Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 4-B, pp. 1309–1316
DOI : 10.47974/JDMSC-2269

Z-small critically compressible modules

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Abstract

For an R-module ($R - mod$) C and a commutative ring (with identity) R, we introduce the notion Z-Small critically compressible modules (An R-mod C is Z-small Critically compressible ($ZSC - C$) where C may not embedded, in proper quotient modules C/N with $0 \neq N \ll_Z C$) as a generalization of Small critically compressible. We also provide the relationship between Z-small monoforms and Z-critically compressible modules in which the module is Z-small compressible.

Subject Classification: *76B03, 76B45.*

Keywords: *Z-small prime sub-module, Z-small compressible module (Z.S.C.), Z-small uniform module (Z.S.U.), Z-small monoform module (ZSM).*

1. Introduction

All modules will be unital left R-modules C, and R will be an associative ring with identity throughout. A sub−module N for $R - mod\, C$ is small If $N + K = C$ implies that $K = C$[1] and denoted by ($N \ll C$). In [2], Amina & Alaa presented the idea of a Z-small sub−module if $N + B = C$ with $B \supseteq Z_2(C)$ implies $B = C$ (briefly $N \ll_z C$), (where $Z_2(C)$ is called the second singular sub−module which is containing $Z(C)$ such that $\frac{Z_2(C)}{Z(C)}$ is the singular sub−module of $\frac{C}{Z(C)}$ [3]).

In [4], presented the idea of a smallprime module & smallprime submodule, when an R-mod C is small prime(ZSP) if $ann\, C = ann\, N$, for each

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nonzero small sub−module N for C. A sub-module N for R-mod is smallprime if whenever $r \in R$, $x \in C\ with\ (x) \ll C$ and $rx \in N$ implies either $x \in N$ or $r \in [N\colon C]$. As a generalization of these concepts, Aya and Alaa in [5] introduce Z-small prime (ZSP) module & Z-small prime (ZSP) sub−module where C named Z-small prime iff $ann\ C = ann\ N$, for each nonzero Z-small sub−mod N of C. And a sub−mod N of an R −mod is Z-small prime if if whenever $r \in R$, $x \in C$ with $(x) \ll_Z C$ and $rx \in N$ implies either $x \in N$ or $r \in [N\colon C]$.

As an extension of small monoform, Z-small monoform (ZSM) was developed in [6]. An $R - mod\ C$ is called Z-small monoform module (ZSM module; for short), if every nonzero partial endo−morphism of C has Z-small kernel.

An $R - mod\ C$ is called Z-small uniform (ZSU module; for short) if every nonzero Z-small sub−mod is essential in C[7].

A module C is called small Critically compressible, if C cannot be embedded, in any proper quotient module $C/_N$ with $0 \neq N \ll C$ [8].

2. Z-small critically compressible module

In the section, a generalization of small critically compressible module was given and studied

Definition 2.1: [7] An $R - \text{mod}\, C$ is called Z-small compressible (ZSC. module; for short), if C can embed in all for it is nonzero Z-small sub−mod. Equivalent to, C be Z.S.C. if ∃ a mono−morphism from C into N whenever $0 \neq N \ll_z C$.

Lemma 2.2: [7]

1) *A Z-small sub−module of a ZSC module is also ZSC*
2) *Every nonzero direct−summand, of a ZSC is also ZSC.*

As generalization of small critically compressible we introduce the concept of Z-small critically compressible.

Definition 2.3: the ZSC modules C named Z-small critically compressible $(ZSC - C$ module; for short), if C can not $embedded$, in proper quotient module $C/_N$ with $0 \neq N \ll_Z C$.

Note 2.4: Let Q and Z be Z-modules and let $N = 3Z$. Then $N \ll Q$, but N is not Z- small of Z and it is clear that $N \cong N$.

Hence, we give the following condition $(*)$:-

If C_1 and C_2 are two modules, let $N_1 \leq C_1$ and $N_2 \leq C_2$ with $N_1 \cong N_2$, then $N_1 \ll_Z C_1$ implies $N_2 \ll_Z C_1$.

Amina and Alaa in [2] prove the following

Proposition 2.5: If N, K are sub−modules in R- mod C s.t $K \leq N$ &$N \ll_Z C$ so, ${}^{N}/_{K} \ll_Z {}^{C}/_{K}$.

Proposition 2.6: If $K \leq N \leq C$ and $N \ll_Z C$, then $K \ll_Z C$.

In the following remark we show the converse is true under certain conditions.

