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# Small 2-prime sub-modules

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**Abstract**

For an D-module (abbreviated D-mod) W and a commutative ring (with identity) D. we introduce the notion of small 2-prime sub-module (S-2-P sub-module; for short ) as a generalization of 2-prime sub-module. We prove many properties for this kind of sub-modules, among these results we prove if V s proper sub-module of a module W over a ring D, then V is small 2-prime sub-module if and only if $[V:_W (d)] = V$ is a small 2-prime sub-module of W, for each d ∈ DD and $d^2 \in [V:_D W]$.

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***Keywords:*** *Prime sub-module, 2-prime sub-module, Small prime sub-module, Small 2-prime sub-module.*

## 1. Introduction

Let D be a ring that is commutative and has unity. [1] introduced prime sub-mod, where a proper sub-mod V of module W over a ring D is said to be prime if $dx \in$V, for d ∈D, $x \in$ W then either $x \in$V or $r \in [V:_D W]$. Mahmood presented the idea of a small-prime sub-mod in [2], which is a generalization of a P. S-mod. A proper S-mod V of a D-mod $W$ is small 2-prime if for d ∈D, $x \in$W, $dx \in$V with $\langle x\rangle \ll$W, then either $x \in$V or $d^2 \in [V: M]$. Where a sub-mod V of W is known as small (notationally V ≪W) incase $V + K =$W for any sub-mods $K$ of W implies $K =$W [3] W. Messirdi introduced in [4], 2-prime ideals of a ring D is 2-prime if $x, y \in I, x, y \in$D then either $x^2 \in I$ or $y^2 \in I$. Jasme and Elewi in [4], generalized this concept and introduced the concept of 2-P.S-

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mod, where the appropriate Sub-mod V of D-mod W if $dx$ ∈V, where d ∈D, $x$ ∈W, then either $x$ ∈V or $d^2 \in [V:_D W]$.

In this paper, we present the following concept of small 2-prime (S-2-P) sub-mods as a expansion of 2-prime sub-mods. Lots of results and properties are studied in this paper. For example (Let W be a finitely generated faithful multiplication D-mod. If V is S-2-P sub-mod of W, then $[V: W]$ is a S-2-P ideal of D.

Where an D-mod W is called multiplication D-mod if for all V sub-mod of W there exists ideal $J$ of D such that $V = J$W [5, 6].

## 2. Small 2-Prime Sub-Modules

As a generalization of 2-prime sub-mods, a small 2-prime ( S-2-P ) sub-mod was generalized in this section with some properties regarding this kind of sub-mods.

**Definition 2.1:** Let V be the appropriate sub-mod of D-mod W. V is called small 2-prime ( S-2-P sub-mod; for short ); if whenever d ∈ D, x ∈W, dx ∈V with ⟨x⟩ ≪W indicate either x ∈V or $d^2 \in [V:_D W]$ (where $[V:_D W] = \{$d ∈ D; dW ⊆ V}.

A proper ideal $I$ in the ring D is named small 2-prime ideal if $I$ is a S-2-P sub-mod of the D-mod D. Equivalently, a proper ideal $I$ of D is S-2-P if and only if whenever $d, s$ ∈D with $\langle S\rangle \ll$ D $and\ d\ \ s \in I\ indicate\ \ s^2 \in I\ or\ d^2 \in I$.

**Examples and remarks 2.2.**

1. Every small prime sub-mod is S-2-P- sub-mod (where the appropriate sub-mod V of D-mod W is known as "small prime" sub module if and only if whenever d ∈D and $x$ ∈W with $\langle x\rangle$ ≪W and $dx$ ∈V implies either $x$ ∈V or d ∈ $[V:_D W]$).ˡ

***Proof:*** Let V be a small prime sub-mod and let $dx$ ∈V with $\langle x\rangle$ˡ ≪ ˡW (ˡd ∈ ˡ$D, x \in$ Wˡ ). By assumption with either $x$ ∈V or d ∈ $[V: W]$. Thus, either $x$ ∈V or $d^2 \in [V: W]$.

