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The class of semi-modules with direct summands are stable

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Abstract

This work aims to introduce and explore the concept of SS-semimodules over semirings to generalize fully stable semimodules and duo semimodules. We call an S-semimodule C is SS-semimodule if every direct summand of C is stable. Equivalent expressions, properties, and related results of SS-semimodules are given. We prove that an S-semimodule C becomes SS-semimodule in case every decomposition $C = H \oplus K$, $Hom_S(H, K) = 0$. Results regarding fully stable semimodules were generalized to SS-semimodules. For example, if C is a regular SS-semimodule, then $End_S(C)$ is fully stable. Known semimodules related to SS-semimodules were investigated. We obtain that, every multiplication semimodule is SS-semimodule. Also, the SS-semimodule property satisfies the summand intersection property. Moreover, we discuss the direct sum property and inherited property for SS-semimodules.

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1. Introduction

Recall that a fully invariant subsemimodule N of an S -semimodule C is one in which $f(N) \subseteq N$ for each $S-endomorphism\ f\ of\ C$ [1]. An $S-semimodule\ C$ is said to be duo if, each subsemimodule of C is fully invariant [1]. A subsemimodule N of a S – semimodule C becomes stable, if $f(N) \subseteq N$ for every $S-$homomorphism $f: N \to C$. C is named fully stable if, each subsemimodule of C is stable. Every fully stable semimodule is duo, but generally the reverse is not true [1].

Let S be a semiring. A left S-semimodule C is a commutative monoid $(C, +, 0)$ for which we have a function $S \times C \to C$ defined by $(d, x) \to dx$ ($d \in$

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S and $x \in T$) (d, x) such that for all $d, d_1 \in S$and $x, x_1 \in T$, the following conditions are satisfy: (1)$d(x + x_1) = dx + dx_1$, (2) $(d, d_1)x = dx + d_1x$, (3) $dd_1(x) = d(d_1x)$, (4) $0x = 0 = x0$. When $1x = x$ holds for each $x \in T$ then a left S-semimodule T is said to be unitary [2]. The semiring S is said to be commutative if its multiplication is commutative [2]. A (left) semimodule C over a semiring S is a commutative additive semigroup which has a zero element, together with a mapping from $S \times C$ into C (then $(s, c) \rightarrow sc$) such that $(s_1 + s_2)c = s_1c + s_2c$, $s(c_1 + c_2) = sc_1 + sc_2$, $s_1(s_2c) = (s_1s_2)c$ and $0_Sc = s0_C = 0_C$ for all $c_1, c_2 \in C$ and $s_1 + s_2 \in S$ [2].

Let C be a semimodule over a semiring S. A subset N of C is said subsemimodule of C if $n_1, n_2 \in N$, and $r \in S$ such that $n_1 + n_2 \in N$ and $n_1r \in N$ [2]. An S-semimodule C is called the direct sum of subsemimodules T_1, T_2 of C, if each element c in C can be uniquely written as $c = c_1 + c_2$ where $c_1 \in T_1$ and $c_2 \in T_2$ and denoted by $C = T_1 \oplus T_2$. Also, T_1 is said to be a summand of C if there is a subsemimodule T_2 of C such that $C = T_1 \oplus T_2$[3]. For short, we will use summand instead of direct summand. A S- semimodule C is named multiplication if, for every subsemimodule N of C, there is an ideal I of S such that, $N = IC$ [4]. An S- semimodule C becomes simple if $C \neq 0$ and the only subsemimodule of C is (0) and C [5]. Also, an S- semimodule C is called semisimple if it is a direct sum of its simple subsemimodules [6]. Moreover, an S- semimodule C is semi-simple if every subsemimodule of C is summand [6]. A semiring S is said to be regular if, for every $x \in S$, there is an element $r \in S$ such that $x = xrx$[7]. A semimodule C is named a weak duo if each summand subsemimodule of C is fully invariant [8]. A subsemimodule K of an S-semimodule C is called subtractive if for each $x, y \in C$ with $x, x + y \in K$ implies $y \in K$ [6]. A non-zero subsemimodule of C is essential if $T \cap H \neq (0)$ for every non-zero subsemimodule H of C. A subsemimodule T of a semimodule C is closed if it has no proper essential extension in C [9].

Extending object and their generalizations play an important role in different categories (see [10], [11],[12]). A S-semimodule of C is said to be extending semimodule if each subsemimodule of C is essential in a direct summand of C [9]. An S- semimodule C is called a strongly extending if, each subsemimodule of C is essential in a stable summand of C [13].

