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Some results on the non-zero divisor graphs of Z_n

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Abstract

In this work, we study several features of the non-zero divisor graphs ($\aleph ZD$- graph) for the ring Z_n of integer modulo n. For instance, the clique number, radius, girth, domination number, and the local clustering coefficient are determined. Furthermore, we present an algorithm that calculates the clique number and draws the non-zero divisor for the ring Z_n.

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1. Introduction

It is thought that investigating the action of algebraic forms on a graph is one of the most intelligible strategies to analyzing algebraic forms. Many studies use a algebraic structure components to construct graphs that could potentially examined to comprehend algebraic behaviour, for instance, see [1-3]. Consider a ring $\mathcal{R}$, and construct an undirected simple graph on $\mathcal{R}$ as follows, the vertex set of this graph are being all the elements non-zero elements of $\mathcal{R}$ except multiplicative identity (if exists) and its additive inverse, moreover, two vertices generate a line if they are the product of the first vertex with the other not equal zero or the other side. This kind of graph is called a non-zero divisor group ($\aleph\mathcal{Z}D -$ graph) and indicated as $\Phi(\mathcal{R})$ and initially established by Kadem *et al*. [4]. In the other hand, the zero divisor, $\mathcal{Z}D -$ graph was initially introduced by Anderson and Livingston [5] after they had modified Beck's [6] description when the study at the time was mostly concerned with graph coloring. However, basic results on $\aleph\mathcal{Z}D -$ graph for Z_6 can be seen in [7]. Throughout this paper, we will suppose that all graphs are undirected, simple graphs without multiple edges or loops. Any vertex in an undirected graph that is linked to every other vertex in the graph is called a universal vertex. Let D be a non-empty subset of the graph vertex set of the graph Γ then D is termed as a dominant set if every vertex in the graph is either in D or has a neighbour who is in D. The size of minimum dominating set within Γ is denoted by the dominance number, $\gamma(\Gamma)$, for deep information see [8]. The primary intent of the current study is to ascertain specific properties of $\Phi(\mathcal{R})$, including its girth, radius, clique number, dominance number, and local clustering coefficient.

2. Preliminaries

Some results related to $\aleph\mathcal{Z}D -$ graph are presented. We first start with the definition of the $\aleph\mathcal{Z}D -$ graph followed by an illustrated example:

Definition 2.1: [4] Let $\mathcal{R}$ be a ring and let $\Phi(\mathcal{R})$ be the graph whose vertices are the non-zero elements of R that are not units, $V(\Phi(\mathcal{R})) = \mathcal{R} \setminus \{0,1,-1\}$. Two distinct vertices a and b in $V(\Phi(\mathcal{R}))$ are linked within $\Phi(\mathcal{R})$ contingent upon $\{ab, ba\} \neq \{0\}$. the graph in this case is called the $\aleph\mathcal{Z}D -$ graph of $\mathcal{R}$.

Example 2.1: $\Phi(Z_{16})$ is illustrated with the following Figure 1:

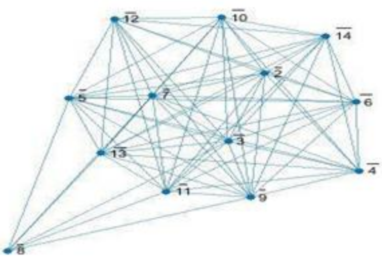

Figure 1
The Structure of $\Phi(Z_{16})$

Theorem 2.1: [4] Assume that $n > 6$ or $n = 4,5$. Then $\Phi(Z_n)$ is connected.

Theorem 2.2: [4] For a any ring R, the Diam(Φ(R))∈{1,2}, under the connectivity condition of $\Phi(\mathcal{R})$ along $\mathcal{R} \ncong Z_2 \times Z_4$.

Corollary 2.1: [4] The $\aleph\mathcal{Z}D -$ graph of Boolean rings is connected.

3. Main Results

We will provide several results related to the $\Phi(Z_n)$. This result includes the girth, radius, clique number, domination number, and local clustering coefficient. For simplicity of studying, we may allow x instead of $\overline{x}$ in Z_n. Also, we should note that N(a) means the neighbour of the vertex a in $\Phi(Z_n)$, that the vertices with distance one with x.

Since Z_5 is a field then the $\aleph\mathcal{Z}D -$ graph ($\Phi(Z_5)$ is complete with 2 vertex namely {2,3}. Thus $\omega(\Phi(Z_5))$ =∞. The next theorem shows that g ($\Phi(Z_n)$) =3, for $n > 5$ as shown in figure 2.

Figure 2
The Structure of Φ(Z$_5$)

Theorem 3.1: *Suppose that* $n > 5$ *,then* $g\left(\Phi(Z_n)\right) = 3$.

