PMU on b_i.

Step 3: Compute the power dominating set $F(b_i)$, representing the vertices monitored by vertex b_i with propagation depth restricted to ≤ 2.

($j = 1$; $j \leq n$; $j++$) Condition is satisfied

Step 4: Place PMU on b_j.

Step 5: Compute power dominating set PDS on $F(b_j)$.

Step 6: Calculate $X_{i,j} = F(b_i) \cap F(b_j). \quad j \geq n$

Step 7: $\cup \quad X_{i,j} = V(G)$ It is a double power dominating set.
Otherwise, it is not a DPDS. $\quad i = 1 + i \;\; i \geq n$

Step 8: end

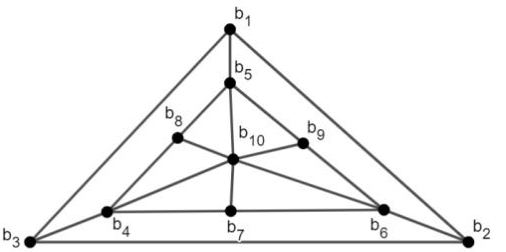


Figure 6
Example for the algorithm

Algorithm 1: Double Power Domination

Data: Graph $G = (V, E)$ where $V = \{b_1, b_2, b_3, \ldots, b_n\}$ and $E = \{e_1, e_2, e_3, \ldots, e_m\}$ with $e = b_i b_j$ for all $i, j = 1, 2, 3, \ldots, m$; $S = \{b_1, b_2, \ldots, b_i\} \subseteq V(G)$. Here S is a maximum degree vertex.
Result: A double power dominating set

Step 1: Set $i = 1$ to n where n is the number of vertices in $V(G)$. **while** *(*$i = 1$*;* $i \leq n$*;* $i++$*)* **do**
  if *Condition is satisfied* **then**
    Step 2: Place PMU on b_i.
    Step 3: Compute the power dominating set $F(b_i)$, representing the vertices monitored by vertex b_i with propagation depth restricted to ≤ 2.
    while *(*$j = 1$*;* $j \leq n$*;* $j++$*)* **do**
      if *Condition is satisfied* **then**
        Step 4: Place PMU on b_j.
        Step 5: Compute power dominating set PDS on $F(b_j)$.
        Step 6: Calculate $X_{i,j} = F(b_i) \cap F(b_j)$.
    if $j \geq n$ **then**
      Step 7: $\bigcup X_{i,j} = V(G) \implies$ It is a double power dominating set. Otherwise, it is not a DPDS.
  $i = 1 + i$ **if** $i \geq n$ **then**
    Step 8: end

Example 3.1: The graph G be represented in Fig.6, $V(G)$= $\{b_1, b_2, \ldots . b_{10}\}$ be the vertices and $e = b_i b_j$ for all $i, j = 1,2,3, \ldots ,10$ be the edges of the graph. Here vertex b_{10} degree is 6 and b_5, b_6, b_4 is of degree 4. These are the maximum degree vertex in this graph. Consider $S = \{b_{10}, b_4, b_5 b_6\} \subseteq V(G)$.

Step 1: For $i = 1,2,3, \ldots ,10$, identify the vertex with the maximum degree, which monitors more vertices. Choose $i = 10$.

Step 2: Place a PMU on b_{10}.

Step 3: Vertex b_{10} monitors vertices within a propagation depth of 2. The feasible region is:

$$F(b_{10}) = \{b_{10}, b_4, b_5, b_6, b_7, b_8, b_9, b_3, b_2, b_1\}.$$

Step 4: Consider $j = 1,2,3, \ldots ,9$. The vertex with the higher degree monitors the largest number of vertices. Consequently, the complete number of step count is reduced, so we only need to consider $j = 4,5,6$.

Step 5: The feasible region of b_4 is:

$$F(b_4) = \{b_4, b_3, b_7, b_{10}, b_8, b_6, b_5, b_9\}.$$

Step 6: Calculate the intersection:

$$X_{i,j} = F(b_{10}) \cap F(b_4) = \{b_4, b_3, b_7, b_{10}, b_8, b_6, b_5, b_9\}.$$

Step 7:

$$\cup\, X_{i,j} = F(b_{10}) \cap F(b_4) \neq V(G)$$

Since this is not a DPDS, go back to Step 4.

Step 4: consider the next highest degree vertex $j = 5$.

Step 5: The feasible region of b_5 is:

$$F(b_5) = \{b_5, b_1, b_8, b_{10}, b_9, b_4, b_6, b_7\}.$$

Step 6: Calculate the intersection:

$$X_{i,j} = F(b_{10}) \cap F(b_5) = \{b_5, b_1, b_8, b_{10}, b_9, b_4, b_6, b_7\}.$$

Step 7: Evaluate the union of $X_{i,j}$

$$\cup\, X_{i,j} = (F(b_{10}) \cap F(b_4)) \cup (F(b_{10}) \cap F(b_5)) \neq V(G).$$

Since this is not a DPDS, go back to Step 4.