Remark 2.7: The converse of proposition (2.5) hold if $Z_2\left(\frac{C}{K}\right) = \frac{Z_2(C)+K}{K}$ (named this condition by $\left(*^{/}\right)$) for any $K \leq C$.

Proof: Let $N + L = C$ $with$ $Z_2(C) \subseteq L$. Thus $\frac{N}{K} + \frac{L+K}{K} = \frac{C}{K}$, $\frac{L+K}{K} \supseteq \frac{Z_2(C)+K}{K} = Z_2\left(\frac{C}{K}\right)$. But $\frac{N}{K} \ll_Z \frac{C}{K}$ so $\frac{L+K}{K} = \frac{C}{K}$ thus $L + K = C$. Since $Z_2(C) \subseteq L$ and $K \ll_Z C$ thus $L = C$ i.e. $N \ll_Z C$.

The following, theorem gives the relation between Z-small critically compressible and Z-small monoform when the module is ZSC.

Theorem 2.8: *Suppose condition* $(*)$ *and* $\left(*^{/}\right)$ *hold and let* C *be a ZSC* R*-mod. Then, the statements that follow are equivalent:-*

1. C *is Z-small critically compressible* $(ZSC - C)$.
2. C *is monoform (and hence ZSM)*

Proof: (1) $\Longrightarrow$ (2) Let C be a $(ZSC - C\,)$. To prove C is ZSM: let $0 \neq N \ll_Z C$ and $0 \neq f \in \mathrm{Hom}(N, C)$. To prove that $\mathrm{kernal}\, f = (0)$. Suppose that kernal f $\neq (0)$ then, $\frac{N}{\mathrm{kernal}\, f} \simeq f(N)$and since $N \ll_Z C$ thus $f(N) \ll_Z C$ by [2]. And also there exist an isomorphism g such that $g : f(N) \longrightarrow \frac{N}{\mathrm{kernal}\, f}$ but C is a ZSC. So $\exists$ a mono−morphism $h\colon C \longrightarrow f(N)$. Now consider the following diagram $C \xrightarrow{h} f(N) \xrightarrow{g} \frac{N}{\mathrm{kernal}\, f} \xrightarrow{i} \frac{C}{\mathrm{kernal}\, f}$ where i is the inclusion mapping then the composition $(iogoh)$ is mono−morphism from C to $\frac{C}{\mathrm{kernal}\, f}$. But C can be embedded in $\frac{C}{\mathrm{kernal}\, f}$ which is contradiction with our assumption, thus $\mathrm{kernal}\, f = (0)$. In the other word C is monoform and hence ZSM.

(2) $\Longrightarrow$ (1) suppose C is not $(ZSC - C\,)$ module. Thus $\exists$ a non−zero Z-small sub−mod N of C, such that C embeddedin ${}^{C}/_{N}$. This means $\exists$ a mono−morphism $h : C \longrightarrow {}^{C}/_{N}$. Thus $C \simeq \frac{L}{N}$, for some $\frac{L}{N} \leq {}^{C}/_{N}$. Hence there exist an isomorphism $h_1 : \frac{L}{N} \longrightarrow C$. On other hand since C is ZSC, so $\exists$ a mono−morphism $g : C \longrightarrow N$ thus $goh_1\colon \frac{L}{N} \longrightarrow N$ is mono−morphism. That is $\frac{L}{N} \simeq N_1$for some $N_1 \leq N$. But $N \ll_Z C$ so by [2] $N_1 \ll_Z C$ thus by condition $(*)$ $\frac{L}{N} \ll_Z \frac{C}{N}$ and by condition $\left(*^{/}\right)$ $L \ll_Z C$ since $N \ll_Z C$. Consider the diagram L

$\underset{onto}{\xrightarrow{\pi}} \frac{L}{N} \underset{isomo}{\xrightarrow{g}} C$ where π is the natural epimorphism. Thus $(h_1 o\, \pi) : L \longrightarrow C$ and $(h_1 o\, \pi) \neq (0)$. But $L \ll_Z C$, since C is ZSM we get $kernal(ho\pi) = (0)$. However for each $n \in N$, $(h_1 o\, \pi)(n) = h_1(n + N) = h(0) = 0$; that is $N \leq$ kernal$(h_1 o\, \pi) = 0$ which is contradiction. Thus C is $(ZSC - C\,)$.

Proposition 2.9: A nonzero Z- small sub−module of a $(ZSC - C\,)$ module be $(ZSC - C\,)\ also$.