2. SS – semimodules.

As a generalization of fully stable semimodules as well as duo semimodules the following concept will be introduced:

Definition 2.1: An S – semimodule is said to be SS – semimodule if, each summand of C is stable. Also, we call a semiring S is right (left) SS – semiring if, S is $SS -$ semimodule as right (left) $\mathrm{S} -$ semimodule.

Remarks and Examples 2.2:

1- Every fully stable semimodule is SS – semimodule. The converse is not true (see the examples below in (2), (3)).

2- Every uniform semimodule is SS- semimodule since the only summand of a uniform $S-s$emimodule C are $(0) and\ C$, so they are stable of C. In particular, $\mathbb{Z}^+$- semimodule Q is SS- semimodule, while. $Q_{\mathbb{Z}^+}$ is not fully stable ($\mathbb{Z}^+$- semimodule Q of rational numbers is not fully stable, consider $N = \left\{\frac{2n}{q}\middle| n, q\ is\ odd\right\}, N\ is\ a\ \mathbb{Z}^+$- semimodule Q, define $\pi: N \to Q\ by\ \pi(x)$ $= (\frac{3}{2})x$ for each $x \in N, \pi\ is\ a\ \mathbb{Z}^+ -$ homomorphism while $\pi\left(\frac{2}{5}\right) = \frac{3}{5} \notin N$, hence $Q_{\mathbb{Z}^+}$ is not fully stable.

3- The $\mathbb{Z}^+$- semimodule $\mathbb{Z}^+$ is uniform (by [9]), and so (by the above (2)) it is SS- semimodule, while the $\mathbb{Z}^+$- semimodule $\mathbb{Z}^+$ is not fully stable. If $h: 7\mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by $h(7x) = 5x$ for any $x \in \mathbb{Z}^+$. Obviously, h is $\mathbb{Z}^+$- homomorphism. Then $7\mathbb{Z}^+$ is not a stable subsemimodule of $\mathbb{Z}^+$ since $h(7\mathbb{Z}^+) \nsubseteq 7\mathbb{Z}^+$. So $\mathbb{Z}^+$- semimodule $\mathbb{Z}^+$

4- The reverse of (2) is generally not true. As an example, $\mathbb{N}$- semimodule ${\mathbb{Z}^+}_6$ is $SS-$ semimodule but is not uniform.

5- Following [13], every direct summand fully invariant subsemimodule is stable. So, we have SS- semimodules and weak duo semimodules equivalent concepts. In other words, C is SS-semimodule if and only if every summand of C is fully invariant.

6- Directly, from (5), every duo semimodule is $SS-$ semimodule, and hence the following implications hold: fully stable semimodules ⇒ duo semimodules ⇒ SS-semimodules. Note that Q as $\mathbb{Z}^+ -$ $s$emimodule is SS- semimodule which is not a duo. Also, $\mathbb{Z}^+$- semimodule $\mathbb{Z}^+$ is a duo but is not fully stable [1].

Remarks 2.3:

1- From ([13], Proposition 2.10), every closed subsemimodule (and so summand) is stable. So, we have every strongly extending semimodule is SS- semimodule. It is not like that in extending semimodules,

2- If C is a semi-simple S- semimodule, the following assertions are considered equal:

a) C is strongly extending semimodule;
b) $C\ is\ SS-\ semimodule$;
c) C is a duo semimodule;
d) C is fully stable.

Proof: $(a) \Rightarrow (b)$. From remark (2.4) (1). $(b) \Rightarrow (c)$. Directly, since each subsemimodule of C is direct summand, $(c) \Rightarrow (d)$. From ([13], Corollary (2.3)). $(d) \Rightarrow (a)$. Since every semi-simple $S-$ semimodule C is extending so fully stability of C implies that C is strongly extending.

3- Every indecomposable semimodule is SS- semimodule. The reverse is not true, as an example, $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule is fully stable [1] and hence it is SS- semimodule while it is not indecomposable.

Lemma 2.4: *Let C be an S- semimodule with subsemimodules $K \subseteq N \subseteq C$. If K is a summand of N and N is a summand of C then K is a summand of C.*

Proof: Since N is a summand of C, there is a subsemimodule D of C such that $C = N \oplus D$. Since K is a summand of N, then there exists subsemiodule H of N such that $N = K \oplus H$. By using (Modular law) ([14], lemma (2.3)). $C = (K \oplus H) \cap D = K \oplus (H \cap D) = K \oplus H$. Then K is a summand of C.

