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Graph-theoretic and geometric methods for social network influence analysis

Neha Janu [@]
Department of Computer Science and Engineering
Manipal University Jaipur
Jaipur 303007
Rajasthan
India

Payal Garg [†]
Department of Computer Science and Engineering
GL Bajaj Institute of Technology and Management
Greater Noida 201306
Uttar Pradesh
India

Komal Mehta Bhagat [§]
Department of Electronics & Communication Engineering
Bhagwan Parshuram Institute of Technology
New Delhi 110089
India

Blessy Thankachan [‡]
School of Computer Applications
JECRC University
Jaipur 303905
Rajasthan
India

[@] *E-mail:* **neha.janu@jaipur.manipal.edu**
[†] *E-mail:* **payalgarg.cs@gmail.com**
[§] *E-mail:* **komalbhagat@bpitindia.com**
[‡] *E-mail:* **blessy.thankachan@jecrcu.edu.in**

Susheela Vishnoi *
Department of Computer Science and Engineering
Manipal University Jaipur
Jaipur 302012
Rajashtan
India

Pankaj Dadheech #
Department of Computer Science and Engineering
Swami Keshvanand Institute of Technology, Management & Gramothan
Jaipur 302017
Rajasthan
India

Abstract

The growing role we play in influencing and being influenced by those around us, for better or worse, underscores the importance of having proven ways to analyze and model the propagation of influence. The use of such techniques or traditional graph theoretic approaches alone do not capture the complex, non-linear nature of many real-world social networks. In response to this problem, in this paper we present a new framework combining graph and geometry approaches for full analysis of social network influence. We mix traditional centrality statistics like degree, betweenness, clustering coefficients, with more sophisticated lower-level geometric features, such as geodesics and curvature analysis obtained at the price of using Riemannian geometry. This Hybrid model also allows us to infer the key players who provide us with further detailed information about the structural strength and interaction of a network. Drawing on geometric characteristics, our approach successfully captures correlation of nonlinear effect, resulting in more accurate influence spread estimation. We use the real-world social network data to conduct a case study and show that the proposed framework not only enhances the effectiveness of critical node-oriented attacks in target identification but also leads to a more accurate discovery of bottlenecks and weakly connected regions. The presented method alleviates the difficulties of the established techniques by providing a mathematically structured, yet computationally efficient framework that provides interpretability offered from graph theory with the level of detail from geometric analysis. This work can inform targeted marketing, misinformation control and community behavior dynamics, and lends novel insights to the study of influence dynamics in social networks more broadly.

Subject Classification: *05C82.*

Keywords: *Social network analysis, Graph theory, Riemannian geometry, Centrality measures, Curvature analysis, Geodesic paths.*

* *E-mail:* **susheela.vishnoi@jaipur.manipal.edu (Corresponding Author)**
# *E-mail:* **pankajdadheech777@gmail.com**

1. Introduction

Today, social networks have become a structurally important aspect of communication and are intensively being used to affect decision-making, opinion making, as well as trends in behavior. Modeling and measuring influence propagation in such networks are important problems, due to the intricate underlying interactions and ties. However, they are based on graph theory only, and not on a non-linear, dynamic model of a social network. As a result, it is important to design and formulate new mathematical models, which can describe in a comprehensive way local as well as global network.

A key issue of the social network analysis is to accurately target key influencers playing crucial roles for the information propagation. Although centrality metrics like degree, closeness and betweenness are used to rank nodes, they only capture shallow structural measure, such as connectivity and robustness under random failure [3]. Moreover, the propagation of influence is inherently non-linear, and the dependence and intensity of the relationship are usually at different levels. Such issues render influence modeling in the network more challenging that simply being a graph theoretic phenomenon and thus require geometric tools to assist with both the discovery of latent relationships and underlying network structure.

