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An exponential Diophantine equation $x^2 + 3^a 73^b = y^n$

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Abstract

This paper aims to identify all solutions in positive integers x and y (where $x, y \geq 1$), with nonnegative exponents a and b, and an integer $n \geq 3$, that satisfy the Diophantine equation $x^2 + 3^a 73^b = y^n$ under the condition x and y are coprime.

Subject Classification (2020): *11D61, 11D41, 11Y50.*

Keywords: *Integer solution, Diophantine equations, S-integers, Primitive divisor, Lucas sequence.*

1. Introduction

Consider the Diophantine equation of the form

$$x^2 + \mathcal{F} = y^n \tag{1.1}$$

where $x, y \geq 1, n \geq 3$ and $\mathcal{F}$ is a fixed positive integer. The initial discovery of positive integer solutions for the given equation dates back almost 17 decades [24]. It has been proven that the equation invariably has only a countable number of positive integer solutions [22]. Early investigations focused on Eq.

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(1.1) when $\mathcal{F} = \mathcal{f}_0$ is a constant integer [21,33,34]. In [17], Cohn provided a solution to (1.1), assuming that x, y are coprime and considering the parameter $\mathcal{F}$ lies between 1 to 100, leave out few values of $\mathcal{F}$. The work in [29] addressed additional $\mathcal{F}$ values within the same range, and the remaining values were addressed in [11]. Over the years, researchers have explored various instances involving the parameter $\mathcal{F} = \mathcal{p}^k$, where $\mathcal{p}$ is a specific prime number, as documented in several studies [1-4,6,16,28]. Furthermore, the case of a general prime number $\mathcal{p}$ has also been investigated in several works [7,9,23, 35,41].

Consider a set of primes $\mathcal{S} = \{\mathcal{p}_1, \mathcal{p}_2, \dots, \mathcal{p}_k\}$. Recent studies have focused on Equation (1.1), particularly when $\mathcal{F}$ is the product of the main powers $\mathcal{p}^k$, where $\mathcal{p} \in \mathcal{S}$ and k is a nonnegative integer [5,10,13,14,18-20,25-27,30-32,36-40]. Furthermore, in [42], the researchers are looking at natural number solutions to equation (1.1) in a more comprehensive framework, considering $\mathcal{F} = 2^a \mathcal{p}^{\mathrm{b}}$, where $\mathcal{p}$ is any odd prime.

The primary focus of this paper is the Diophantine equation given by

$$x^2 + 3^a 73^b = y^n \tag{1.2}$$

where, $n \geq 3, a, b$ are non-negative integers and $x, y \in \mathbb{N}$ with $gcd(x, y) = 1$.

Now, we proceed to establish the subsequent result.

Theorem 1.1: *Equation (1.2) has the following solutions:*

$$(x, y, n, a, b) = (1430,181,3,6,2), (82,55,3,7,1),$$
$$(4744,283,3,7,1), (782,127,3,9,1), (5570,319,3,9,1),$$
$$(46,13,3,4,0), (10,7,3,5,0), (532,67,3,5,1)$$

2. Preliminaries

Consider algebraic integers ξ and $\bar{\xi}$. A Lucas pair, represented as $(\xi, \bar{\xi})$ is defined by the conditions that $\frac{\xi}{\bar{\xi}}$ is not a root of unity and $\xi + \bar{\xi} \neq 0$ and $\xi\bar{\xi} \neq 0$ are relatively rational integers.