Lemma 2.5: *Let $C = H \oplus K$ be an SS- semimodule, where H and K are subsemimodules of C. Then H is fully invariant in C if and only if $Hom_S(H, K) = 0$.*

Proof: For the first direction. Let π be the projection homomorphism of $C = H \oplus K$ onto K with kernel H and let $\varphi \in End_S(C)$. If $Hom_S(H, K) = 0$, then the restriction of $\pi\varphi$ to H must be trivial. But this means that $\varphi(H) \subseteq H$. Thus H is fully invariant of C. Conversely, assume that $\varphi_1 \in Hom_S(H, K)$. Then φ_1 extends a map $\varphi \in End_S(C)\ by\ \varphi(h + k) = \varphi_1(h)$. If H is fully invariant of C, then for all $h \in H, \varphi_1(h) = \varphi(h) \in H$. But $\varphi_1(h) \in K\ and\ H \cap K = 0$. Thus $Hom_S(H, K) = 0$.

Directly by (Remark and example (2.2) (5)), we have the next result.

Proposition 2.6: Suppose that $C = H \oplus K$ be an SS- semimodule, where H and K are subsemimodule of C. Then H is stable in C if and only if $Hom_S(H, K) = 0$.

We provide a useful characterization of SS- semimodules.

Proposition 2.7: An S- semimodule C is SS- semimodule if and only if every decomposition $C = H \oplus K, Hom_S(H, K) = 0$.

Proof: For the first direction, it is clear by Remarks (2.2) (6), and lemma (2.5). Conversely, let H be a summand of C, thus there is a summand K of C such that $C = H \oplus K$. By hypothesis, $Hom_S(H, K) = 0$, and by lemma (2.5), H is fully invariant of C. Therefore, by Remarks (2.2) (5), C is SS- semimodule.

Recall that, an S- semimodule C has the summand intersection property (SIP) if the intersection of any two summands of C is a summand [8]. Motivated by proposition (2.6), the next result asserts that the class of $\mathrm{SS} -$ semimodule is contained in the class of semimodules with the (SIP) property.

Lemma 2.8: *The S-semimodule C has the summand intersection property if and only if, for every decomposition $C = K \oplus D$ and every $\varepsilon: K \to D$, the kernel of ε is a summand.*

Proof: Assume K and D be summands of C, let $C = K \oplus K_1 = D \oplus D_1$, and let $\alpha: C \to K$ and $\beta: C \to D$ be the canonical projections along K_1 and D_1, respectively. Define $\varepsilon: ((\alpha - 1) \circ \beta | K$. Then $\varepsilon: K \to K_1$, so that kernel ε is a summand of K. As kernel $\varepsilon = (K \cap D) \oplus (K \cap D_1)$

Proposition 2.9: Every SS- semimodule has the (SIP) property.

Proof: Assume that C is an SS- semimodule and let $C = A \oplus B$ be a decomposition. By proposition (2.4), we have $Hom_S(A, B) = 0$ (i.e.) every homomorphism $\alpha: A \to B$ is trivial, so the kernel of α is A, hence is a summand of C. Therefore, by the above motivation C has the (SIP) property.

Corollary 2.10: *Every fully stable semimodule has the (SIP) property.*

The author in [9], proved that every non-singular extending semimodule has the SIP property. Here, we show that strongly extending semimodules have the SIP property without extra condition.

Corollary 2.11: *Every strongly extending semimodule has the (SIP) property.*

Remarks and Examples 2.12:

1- The reverse of proposition (2.7) generally not true. As an example, $V = \mathbb{R}^{(2)}$ over the field $\mathbb{R}$ is semi-simple $\mathbb{R}$- semimodule and clearly it has the (SIP) property, but $V = \mathbb{R}^{(2)}$ is not $\mathrm{SS}-$ semimodule.

2- As an application of proposition (2.5), we have the following example: Consider $C = Z_P^{\infty} \oplus Z_P^{\infty}$ as $\mathbb{Z}^+$- semimodule. Thus, C is not SS-semimodule since we can define a $\mathbb{Z}^+ - homomorphism\ f: Z_P^{\infty} \to Z_P^{\infty}$ as follows $f\left(\frac{n}{p^t} + \mathbb{Z}^+\right) = \frac{n}{p^{t-1}} + \mathbb{Z}^+ \ \forall n \in Z, t \in N$. It is clear that $f \neq 0$. But $C = Z_P^{\infty} \oplus Z_P^{\infty}$ is injective $\mathbb{Z}^+$- semimodule and hence it is extending, while $C = Z_P^{\infty} \oplus Z_P^{\infty}$ is not strongly extending $\mathbb{Z}^+$- semimodule (by Remarks (2.2) (9)).