Based on graph-theoretic techniques, the analysis of social networks is well poised to offer tools for quantifying structural features, such as centrality and clustering. But there is an added bonus in the incorporation of geometric techniques. Geometrical properties like the curvature and geodesic help us understand how effectively the information is diffusing or marching across the network. For example, the notion of Ollivier-Ricci curvature has been used to quantitatively evaluate the strength of network connections, geodesics as well can be used to model the shortest paths of influence diffusion [4]. There is more we would like to achieve with these models, mainly, we aim to reconcile structural and dynamic analysis, allowing to understand the full spectrum of influence dynamics.

This paper presents a hybrid framework that combining graph-theoretic and the geometric approaches provides a better precision and understanding for influence analysis in the context of social networks. The goals of the framework are, (i) to discern influential nodes by combining centrality scores with geometric measures (ii) to account for the influence proliferation through geodesic paths in the network and examining its resistance by means of curvature (iii) to synthesize a composite influence score based on both graph theoretic and geometric insights for efficient node ranking and (iv) to demonstrate the framework's efficiency by empirically validating results obtained from real-world social network data. By accomplishing these objectives, the framework hopes to overcome current methodological limitations and provide continued applicability to dynamic influence processes. The findings have implications for targeted advertising, identification of misinformation, and fostering community interactions, further increasing our understanding of social networks as complex, evolving systems.

2. Related Work

The study of influence spread on social networks has received substantial attention in recent literature and several approaches using graph-theoretic or geometric considerations have been proposed to address the problem. This review focuses on these approaches, explaining their merits and limits, and motivates for an integrated graph-geometric framework.

2.1 *Graph-Theoretic Approaches*

Graph methods have been widely used in structural measurements in social networks. Centrality measures, including degree, closeness, and betweenness are basic metrics that are used to detect important nodes in the network.

Degree Centrality only considers the direct contacts of a node, offering a simple indication of its influence. It, however, does not consider the indirect or global influence in the network [1].

$$C_D(v)= deg(v) / |V|-1$$

where deg(v) is the degree (number of direct connections) of node v, and |V| is the total number of nodes in the network.

Closeness Centrality identifies nodes that can quickly disseminate information by measuring their average shortest path to all other nodes. This measure is particularly effective for identifying efficient spreaders in a network [2].

$$C_C(v)= 1/ \sum_{u\in V} d(v,u)$$

Betweenness Centrality quantifies a node's role as a bridge by measuring how often it lies on the shortest paths between other nodes. It is especially useful for detecting bottlenecks but can be computationally expensive for large networks [3].

$$C_B(v)=\sum_{s\neq v\neq t} \sigma_{st}(v)/\ \sigma_{st}$$

Other graph-theoretic methods, such as clustering coefficients and influence maximization techniques, enhance our understanding of network cohesiveness and the optimal selection of influential nodes. The clustering coefficient measures the tendency of a node's neighbors to form a tightly knit group, offering insights into local network structures [4].

$$C(v)= 2|\{eij:vi,vj\in N(v)\}|\ /\ |N(v)|(|N(v)|-1)$$

where N(v) is the set of neighbors of v, and eij are the edges between neighbors.

Influence maximization techniques aim to identify subsets of nodes that maximize the spread of information, often modeled as optimization problems [5].

$$\text{maximize } \sigma(S)\text{subject to } |S|=k$$

where $\sigma(S)$ is the expected spread of influence starting from set S.

2.2 *Geometric Approaches*

Geometric frameworks generalize graph-theoretic techniques by placing networks in geometric domains and studying their geometrical and dynamical properties.

Geometric Embeddings represent vertices into low-dimensional Euclidean or Riemannian spaces while keeping the graph's geodesic distances intact. These embeddings help mitigate the complexity in visualization, and clustering of network nodes [6].

$$\min \sum_{i,j\in V} wij\|f(vi)-f(vj)\|^2$$

where wij is the weight of the edge between nodes i and j, and f(v) is the embedding function.