The sequences of Lucas numbers are defined in correspondence with any given Lucas pair $(\xi, \bar{\xi})$ as follows:

$$\mathfrak{Q}_n(\xi, \bar{\xi}) = \frac{\xi^n - \bar{\xi}^n}{\xi - \bar{\xi}}, \mathrm{n} = 0,1,2, \dots$$

In the context of Lucas sequences, the presence of primitive divisors for $\mathfrak{Q}_n(\xi, \bar{\xi})$ is an important idea. A primitive divisor of $\mathfrak{Q}_n(\xi, \bar{\xi})$ is a prime number q. Let $q | \mathfrak{Q}_n(\xi, \bar{\xi})$ and

$$q \nmid \left(\xi - \bar{\xi}\right)^2 \prod_{i=1}^{n-1} \mathfrak{Q}_i(\xi, \bar{\xi}) \text{ for } n > 1.$$

Furthermore, $q \equiv \left(\frac{(\xi - \bar{\xi})^2}{q}\right) (mod\ n)$ where $\left(\frac{*}{q}\right)$ is a primitive divisor of $\mathfrak{Q}_n(\xi, \bar{\xi})$. The Legendre symbol [15] is indicated by $\left(\frac{*}{q}\right)$. Except for some finite values of parameters $\xi, \bar{\xi}$ and n in [8], all n-th terms of any Lucas sequence $\mathfrak{Q}_n(\xi, \bar{\xi})$ for $n > 4$ and $n \neq 6$ have primitive divisors.

3. Proof of theorem 1.1

The following is an independent investigation of Equation (1.2) for the situations of $n = 3, 4$ and $n \geq 5$:

Lemma 3.1: *If $n = 3$, the Eq. (1.2) has the following solutions:*

$$(x, y, a, b) = (1430,181,6,2), (82,55,7,1), (4744,283,7,1),$$
$$(782,127,9,1), (5570,319,9,1), (46,13,4,0), (10,7,5,0), (532,67,5,1)$$

Proof: When $n = 3$, represent $a = 6a_1 + \mathfrak{i}$ and $b = 6b_1 + \mathfrak{j}$, $0 \leq \mathfrak{i}, \mathfrak{j} \leq 5$. Subsequently, (1.2) takes the form of an elliptic curve

$$L^2 = M^3 - 3^{\mathfrak{i}}73^{\mathfrak{j}}$$

where $L = \frac{x}{3^{3a_1}73^{3b_1}}$ and $M = \frac{y}{3^{2a_1}73^{2b_1}}$. Therefore, determining all $\{3,73\}$-integer points on the 36 elliptic curves for each $\mathfrak{i}$ and $\mathfrak{j}$ is a modified version of finding positive integer solutions for Eq. (1.2). Emphasizing that a $\mathcal{S}$-integer is a rational number $\frac{\mathfrak{r}}{\mathfrak{s}}$ where $\mathfrak{r}$ and $\mathfrak{s} > 0$ are relatively integers and any prime factor of $\mathfrak{s}$ is a member of the set $\mathcal{S}$ is important. This is true for any finite collection of prime numbers $\mathcal{S}$. To locate each $\mathcal{S}$-integral point on the given curves, we now employed the MAGMA function S Integral Points. The following highlights the findings for

$$\mathcal{S} = \{3, 73\}\ [12]$$
$$(M, L, \mathfrak{i}, \mathfrak{j}) = (1, 0, 0, 0), \left(\frac{181}{9}, \frac{1430}{27}, 0,2\right),$$
$$(1825, 77964, 0, 2), (73, 0, 0, 3), \left(\frac{55}{9}, \frac{82}{27}, 1,1\right), \left(\frac{283}{9}, \frac{4744}{27}, 1,1\right),$$
$$\left(\frac{1387}{9}, \frac{42632}{27}, 1,3\right), \left(\frac{5767}{9}, \frac{436978}{27}, 1,3\right), (73,584,2,2), (3,0,3,0),$$
$$\left(\frac{127}{9}, \frac{782}{27}, 3,1\right), \left(\frac{319}{9}, \frac{5570}{27}, 3,1\right), (75,648,3,1), (219,0,3,3),$$
$$(13, 46, 4, 0), (7, 10, 5, 0), (67,532,5,1)$$

Considering that x and y are positive integers with no common factors, it's important to highlight that only eight among these points yield a solution for Eq. (1.2).