3- Consider $\mathbb{Z}_4^+$ and $\mathbb{Z}_2^+$ as $\mathbb{Z}^+$- semimodule. Now, define $f: \mathbb{Z}_4^+ \to \mathbb{Z}_2^+$ with $(x) = 2x \ \forall x \in \mathbb{Z}_4^+$ $.f$ is a non-zero $\mathbb{Z}^+$- homomorphism. Thus, by proposition (2.7), $\mathbb{Z}_4^+ \oplus \mathbb{Z}_2^+$ does not have SS- semimodule property.

We noticed that every fully stable semimodule is $\mathrm{SS}-$ semimodule but the reverse generally is not true (Remarks (2.2) (1)). In the following proposition, we obtain the condition that makes the converse true.

Proposition 2.13: Every regular SS- semimodule is fully stable.

Proof: Let N be a cyclic subsemimodule of SS- semimodule regular S-semimodule C. By regularity of C, N becomes a summand of C [1]. So since C is SS- semimodule, thus N is a stable of C. Therefore by ([1], corollary 1.2.9), C is fully stable

Corollary 2.14: *Let C be a regular S- semimodule. Then, C is fully stable if and only if C is SS- semimodule.*

Remark 2.15: The condition of the regularity of semimodule in corollary (2.11) is necessary because the $\mathbb{Z}^+$- semimodule Q is not regular and $Q_{\mathbb{Z}^+}$ is SS-semimodule but not fully stable.

The following corollary generalizes the result proposition 1.3.4 in [1] to SS- semimodule.

Corollary 2.16: *If C is a regular SS- -semimodule, then $End_S(C)$ is fully stable.*

Following [1], if the condition $End_S(C)$ is a fully stable semiring then C need not be fully stable. The following proposition ensures that this condition is sufficient to make C is SS- semimodule.

Proposition 2.17: If $End_S(C)$ is a fully stable semiring, then C is SS-semimodule

Proof: Suppose that N be a summand of C and let $f: N \to C$ be any S-homomorphism. Consider $I = Hom_S(C, N), I$ is a semiring ideal of $End_S(C)$. Define $g: I \to End_S(C)$ by $g(h) = f \circ h$ for each $h \in I$. Cleary, $g \in End_S(C)$. Also, g is an $End_S(C)$ (C) – homomorphism. Since $End_S(C)$ is fully stable semiring, then $g(I) \subseteq I$, that is for each $h \in I$, $f \circ h \in I$ so $f \circ h: C \to N$. But N is a summand of C, then the natural projection π_N of C onto N is in I, hence $f \circ \pi_N \in I$, that is $f \circ \pi_N: C \to C$ and because π_N is onto, then $f: N \to N$ (i.e.) $f(N) \subseteq N$.

It is known that, if C is a fully stable S - semimodule, then $End_S(C)$ is commutative ([1], corollary 1.3.5). However, generally, the reverse is not true. The fact that if $End_S(C)$ is commutative sufficient to make C is SS- semimodule (i.e.) is asserted with the following result. We get the implications:

C is fully stable $\Rightarrow End_S(C)$ is commutative $\Rightarrow C$ is SS- semimodule

Proposition 2.18: If $End_S(C)$ is commutative, then C is SS- semimodule.

Proof: Assume N is a direct summand of C and $f: N \to C$ by any S-homomorphism. Thus, there exists a subsemimodule $K\ of\ C$ such that $C = N \oplus K$. f can be extended to a S- homomorphism $g: C \to C$ by replacing $g(k) = 0$ for every $k \in K$. Let $h: C \to C$ by $h(n, a) = n$ for each $n \in N$ and $a \in K$. Let $f(n) = a + z$ for some $a \in N$ and $z \in K$. Now $(h \circ g)(w) = (h \circ g)(n + a) = h\big(f(n)\big) = h(a + z) = a$ and on the other hand, $(g \circ h)(w) = (g \circ h)(n + a) = g(n) = f(n) = a + z$. Since $End_S(C)$ is commutative, then $h \circ g = g \circ h$, and so $z = 0$. Then $f(n) \in N$, therefore $f(N) \subseteq N$. So N is a stable subsemimodule and thus C is SS- semimodule.

By using proposition (2.15), corollary (2.13), proposition (2.14), and proposition (2.10), we get the following corollary.

Corollary 2.19: *Suppose that C is a regular S- semimodule then the following statements are equivalent:*

1- $End_S(C)$ is commutative;

2- $End_S(C)$ is fully stable;

3- C is fully stable;

4- C is SS- semimodule.