Curvature-Based Analysis uses metrics like Ollivier-Ricci curvature to evaluate the robustness and efficiency of connections. For instance, a positive curvature indicates tightly connected communities, while negative curvature reveals weakly connected or tree-like structures [7].

$$\kappa(x,y)=1-\ (W1(\mu x,\mu y)\ /\ d(x,y))$$

where W1 is the Wasserstein distance between the local distributions μx and μy and d(x,y) is the shortest path distance between nodes x and y.

Riemannian Metrics allow for the study of non-linear relationships and provide tools for modeling the manifold structures underlying social networks. These metrics are particularly useful in capturing complex patterns of influence propagation [8].

Even though the graph-theoretic and geometric strategies have gone through significant developments, such methods are still restricted in their direct uses. Graph based methods in most of occasions neglect the non-linear and continuous behavior of linkages in a network and concentrate mainly on structural discrete measures [9].

3. Theoretical Background

This work is based on combining graph-theoretic and geometric techniques to study influence diffusion in social networks. Here we present a summary of the main theoretical ideas on which those methods are grounded and their significance for modeling and comprehension of the dynamics in networks.

3.1 *Graph-Theoretic Concepts*

Social network analysis is rooted in graph theory which is employed to model the relationship between entities. A social network can be modeled as a graph, with vertices that correspond to people or entities, and edges that represent interactions or relationships among them. Centrality Measures Some key network indicators, such as betweenness centrality, closeness centrality, and degree centrality, make it possible to identify strategic nodes of the network. Degree centrality is about direct connections, the people with the most connections in fact. Closeness centrality estimates how fast a node can transmit information around the network, whereas betweenness centrality finds the nodes that function as bridges for information could move from one side of the network to another. These measurements are structural by nature and allow to see the role of and importance of node within the topology of a network. Besides centrality measures, the clustering coefficients reveal the existence of clusters or communities of closely connected nodes in the network. Such cohesive entities frequently mirror common interests, behaviors or attributes, and are thus fundamental to the characterization of local interactions and have strong impact on collective phenomena. Influence maximization is another significant study in graph theory which is interested to find out the best set of nodes that can spread a maximum influence in the network [10]. We note practical applications involving this feature in viral marketing, public awareness campaigns and misinformation control.

3.2 *Geometric Concepts*

Geometric methods are beyond traditional graph-theoretic methods, which is why geometric approaches are attempting to extend the applicability domain of graph-theoretic techniques by offering methods to cope with the underlying structure and relationships in networks in terms of geometry. Such approaches can be taken to treat networks as if they were contained in a geometric space, which is the fundamental insight of these works towards a richer appreciation of the properties and dynamics of networks. Geometric embeddings project nodes to low-dimensional spaces, such that the distances between nodes in the embedding space are representative of their structure in the original network. This transformation is useful notably for clustering, visualization and discovering hidden patterns. For instance, in a geometric space, nodes that are located nearby are likely to have similar roles or properties in the network.

3.3 *Combining Graph-Theoretic and Geometric Insights*

Although, these graph-theoretical methods provide a comprehensive picture of structural properties of networks, they do not take into account continuous and non-linear connections affecting dynamics. On the other hand, geometric methods are very good in capturing such relationships, but the interpretability of their graph metrics may be missing. The joint use of these two strategies constitutes a promising context which fills the limitations of the two above-mentioned approaches. Combining graph measures like centrality and clustering coefficients with geometrical properties such as curvature and geodesics, the method allows for a full analysis of influence dynamics. This integration allows one to identify both structurally central nodes in the network as well as those that are most important to the overall relational dynamics of the network. It combines the best of both worlds: since it's based on decision trees, it becomes very easy to interpret the structure of the social network, while taking information on the link strength distribution into account. This complement is highly useful for applications like influential nodes detection, information propagation prediction as well as strategies planning for preventing misinformation outbreak or promoting engagement of members in a community.

4. Proposed Framework

Combination of graph-theoretic and geometric methods provide a unified model for ranking influencers and impact spread in social networks. This approach includes considerations about centralities measured on the graph and on the underlying metric space, which allows to observe the structured and dynamic behavior of a network.