Lemma 3.2: *If $n = 4$, the equation (1.2) doesn't have a solution.*

Proof: Assume $n = 4$. Let $a = 4\alpha_1 + \mathfrak{i}$ and $b = 4\beta_1 + \mathfrak{j}$,

$0 \leq \mathfrak{i}, \mathfrak{j} \leq 3$, be the initial expressions. Eq. (1.2) thus has the following form:

$$\mathcal{U}^2 = \mathcal{V}^4 - 3^{\mathfrak{i}}73^{\mathfrak{j}}$$

where $\mathcal{U} = \frac{x}{3^{2\alpha_1}73^{2\beta_1}}$ and $\mathcal{V} = \frac{y}{3^{\alpha_1}73^{\beta_1}}$. To discover all

$\mathcal{S} = \{3, 73\}$-integral points on the corresponding sixteen quartic curves, one must find every integer solution of Eq. (1.2). By employing the

SIntegralLjunggrenPoints, we were able to locate every $\mathcal{S}$-Integral Point on these curves, which resulted in

$$(\mathcal{U}, \mathcal{V}, \mathfrak{i}, \mathfrak{j}) = (\mp 1, 0, 0, 0)$$

Given the condition of x and y, it is evident that Eq. (1.2) does not have any solutions.

Lemma 3.3: *If $n \geq 5$, the Eq. (1.2) possesses no positive integer solutions.*

Proof: Assume that $n \geq 5$. If (1.2) for $n = 2^k$ with $k \geq 3$ has a solution, it may be deduced from solutions for $n = 4$ as follows: $y^{2^k} = \left(y^{2^{k-2}}\right)^4$. Therefore, there are no solutions to Eq. (1.2) with $n = 2^k$ and $k \geq 3$. Similarly, Eq. (1.2) does not have a solution for $n = 3^k$ with $k \geq 2$. Therefore, we assume that n is an odd prime to preserve generality.

We now begin to factorization of Eq. (1.2) in the field $\mathfrak{K} = \mathbb{Q}\left(\sqrt{-d}\right)$ as follows

$$\left(x + \mathfrak{e}\sqrt{-d}\right)\left(x - \mathfrak{e}\sqrt{-d}\right) = y^n$$

where α, β are some non-negative integers, $\mathfrak{e} = 3^{\alpha}73^{\beta}$ and

$$d \in \{1, 3, 73, 219\}.$$

Since Eq. (1.2) requires that x must be odd, assuming that y is even results in a contradiction, with $1 + 3^a \equiv 0\ (mod\ 8)$. Therefore, y must be odd. Consequently, $\left(x + \mathfrak{e}\sqrt{-d}\right)$ and $\left(x - \mathfrak{e}\sqrt{-d}\right)$ are generating the ideals and relatively prime in the field $\mathfrak{K}$.

There are only two possible values for the class number $\mathfrak{h}(\mathfrak{K})$, depending on the particular choice of d : 1 or 4.

Consequently, we can say that n and $\mathfrak{h}(\mathfrak{K})$ are coprime.

Now, consider an algebraic integer $\xi \in \mathfrak{K}$ along with units $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{O}_{\mathfrak{K}}$. We can express this as follows:

$$x + \mathfrak{e}\sqrt{-d} = \xi^n \mathfrak{u}_1$$
$$x - \mathfrak{e}\sqrt{-d} = \bar{\xi}^n \mathfrak{u}_2$$

Given that the units $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are coprime to n and that the orders of the multiplicative group of units in the ring of algebraic integers of $\mathfrak{K}$ are 2,4, or 6 depending on the value of d, we can exclude them from the equations. The units $\mathfrak{u}_1$ and $\mathfrak{u}_2$ can be included into the factors ξ^n and $\bar{\xi}^n$. Subsequently, we will analyze the two scenarios individually: when $d \in \{1,73\}$ or $d \in \{3,219\}$, which relate to different integral bases for ring of integers of $\mathfrak{K}$, specifically $\left\{1, \frac{1+\sqrt{-d}}{2}\right\}$ and $\left\{1, \sqrt{-d}\right\}$ respectively.