Step 4: Next, consider the highest degree vertex $j = 6$.

Step 5: The feasible region of b_6 is:

$$F(b_6) = \{b_6, b_2, b_1, b_{10}, b_3, b_9, b_7, b_5, b_4, b_8\}.$$

Step 6: Calculate the intersection:

$$X_{i,j} = F(b_{10}) \cap F(b_6) = \{b_6, b_2, b_1, b_{10}, b_3, b_9, b_7, b_5, b_4, b_8\}.$$

Step 7: Evaluate the union of all $X_{i,j}$

$$\cup X_{i,j} = (F(b_{10}) \cap F(b_4)) \cup (F(b_{10}) \cap F(b_5)) \cup (F(b_{10}) \cap F(b_6)) = V(G).$$

Thus, the given set $S = \{b_{10}, b_4, b_5, b_6\}$ is a Double Power Dominating Set (DPDS).

The complexity of double power domination in arbitrary graphs is NP-complete. When the number of vertices is small, the problem is classified as NP-Problem. However, in larger networks, DPDS becomes a computationally challenging issue and is classified as NP-Hard. Nevertheless, it remains a non-deterministic polynomial-time problem, meaning that any proposed solution can be verified in polynomial time. Hence, the double power dominating set is an NP-complete problem. Exponential time algorithms are used we have to solve this problem for practical purposes, which operate with a complexity of 2^n. These algorithms are effective for assessing network properties such as connectivity, the presence of cycles, and other structural characteristics.

A Comparison between power domination, restricted propagation depth of two, and double power domination is shown in table 1. The table shows that double power domination and restricted propagation depth of two necessitate more vertices than power domination in regular graphs. The table also explains that the lower bound of double power domination is two, and its upper bound always depends on the nature of the graph. If the diameter of a graph is two, its maximum propagation depth is two. The ultimate finding is that double power domination offers greater efficiency compared to power domination in security-based solutions.

Table 1

A Comparison between power domination, restricted propagation depth of two, and double power domination

Graph (G)	Diameter of G	Power Dominating Set	Propagation Depth Restricted to 2	Double Power Dominating Set
P_n	$diam(P_n)$	1	$\lceil n/5 \rceil$: for $n \geq 3$	$\lceil n/2 \rceil$; $n \not\equiv 2 \pmod 4$ $\lceil n/2 \rceil + 1$; $n \equiv 2 \pmod 4$
K_n	1	1	1	2

Contd...

C_n	$n\backslash 2$	1	$\lceil n/5 \rceil$: for $n \geq 3$	$\lceil 2n/5 \rceil$: for $n \geq 3$
$K_{n,m}$	2	1	1	2
$K_{n_1,n_2,n_3,\ldots,m}$	2	1	2	2
$F_{1,n}: n \geq 3$	2	1	1	2
$W_n: n \geq 5$	2	1	1	2
Helen graph (H_n)	4	1	2	$n-1$
Butterfly graph	2	1	1	3
Diamond graph	2	1	1	2

Using the double power domination condition, we have to arrange the PMUs in electrical power networks to become more efficient than the power domination networks. When the PMUs count increases, the observability time decreases, the system becomes more efficient than the minimum number of PMUs. Minimum power domination is economic-based observability, and Double power domination is mainly concentrated on security-based observability. The $\gamma_{DP}(G)$-set is not unique in any of the networks. The least double power dominating sets of 14-IEEE bus is $B_1 = \{v_2, v_5, v_6, v_9, v_7\}$, $B_2 = \{v_7, v_4, v_{11}, v_6, v_2\}$, $B_3 = \{v_4, v_5, v_{11}, v_9, v_7\}$. Consider any of the sets where all the vertices of the 14-IEEE Bus are controlled at least twice, meaning all of the busâ€™s vertexes are controlled by two PMUs. The observability propagation depth of a node is the distance from its nearest PMUs. Here we have to take the maximum observability propagation depth is two.

4. Conclusion

In this research, a recently developed variant of power domination, called double power domination is derived and explained. The set of vertices of S is monitored at least twice each of the vertices of G. Then S is called a double power dominating set and is denoted as $\gamma_{DP}(G)$. A few elementary findings are also included. Some fundamental graphs illustrate a comparison study of power domination, restricted propagation depth, and double power domination.

The value of the double power domination does not fall within the range $\gamma_P(G) \leq \gamma_{DP}(G) \leq \gamma(G)$ as shown by the star graph, Peterson graph, etc. The double power domination value is always greater than $\gamma(G)$. Our future work will focus on that type of graph classification. The next milestone is to focus on characterizing the double power domination and simulation methods used to derive the efficiency of power networks.

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