2. In general, the opposite of (1) is not true. For instance: The submodule $V = \langle\overline{4}\rangle$ of the $\mathbb{Z}$-mod $\mathbb{Z}_{24}$ is small 2-prime sub-mod. To check this, the small sub-mods of $\mathbb{Z}_{24}$ are $\langle\overline{0}\rangle$, $\langle\overline{6}\rangle$, $\langle\overline{8}\rangle$, and $\langle\overline{12}\rangle$. Now, $\langle\overline{4}\rangle = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\}$. $0 \in N$, $\overline{0} = 2 \cdot \overline{12} = 4 \cdot \overline{6} = 6 \cdot \overline{4} = 8 \cdot \overline{3} = 3 \cdot \overline{8}$
   i. $\overline{0} = 2 \cdot \overline{12}$, $\langle\overline{12}\rangle \ll \mathbb{Z}_{24}$ , $\overline{12} \in \mathbb{N}$.
   ii. $\overline{0} = 6 \cdot \overline{4}$, but $\langle\overline{4}\rangle$ is not a small sub-mod of $\mathbb{Z}_{24}$.
   iii. $\overline{0} = 4 \cdot \overline{6}$, $\langle\overline{6}\rangle \ll \mathbb{Z}_{24}$, $\overline{6} \notin \mathbb{N}$, but $4^2 \in [\mathbb{N}: \mathbb{Z}_{24}] = 4\mathbb{Z}$.
   iv. $\overline{0} = 8 \cdot \overline{3}$, but $\langle\overline{3}\rangle$ is not a small sub-mod of $\mathbb{Z}_{24}$.
   v. $\overline{0} = 3.\overline{8}$, $\langle\overline{8}\rangle \ll \mathbb{Z}_{24}$ while $\overline{8} \notin \mathbb{N}$.

We test the elements $\overline{4}, \overline{8}, \overline{1}2, \overline{1}6$ and $\overline{2}0$ of $\mathbb{N}$ in the same way. So $\mathbb{N}$ is S-2-P sub-mod.

3. Every 2-prime sub-mod is S- 2-P sub-mod.

***Proof:*** Clear.

4. In general, the opposite of (3) is incorrect. For instance: $8\mathbb{Z}$ in the $\mathbb{Z}$-mod $\mathbb{Z}$ is not 2-P sub -module, since $2 \cdot 4 \in 8\mathbb{Z}$, but $4 \notin 8\mathbb{Z}$ and $2^2 \notin [8\mathbb{Z}: \mathbb{Z}] = 8\mathbb{Z}$. While $8\mathbb{Z}$ is S-2-P sub-mod by logic, since there is no small sub-mod in $\mathbb{Z}$ only (0).
5. Every prime sub-mod is S- 2-P sub-mod.

***Proof:*** Clear, because every prime sub-mod is 2-prime sub-mod [4], and by (3) we get the result.

6. The converse of (5). This is mostly untrue. According to the example that follows: The sub-mods of $4\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a S- 2-P sub-mod.
   Sub-mod but it is not prime because $2 \cdot 2 = 4 \in 4\mathbb{Z}$, but $2 \notin 4\mathbb{Z}$.
7. Every appropriate sub-mod of the $\mathbb{Z}$-mod $\mathbb{Z}_6$ is S-2-P (because only small sub-mod in $\mathbb{Z}_6$ is $(\overline{0})$ so by logic, we get the result).
8. A sub-mod $\langle\overline{12}\rangle$ of $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is not S-2-P sub-mod since $4 \cdot \overline{6} = 0 \in \langle\overline{12}\rangle$ with $\langle\overline{6}\rangle \ll \mathbb{Z}_{24}$. But neither $\overline{6} \in \langle\overline{12}\rangle$, non $4 \in [\langle\overline{12}\rangle: \mathbb{Z}_{24}] = 12\mathbb{Z}$.
9. If W is a vector space over a field $F$ (i.e., W is a module over a field $F$). So, each proper sub-mod V of W is a S-2-P sub-mod of type (0). Since by [6, Remarks and Examples 2.1.2] all proper sub-mod V of W is a prime sub-mod of type (0) and by (5) we get the required result.
10. The set $T(W) = \{w \in W: \exists a \in D, aw = 0\}$ is aesub-modeofeW and isecalledeTorsion sub-mod [7, 11]. $T(W)$ is small 2-prime sub-mod, since it is prime [6, Remarks and Examples 2.1.2] and by (5) it is small 2-prime sub-mod.
11. Every 2-prime ideal is small 2-prime ideal.
12. Every prime ideal is small 2-prime ideal.

**Remark 2.3:** Not every module has a S-2_P sub-mod. As next example shows: Every proper sub-mod in the $Z$-mod $Z_{p^\infty}$ is of the form $\left\langle\frac{1}{p^n} + \mathbb{Z}\right\rangle$, where $n \in \mathbb{Z}$. And every proper sub-mod of $\mathbb{Z}_{p^\infty}$ is small [8, Remarks and Examples 3.2.2 (5)]. Now, let $\mathbb{N} \subsetneq \mathbb{Z}_{p^\infty}$. So $\mathbb{N} = \left\langle\frac{1}{p^i} + \mathbb{Z}\right\rangle$, where $i \in \mathbb{Z}_+$. It is clear that $p\frac{1}{p^{i+1}} + \mathbb{Z} = \frac{1}{p^i} + \mathbb{Z} \in \mathbb{N}$, and $\left\langle\frac{1}{p^{i+1}} + \mathbb{Z}\right\rangle \ll \mathbb{Z}_{p^\infty}$. But $\frac{1}{p^{i+1}} + \mathbb{Z} \notin \mathbb{N}$ and $P^2 \notin [\mathbb{N}: \mathbb{Z}_{p^\infty}] = 0$.