To begin with, let us assume $d \in \{1,73\}$. Consequently,

$$x + \mathfrak{e}\sqrt{-d} = \xi^n = \left(s + t\sqrt{-d}\right)^n$$
$$x - \mathfrak{e}\sqrt{-d} = \bar{\xi}^n = \left(s - t\sqrt{-d}\right)^n$$

and $y = s^2 + dt^2$ for some integers s and t. By analyzing these equations, we can derive that

$$\mathfrak{e} = \mathfrak{L}_n t$$

The Lucas sequences devoid of primitive divisors are methodically catalogued in [8], and it has been determined that $\mathfrak{L}_n$ does not align with any of these sequences. Therefore, we investigate the possible existence of a primitive divisor for $\mathfrak{L}_n$. Let q denote any primitive divisor of $\mathfrak{L}_n$. If a divisor exists, it must be either 3 or 73. Since primitive divisors are equivalent to $\pm 1\ modulo\ n$, we exclude $q = 3$ for $n \geq 5$. Consequently, we continue with $q = 73$. According to the concept of a primitive divisor, q does not divide $\left(\xi - \bar{\xi}\right)^2 = -4dt^2$, indicating that $d = 1$. Furthermore, given that $\left(\frac{-4dt^2}{q}\right) = \left(\frac{-1}{73}\right) = 1$, it may be concluded that $73 \equiv 1\ (mod\ n)$. The sole permissible options for n are 2 and 3, neither of which exceeds 5, resulting in the conclusion that the scenario of $d \in \{1,73\}$ lacks a solution.

Next, we examine the case where $d \in \{3,219\}$. In such cases, $\left\{1, \frac{1+\sqrt{-d}}{2}\right\}$ is the integral basis for ring of integers of $\mathfrak{K}$ allowing us to express

$$x + \mathfrak{e}\sqrt{-d} = \xi^n = \left(\frac{s+t\sqrt{-d}}{2}\right)^n$$

$$x - \mathfrak{e}\sqrt{-d} = \bar{\xi}^n = \left(\frac{s-t\sqrt{-d}}{2}\right)^n$$

with $s \equiv t(mod\ 2)$. Then

$$2\mathfrak{e} = \mathfrak{L}_n t$$

where $\mathfrak{L}_n = \frac{\xi^n - \bar{\xi}^n}{\xi - \bar{\xi}}$. It is observed that $\mathfrak{L}_n$ is a Lucas sequence. If there were no primitive divisor for $\mathfrak{L}_n$, the sequence would match one of the Lucas sequences in [8]'s table. For $d \in \{3,219\}$, this is not the case.

Therefore, $\mathfrak{L}_n$ must have a primitive divisor, denoted as q, where q can either be 3 or 73.

Using the properties $q \nmid \left(\xi - \bar{\xi}\right)^2 = -dt^2$ and

$q \equiv \left(\frac{-dt^2}{q}\right)(mod\ n)$, it follows that $q = 73$ and $d = 3$, with the possible values for n being 2 or 3. Since these values are not greater than 5, no solution exists for $n \geq 5$. This concludes the proof.

4. Conclusion

This article shows that the exponential Diophantine equation

$x^2 + 3^a 73^b = y^n$ has a solution in $n = 3$ and no solution in $n \geq 4$. We make the following conjecture.

Conjecture 4.1: Let p be an odd prime, $p \equiv 1(mod\ 8)$,

$p > 113, a, b$ are non-negative integers, x, y are natural numbers and $n \geq 3$ with $gcd(x, y) = 1$. for the equation

$$x^2 + 3^a p^b = y^n \tag{4.1}$$

has a solution if $n < 5$ and no solution if $n \geq 5$.

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