In order to combine the benefits of graph-theoretic and geometric methods, a combined influence score I(v) is considered. This score combines betweenness centrality CB(v), curvature(κ(v)) and geodesic distance (d(v)), along with weigth W to calibrate their relative importance:

$$I(v)=\alpha C_B(v)+\beta\kappa(v)+\gamma d(v)$$

Here:

α represents the weight assigned to betweenness centrality, emphasizing structural connectivity.

β captures the significance of curvature, reflecting robustness and community structure.

γ denotes the influence of geodesic distance, highlighting efficiency in information spread.

The weights (α,β,γ) can be adjusted based on the specific application or context of the analysis. For instance, in scenarios where structural bridging is critical, a higher α may be preferred, while networks requiring robustness insights may prioritize β.

The architecture in Figure 1 combines graph-theoretic and geometric approaches to the study of influence propagation in social networks in a

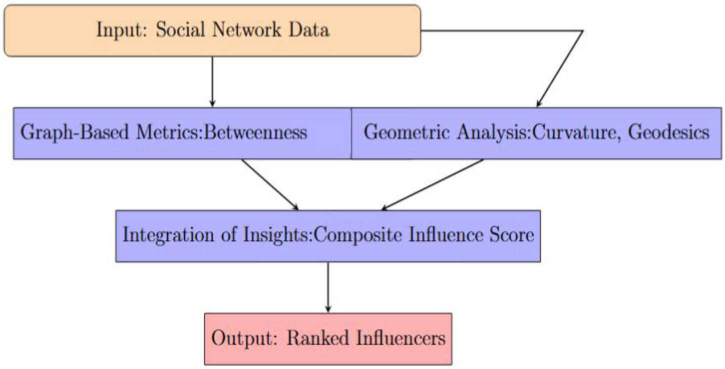

Figure 1

Workflow Architecture of proposed framework

global manner. The framework is built starting from the Input - social network data is modeled as a graph: nodes represent entities and edges represent interactions. The data is analyzed in two parallel threads: Graph-Based Metrics and Geometric Analysis. Graph theoretic measures, like betweenness and clustering coefficients, identify structurally salient nodes from their function in the connectivity and information flow of the network. At the same time, geometric analysis makes use of measures like curvature to quantify the strength of ties and geodesic distances to weigh the "quality" of spreading paths. The joint insights are integrated in the so-called Integration of Insights step, in which a combined influence score is computed by aggregating graph- and geometry-based measures with weights modelling their relative importance. The ultimate Output is an ordered list of influencers, serving as a practical reference to understand the dynamics of the network. Our architecture strikes a good balance between structural (i.e. continuous) and dynamic (i.e. discrete) analysis, while providing a strong and interpretable approach to influence modeling.

5. Conclusion

The embeddability combines graph-theoretic and geometric approaches to facilitate a complete investigation of influence diffusion within social graphs. Integrating structural information derived from graph-based metrics (e.g. betweenness, clustering coefficients) with strength and efficiency that are measured by the coefficients derived in geometric analysis (e.g. curvature, geodesic distances), the proposed framework provides a comprehensive measure of node's influential behavior. The composite influence score integrating these heterogeneous measurements make influencer ranking more accurate and interpretable. In combining these the hybrid approach, we propose overcomes limitations in traditional approaches due, in part, to its incorporation of local as well as global relationships among network links and because it can capture complex non-linear dynamics that often remain unseen. Modularity The modularity of the framework gives flexibility in the weights given to metrics from different domains, to accommodate the needs of a wide variety of applications, for targeted content marketing, misinformation suppression, and community engagement strategies. Future work might aim at generalizing this framework to multi-layered or temporal networks with, for instance, temporal dynamics and cross-network couplings, which make the analysis more complex. Finally, to be practical, the current approach needs to be extended to extremely large datasets and real-time analysis. In general, the incorporation of graph-theoretic and geometric considerations provides a huge leap ahead of traditional social network analysis, as it permits full insight into the mechanisms of influence and thus guarantees more sound decision-making in networked systems.

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