Recall that a semiring S is right (left) SS-semiring if, every right (left) ideal of S which is a summand of S is stable. Equivalently from (remarks and examples 2.2 (6)), a semiring S is right (left) SS- semiring if and only if every right (left) ideal which is summand two-sided ideal of S.

It is known that the endomorphism semiring of a fully stable semimodule need not be fully stable [1], following proposition shows it is SS- semiring.

Proposition 2.20: If C is a fully stable S- semimodule, then $End_S(C)$ is SS-semiring.

Recall that an S- semimodule C is multiplication if, each subsemimodule of C is of the form IC for some ideal I of S [9]. In [1], the author stated there is no direct relation between fully stable semimodules and multiplication semimodules.

The following proposition proves that the class of multiplication semimodules is contained in the class of SS- semimodule.

Proposition 2.21: Every multiplication semimodule is SS- semimodule.

Proof: Let N be a summand of a multiplication S- semimodule C and $f: N \to C$ be any S- homomorphism. Since C is multiplication, then $N = IC$ for some ideal I of S. But N be a summand of C, thus f can be extended to an S-homomorphism $g: C \to C$. Now, $f(N) = g(N) = g(IC) = Ig(C) \subseteq IC = N$. So, N is a stable of C and therefore, C is SS-semimodule.

Remark 2.22: The converse of proposition (2.19) is not true. For example, Q as $\mathbb{Z}^+$- semimodule is SS-semimodules and is not multiplication.

From [15], every cyclic semimodule over commutative semiring is multiplication. So, we have.

Corollary 2.23: *Every cyclic semimodule over commutative semiring is SS-semimodule.*

Corollary 2.24: *Every commutative semiring is SS- semiring.*

As an application of proposition (2.9), we get the next result due to ([1], Examples and Remarks (1.2.11)).

Corollary 2.25: *Every commutative regular semiring is fully stable.*

Proposition 2.26: Every summand of SS- semimodule is SS- semimodule.

Proof: Let C be a SS- semimodule and N be a summand of S. Let K be a summand of N and let $f: K \to N$ be any homomorphism. N is a summand of C by (lemma 2.4). By SS- semimodule of S, we have K is a stable subsemimodule of S. Thus $(i \circ f): K \to S$ where $i: N \to S$ is the inclusion mapping and so $(i \circ f)(K) \subseteq K$ (i.e.) K is stable subsemimodule of N. Hence N is SS-semimodule.

Remark 2.27: The direct sum of SS- semimodules need not be SS- semimodule. For example, consider $\mathbb{Z}^+$ and $\mathbb{Z}^+{}_p$ as $\mathbb{N}$- semimodule (where p is a prim number). Since $\mathbb{Z}^+$ and $\mathbb{Z}^+{}_p$ are uniform $\mathbb{N}$- semimodules, then they are SS-semimodule. But $S = \mathbb{Z}^+ \oplus \mathbb{Z}^+{}_p$ as $\mathbb{Z}^+$- semimodule is not SS- semimodule. But $S = \mathbb{Z}^+ \oplus \mathbb{Z}^+{}_p$ as $\mathbb{Z}^+$- semimodule is not SS- semimodule. In fact, by uniformity of $\mathbb{Z}^+$ and $\mathbb{Z}^+{}_p$ the only summands of S are $0 \oplus \bar{0}$, $\mathbb{Z}^+ \oplus \bar{0}$, $0 \oplus \mathbb{Z}^+{}_p$ and C. But $\mathbb{Z}^+ \oplus \bar{0}$ is not a stable subsenimodule of S. For define $f: \mathbb{Z}^+ \oplus \bar{0} \to C$ by $f\big((x, \bar{0})\big) = (0, \bar{x})$ for each $(x, \bar{0}) \in \mathbb{Z}^+ \oplus \bar{0}$. f is a $\mathbb{Z}^+$- homomorphism. But $f(\mathbb{Z}^+ \oplus \bar{0}) \nsubseteq \mathbb{Z}^+ \oplus \bar{0}$.

3. Conclusions

This work introduces SS-semimodules as a generalization of fully stable and duo semimdoules. It is generalized various results of fully stable semimodules to SS-semimodules. It is discussed the hereditary property and direct sum property for SS-semimodules. This paper gives source examples of SS-ring and show that every commutative semiring is SS-ring. Also, unlike fully stable semimodule, it is proved the class of SS-semimodule is contained in the class of multiplication semimodules.

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