Proof: **Case 1:** If $n \geq 7$ is odd or $n = 10$ then the vertices {2,4,6} form a cyclic of length 3, thus $g\left(\Phi(Z_n)\right) = 3$.

Case 2: If n is even and if $n \neq 0$ in this case the vertices {2,5,3} generated a cyclic of length 3, thus $g\left(\Phi(Z_n)\right) = 3$.

Theorem 3.2: *The radius of* $\Phi(Z_n)$ *equal one.*

Proof: Consider P a prime number in which $P \times$ n. Then if there is $\ell \in \{2,3, \ldots, n-1\}$ with $\ell \neq p$ and $\ell . P = nk$ for some $\in Z$. So that $P/n.k$,but P is a prime number and $P \times$ n, hence $P\ /\ k$. Therefore, there exists $s \in Z$ such that

$$k = P.s\ .$$

Thus, we have

$$\ell . P = n.P.s$$

Then,

$$\ell = n.s = 0$$

which is impossible as $\ell \in \{2,3, \ldots, n-1\}$, therefore, $N(P) = V\left(\Phi(Z_n)\right) - 1$ so that, radius=1.

Remark 3.1: Let $n = p_1^{\ell_1}, p_2^{\ell_2},\dots, p_k^{\ell_k}$ where $p_1, p_2,\dots, p_k$ are distinct prime factors.

We denote by $f_{p_i} = \frac{n}{p_i^{\ell}}$ where ℓ =ceiling $\left(\frac{\ell_i}{2}\right)$=$\left\lceil\frac{\ell_i}{2}\right\rceil$.

Theorem 3.3: *Let $n = p_1^{\ell_1}, p_2^{\ell_2},\dots, p_k^{\ell_k}$ where $p_1, p_2,\dots, p_k$ are distinct prime numbers then the clique number of connected $\Phi(Z_n)$ equal to*

$$w(\Phi(Z_n)) = (n-3) - \min(f_{p_i} - 1).$$

Proof: Consider the set of vertices $p_i^{\ell}, 2p_i^{\ell}, \dots, (f_{p_i} - 1)p_i^{\ell}$ in $V(\Phi(Z_n))$ where ℓ=$\left\lceil\frac{\ell_i}{2}\right\rceil$. If we remove the above set of vertices from $V(\Phi(Z_n))$ we get a clique graph. This is because any two vertices in such a clique has a product not equal to 0 after exclusion the factors $p_i^{\ell}, 2p_i^{\ell}, \dots, (f_{p_i} - 1)p_i^{\ell}$. Now if we remove the smallest such vertices we get the maximal clique. So that we obtain a clique with size $(n-3) - \min(f_{p_i} - 1)$.

To show that the above clique is maximal, let C_w be another clique with a size greater than $(n-3) - \min(f_{p_i} - 1)$. Then in C_w we should have vertices from

$$p_i^{\ell}, 2p_i^{\ell}, \dots, (f_{p_i} - 1)p_i^{\ell}$$

But this means there are at least two vertices with product zero which is a contradiction then

$$w(\Phi(Z_n)) = (n-3) - \min(f_{p_i} - 1).$$

Example 3.1: The clique number of $\Phi(Z_n)$, for n=8,9,…,40 are equal respectively{4,5,6,8,6,10,10,10,11,14,12,16,14,16,18,20,16,19,22,22,22,26,22,28,26, 28, 30, 28, 27,34,34,34,30}.

Proposition 3.1: The universal vertex in $\Phi(Z_n)$ are three the prime number in $V(\Phi(Z_n))$ which not divide n. consequently, the domination number $\gamma((\Phi(Z_n))=1$.

Proof: Consider P a prime number in which *p* does not divide *n* then by the definition of the non-zero divisor this vertex must be a universal vertex, and as a result, we get $\gamma((\Phi(Z_n)) = 1$.

Example 3.2: The universal vertex of $\Phi(Z_8)$ are 3 and 5. Furthermore, the vertices {3,7} are the universal vertex of $\Phi(Z_{10})$.