Proof: Assume C is $(ZSC - C\,)$ module&$0 \neq N \ll_Z C$. Hence lemma (2.2) N is ZSC. Let $0 \neq K \ll_Z N$. Then $K \ll_Z M$ and $N/_K \ll_Z C/_K$ by[2]. Assume that ∃ a mono−morphism say $\alpha: N \longrightarrow N/_K$. But C is ZSC means that ∃ a mono−morphism say $f: C \longrightarrow N$. So a composition $C \xrightarrow{f} N \xrightarrow{\alpha} N/_K \xrightarrow{i} C/_K$ gives a mono−morphism from C into $C/_K$ which is a contradiction. Thus, N is $(ZSC - C\,)$.

Proposition 2.10: A direct−summand of a $(ZSC - C\,)$ is $(ZSC - C\,)$.

Proof: Let $C = A \oplus B$ be a $(ZSC - C\,)$ module. Then C is ZSC & from (2.2), A also ZSC. suppose $0 \neq K \ll_Z A$. so $K \simeq K \oplus 0 \ll_Z C$. if $f: C \longrightarrow K$ mono−morphism & suppose ∃ is a mono−morphism say, $g: A \longrightarrow A/_K$. So a composition $C \xrightarrow{f} K \xrightarrow{i} A \xrightarrow{g} A/_K \xrightarrow{j} C/_K$ be mono−morphism (when i & j is inclusion homomorphism's). Therefore, a contradiction. Hence N is $(ZSC - C\,)$.

3. *Z*-small compressible related concepts

In the section, the relationship between the concepts was studied (Z-small compressible (ZSC), Z-small prime (ZSP) and Z-small monoform (ZSM)).

Recall that an R-mod C a fully stable if $N \leq C$, N is stable; i,e for each $f \in Hom(N, C). f(N) \subseteq N$, see [9].

The following proposition give connected between ZSC module and ZSP and ZSU module.

Theorem 3.1: *Let C be a fully stable R-mod. If C is ZSP and ZSU, then C is ZSU.*

Proof: Let $0 \neq N \ll_Z C$ and let $f \in Hom\,(N, C)$ such that $f \neq 0$. To prove that $kernal\ f = (0)$. Suppose $kernal\, f \neq (0)$. Since kernal$\,f \leq N$ and $N \ll_Z C$, thus by lemma [2]we have kernal $f \ll_Z C$. But C is ZSU so kernal $f \leq_e C$. Thus $(x) \cap$ kernal$\,f \neq (0)$, for every $0 \neq x \in N$. Hence there exist $r \neq 0$, $0 \neq rx \in$ kernal$\,f$, this implies that $0 = f(rx) = rf(x)$. But $f(x) \subseteq N \ll_Z C$, so $f(x) \ll_Z C$ by [2]. Since C is ZSP module and $rf(x) = 0$ we get either

$f(x) = 0$ or $r \in ann\, C$. But $x \notin$ kernal f, therefore $r \in ann\, C$, so $rx = 0$ which is contradiction. Therefore kernal $f = 0$ i.e. C is ZSM.

Proposition 3.2: *If C is ZSC R-mod, then C is ZSP and C is ZSU.*

Proof: Let $0 \neq N \ll_Z C$. To prove that C is ZSP module we must show that $ann\, C = ann\, N$. Put $K = ann_R N$, thus $kN = (0)$. By hypothesis C is ZSC so there is a mono−morphism $\emptyset: C \longrightarrow N$. Now, $f(kC) = kf(C) \subseteq kN = (0)$, this means that $k \subseteq ann_R C$, so $ann_R C = ann_R N$. Therefore C ZSP.

To prove that C is ZSU: let $0 \neq L \ll_Z C$, so let $0 \neq x \in L$. By [2], $Rx \ll_Z C$. But is C ZSC so there exist a mono−morphism $f: C \longrightarrow Rx$. Thus $f(x) = rx \neq 0$, for some $r \in R$, $r \neq 0$. Now, let $0 \neq a \in C$ and $f(a) = tx$, for some $t \in R$. As f is mono−morphism, thus $tx \neq 0$. So $f(ra) = rf(a) = rtx = trx = tf(x) = f(tx)$. Since f is mono−morphism $ra = tx$. But $tx \neq 0$, so $ra \neq 0$ and $ra \in N$, that is $N \leq_e C$ and hence C is ZSU.