The clustering coefficient quantifies the degree to which one's friends are also friends of one another. This measure gained popularity as a result of Watts and Strogatz's study in 1998 [9]. That is the local clustering coefficient is the number of pairs of neighbors connected by edges divided by the number of pairs of neighbors. For more information see [10-11]. Some results related to the local clustering coefficient of $\Phi(Z_n)$ are given below:

Lemma 3.1: *Let* $n = p_1^{\ell_1}, p_2^{\ell_2}, \dots, p_k^{\ell_k}$ *where* $p_1, p_2, \dots, p_k$ *are distinct prime numbers and x in V(Φ(Zn)) then we have the following:*

i. If x is prime not divide n, then N(x)= {V(Φ(Z_n)) \\{x}}, and |N(x)|=n-4.

ii. If x $= s.p_{i1}^{\ell_{i1}} \cdot p_{i2}^{\ell_{i2}} \dots p_{ik}^{\ell_{ik}}$ where $s = q_1^{\ell_1}, q_2^{\ell_2}, \dots, q_r^{\ell_r}$, q_i^s are distinct prime and g.c.d (s,n) = 1, then N(x)={V(Φ(Z_n)) \\{w. ($p_{j1}^{\ell_{j1}}\ p_{j2}^{\ell_{j2}} \dots p_{jt}^{\ell_{jt}}$)): $w.\ p_{j1}^{\ell_{j1}}\ p_{j2}^{\ell_{j2}} \dots p_{jt}^{\ell_{jt}}.x$}=0} where and $p_{j1}^{\ell_{j1}}\ p_{j2}^{\ell_{j2}} \dots p_{jt}^{\ell_{jt}}$ are all distinct and s = $q_{j1}^{\ell_{j1}}, q_{j2}^{\ell_{j2}} \dots q_{jv}^{\ell_{jv}}$, are distinct prime and g.c.d (w,n) =1, and | N(x) | = (n-4)-((x-1)($p_{j1}^{\ell_{j1}}\ p_{j2}^{\ell_{j2}} \dots p_{jt}^{\ell_{jt}}$)).

Proof: By **Proposition 3.1** when x is prime not divided n we have x as a universal vertex thus we get the proof of (**i**). To prove (**ii**) let $x_1 = p_{i1}^{\ell_{i1}} \cdot p_{i2}^{\ell_{i2}} \dots p_{ik}^{\ell_{ik}}$ then there (x-1) ($p_{j1}^{\ell_{j1}}\ p_{j2}^{\ell_{j2}} \dots p_{jt}^{\ell_{jt}}$) such that ($p_{j1}^{\ell_{j1}}\ p_{j2}^{\ell_{j2}} \dots p_{jt}^{\ell_{jt}}$) . x_1=0. Now if we multiply x_1 by s or ($p_{j1}^{\ell_{j1}}\ p_{j2}^{\ell_{j2}} \dots p_{jt}^{\ell_{jt}}$) by w the same result we will end up with. This will complete the proof of (**ii**).

By using the above lemma one can calculate the local clustering coefficient for certain Φ(Z_n):

Example 3.3:

The non-zero graph	**Neighbour**	**Local clustering coefficient**
Φ(Z_8)	N(2)=3,5,6	1
	N(3)=2,4,5,6	11/12
	N(4)=3,5	1
	N(5)= 2,3,4,6	11/12
	N(6)= 2,3,5	1
Φ(Z_{10})	N(2)=3,4,6,7,8	1
	N(3)=2,4,5,6,7,8	11/15
	N(4)=2,3,6,7,8	1
	N(5)= 3,7	1
	N(6)= 2,3,4,7,8	1
	N(7)=2,3,4,5,6, 8	11/15
	N(8)= 2,3,4,6,7	1

4. Computational implementation

In this section, we construct an algorithm with a goal of drawing Φ(Z_n) for all n>3 and computing the graph's clique number using a MATLAB program see [12-13]. Below is a summary of this algorithm:

Algorithm 1: Create the non- zero-divisor graph Φ(Z_n)

Input: A positive integer n for ring Z_n

Step 1: Generating the graph edges

for each two vertices i and j (except {0,1,n-1}

If $i * j$ is not divisible by n then connect $i \ and \ j$ by an edge

Step 2: Generate the adjacency matrix A

Step 3: Plot the resulting graph

Algorithm 2: Calculate the clique number

Step 1: defines a function maximal Cliques takes three input arguments:

MC: The current clique being explored.

V: The potential vertices that can be added to MC.

E: The excluded vertices

Step 2: If V and E are both empty, then the current clique MC is maximal. If MC is larger than the current largest clique, then replace the current largest clique with MC.

Step 3: Otherwise, iterate over all vertices in V

Step 4: Recursively call the maximal Cliques function with the following parameters:

MC: The current clique, with v added.

V: The potential vertices, with v removed and all neighbours of v added.

E: The excluded vertices, with all neighbours of v added.

Step 5: Remove v from V and add it to E.

Step 6: Repeat steps 3 and 4 until V is empty.

5. Conclusion

In this study, we sought to reveal the features of the $\aleph ZD -$ graph for the group Z_n. For example, this investigation was able to provide systematic rules for calculating the most noteworthy traits of a graph such that, the local clustering coefficient, girth, domination number, and more.

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