Corollary 3.3: *If C a fully stable R-mod, If C is ZSC module. so C is ZSM.*

Proof: It is clear from (3.2) and (3.1).

Proposition 3.4: If C a nonsingular R-mod, if C is ZSU. Then $C$is ZSM.

Proof: Let $0 \neq N \ll_Z C$ and let $0 \neq f \in Hom(N, C)$. To prove kernal $f = 0$. By 1^{st}Fundament theorem $N/_{\text{kernal} f} \cong f(N)$. Since $f(N) \subseteq C$ and C is nonsingular, thus by [10], $f(N)$ is nonsingular. Thus $N/_{\text{kernal} f}$ is nonsingular. But $f(N) \subseteq C$ and $N \ll_Z C$ so by [2] kernal $f \ll_Z C$. But C is ZSU so kernal $f \leq_e C$ and hence kernal $f \leq_e N$ by [10]. Since C is non-singular $N \leq C$, so by [10] , N is non-singular, hence $N/_{kernal\, f}$ is singular by [10]. Thus $N/_{\text{kernal} f}$ singular and nonsingular. It follows that $N/_{\text{kernal} f} = 0$, and so that $f = 0$ which is a contradiction. Then kernal $f = 0$. Thus $C$is ZSM.

Theorem 3.5: *If C is non−singular and C is ZSC. Then C is ZSM.*

Proof: By proposition (3.2), C is ZSU, and by proposition (3.4) then $C$is ZSM.

The following theorem is a partied answer for the converse of proposition (3.3)

Theorem 3.6: *Let $C = Rm_1 \oplus Rm_2 \oplus \dots \oplus Rm_k$, when $m_1, m_2, \dots, m_k \in C$. Assume C be ZSU & ZSP, so C be ZSC module.*

Proof: Let $C = Rm_1 \oplus Rm_2 \oplus \dots \oplus Rm_k$, where $m_1, m_2, \dots, m_k \in C$, let $N \ll_Z C$, $N \neq 0$. By assumption C is ZSU so $N \leq_e C$. Thus for $m_i \in C$ $(i = 1, 2, \dots, k)$, $\exists\, t_i \in R - \{0\}$ such that $0 \neq t_i m_i \in N$. Put $t = t_1, \dots, t_k$.

Claim $t \neq 0$. Assume $t = 0$, hence $t_1, \dots, t_k m_k = 0$, thus $(t_1, \dots, t_{k-1})$ $(t_k m_k) = 0$.But $\langle t_k m_k \rangle \leq N \ll_Z C$, so by [2], $\langle t_k m_k \rangle \ll_Z C$ and since C is ZSP, we get $t_1, \dots, t_{k-1} \in ann\, C$. Then $(t_1, \dots, t_{k-1}) m_k = \langle 0 \rangle$, hence $(t_1 t_2, \dots, t_{k-2})(t_{k-1}\, m_{k-1}) = 0$. This implies $t_1 t_2, \dots, t_{k-1} \in ann\, C$. Since $\langle t_{k-1} m_{k-1} \rangle \ll_Z C$ and C is ZSP. Thus $t_1 t_2, \dots, t_{k-2} m_{k-2} = 0$. By repeating this process we get $t_1 m_1 = 0$ hence, which is a contradiction. Therefore $t = 0$. Now, define $\psi : C \longrightarrow N$ by $\psi(m) = tm$, for each $m \in C$. Clearly $tm \in N$, for $each\ m$.

$\ker f = \langle 0 \rangle$ since: assume $f(m) = \langle 0 \rangle$for some $m \in C$. Since $m = r_1 m_1 + \cdots + r_k m_k$ for some $r_i \in R$, $\forall i = 1, \dots, k$. Hence $t r_i m_i = 0$, for each $i = 1, \dots, k$. But $t r_1 m_1 = 0$ so we get $(t_1 t_2, \dots, t_k)(r_1\, m_1) = 0$ and hence $(t_2 t_3, \dots, t_k r_1)(t_1\, m_1) = \langle 0 \rangle$ since $\langle t_1 m_1 \rangle \ll_Z N$ and C is ZSP, we get $t_2, \dots, t_k r_1 \in ann\, C$ and so $(t_2, \dots, t_k r_1)(\, m_2) = 0$. Thus by a similar argument, we get $t_3, \dots, t_k r_1 \in ann\, M$. Repeating this process, we have $r_1 \in ann\, C$. Therefore $r_1 m_1 = 0$, similary $r_i m_i = 0$, $i = 2, \dots, k$. Therefore $m = 0$, that is $\ker f = \langle 0 \rangle$ and hence C is ZSC module.

Theorem 3.7: *If N proper sub−module in R-mod C. If $C/_N$ is ZSC, then N is a ZSP sub−module of C.*

Proof: Let $rx \in N, r \in R, x \in C$ with $\langle x \rangle \ll_Z C$. Suppose $x \notin N$, then $N \subsetneq N + \langle x \rangle$. We claim that $\frac{N+\langle x \rangle}{N} \ll_Z \frac{C}{N}$. Suppose that $\frac{N+\langle x \rangle}{N} + \frac{L}{N} = \frac{C}{N}$ with $\frac{L}{N} \supseteq Z_2\left(\frac{C}{N}\right)$, for some $L \leq C$ containing N. Hence $\frac{N+\langle x \rangle+L}{N} = \frac{C}{N}$, so $N + \langle x \rangle + L = C$ since $\frac{L}{N} \supseteq Z_2\left(\frac{C}{N}\right)$ so $Z_2(C) \subseteq L$ [2, lemma (2.2.6), $P32$]. Since $\langle x \rangle \ll_Z C$ thus $N + L = C,$as $N \subseteq L$ so $L = C$ and $\frac{L}{N} = \frac{C}{N}$ This implies that $\frac{N+\langle x \rangle}{N} \ll_Z \frac{C}{N}$. By assumption $\frac{C}{N}$ is ZSC, so there exist $f : \frac{C}{N} \longrightarrow \frac{N+\langle x \rangle}{N}$ a mono−morphism. Now, we must prove that $rf\left(\frac{C}{N}\right) = N$. Let $t \in f\left(\frac{C}{N}\right)$, thus $t = f(m + N)$, $m \in C$. since $f(m + N) \in \frac{N+\langle x \rangle}{N}$ hence $t = (n + r_1 x) + N$, for some $n \in N$, $r_1 \in R$. But $rt \in rf\left(\frac{C}{N}\right)$ and $rt = r\, f(m + N) = r(r_1 x + N) = r_1(rx) + N = N$ (since $r_1 x \in N$). Therefore $rf\left(\frac{C}{N}\right) \subseteq N$ and hence $rf\left(\frac{C}{N}\right) = N = f\left(r \frac{C}{N}\right)$, then $r \frac{C}{N} = N = \frac{rC+N}{N}$ and $rC + N = N$, that is $rC \subseteq N$ so $r \in [N \colon C]$ which is means that N is ZSP sub-module.

Corollary 3.8: *Let C is a ZSP R-mod such that $ann\ C \not\supseteq [k \colon C]$ for all nonzero sub-module k of C and every cyclic sub−module of C is Z-small in C. Then $C$is ZSC.*

Proof: Since C is a ZSP, so $\langle 0 \rangle$ is a ZSP sub−module of C [5] and since $ann\ C = [0 : C] \not\supseteq [k : C\ ]$ thus by theorem (3. 6), we get $M$is ZSC.

Corollary 3.9: *Let C be an R-mod such that $ann\, C \not\supseteq [k{:}\, C]$ for each sub$-$module k of C and every cyclic, sub$-$module of C a Z-small in C. Then $C$is ZSP if and only if $C$is a ZSC.*

Remember that "If sub$-$module N in C $\exists$ the ideal I in R s.t, $N = IC$, then an $R - mod\ C$ is a multiplication module." It is referred to as a multiplication module in C. For every submodule N in C, if $[N{:}\, C]C = N$"[11].

Corollary 3.10: *Let C be a Multiplication $R - mod$, N is proper submodule in C & each cyclic submodule in C be Z-small in C. Then $C/_N$ is a ZSC iff N be ZSP submodule for C.*

Proof: Since C is Multiplication module, then $[N : C] \not\supseteq [k : C]$ for all sub$-$module k of C containing N properly. Thus, according to theorem (3.6), and (3.7) we get the result.

Corollary 3.11: *Let R be a ring s.t every principle ideal in R is Z-small in R and I be an ideal of R. Then $R/_I$ is a ZSC if and only if I is a ZSP ideal of R.*

Conclusion

In this research, we generalized the idea for the small critically compressible module, to the Z-small critically compressible (ZSC-C) module and obtained several results. We also studied the relationship between the Z-small critically compressible module and the Z-small monoform. We prove many properties for this kind of sub-modules, among these results we prove if C is non-singular and C is Z-small compreesible. Then C is Z-Small Monoform.

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Received February, 2025