It is widely known that if W is a multiplication finitely generated D-mod and $A, B$ are ideals of $R$, so $AW \subseteq BW$ if and only if $A \subseteq B + ann(W)$ [9].

By using the above information, we prove the following proposition:

**Proposition 2.4:** Let $W$ be a finitely generated faithful multiplication D-mod. If V is S-2-P sub module of W, then $[V: W]$ is a S-2-P ideal of D.

***Proof:*** Let $x \cdot y$e $\in [V:_D W]$ with $\langle x \rangle$e $\ll$ eD, where $x, y \in$ D. Thus $xy$W $\subseteq$V. Since $\langle xy \rangle \leq \langle x \rangle \ll$D. This implies that $\langle xy \rangle \ll$D ([10, Proposition 1.18]). But W is finitely generated faithful multiplication $R$-mod, so $\langle xy \rangle \ll M$ [10, Proposition 1.18] and since $N$ is small 2-prime, Thus either $x^2 \in [V\text{e}:_D W]$ or $y$w $\in$V, for every w $\in$W. Therefore either $x^2 \in [V:_D W]$ or $y^2 \in [V: W]$ hence $[V:_D W]$ isesmall 2-prime ideal of D.

**Remark 2.5:** The opposite of Proposition 2.4 is generally untrue. For instance: Let $W = \mathbb{Z}_{p^\infty}$ as D-mod, and let $V = \left\langle \frac{1}{p^i} + \mathbb{Z} \right\rangle$, $i \in \mathbb{Z}_+$. Then $[V:_{\mathbb{Z}} W] = \langle 0 \rangle$ is prime ideal, so by (Remarks and Examples 2.2 (12)), $[V:_{\mathbb{Z}} W]$ is S- 2-P ideal, but V as we show in Remark 2.3 is not S- 2-P sub-mod.

**Proposition 2.6:** Let W be an D-mod. If V is S-2-P sub-mod of W, then $[V:_W I]$ is S 2-S sub-mod of W for all ideal of D.

***Proof:*** Let $rx \in [V:_W I]$, with $\langle x \rangle \ll$W, where $r \in D, x \in$W. Thus, $rxI \subseteq$V, i.e., $trx \in$V, for every $t \in I$. But $\langle tx \rangle \leq \langle x \rangle$ and $\langle x \rangle \ll$W so by [8, Proposition 3.2.4 (2)], $\langle tx \rangle \ll$W. But V is S- 2-P sub-mod, so either $tx \in$V or $r^2 \in [V:_D W]$. Again, since V is S- 2-P sub-mod, so either $x \in$V or $t^2 \in [V:_D W]$. In other words, $t^2$W $\leq$V. But V $\subseteq [V:_W I]$, thus $t^2$W $\subseteq [V:_W I]$. Hence $t^2 \in [[V:_W I]: W]$ and therefore $[V:_I W]$ is S- 2-P sub-mod.

**Remark 2.7:** The opposite of Proposition 2.6 is generally untrue. As an illustration, let $W = \mathbb{Z}_{24}$ as R-module, $I = 2\mathbb{Z}$ and $V = \langle \overline{12} \rangle$. Then $[V:_W \text{I}] = \langle \overline{6} \rangle$ is a small prime sub-mod of $\mathbb{Z}_{24}$ [2]. Thus, by Remarks and Examples 2.2 (1), $N$ is S-2-P sub-mod. However, V is not S-2-P sub-mod since $2 \cdot \overline{6} \in N$, with $\langle \overline{6} \rangle \ll \mathbb{Z}_{24}$, but neither $\overline{6} \in N$ nor $2^2 \in [V:_{\mathbb{Z}} W] = 12\mathbb{Z}$.

**3. Some Properties of S-2-P-Sub-Mods**

In this section some properties of S-2-P sub modules are given, we start with the following properties:

**Proposition 3.1:** If V proper sub-mod of D-module $W$. Then V is S- 2-P sub-mod if and only if $[V:_V (r)]$ =V, for every $r \in$D and $r^2 \notin W$.

***Proof:*** Let $a$e $\in D, x$e $\in$W with $ax \in [V:_W \langle r \rangle]$ with $\langle x \rangle \ll$W. Thus $axr \in$V, since $\langle\ rx\ \rangle \subseteq \langle x \rangle \ll$W so by [9, Proposition 3.2 (2)] we have $\langle rx \rangle \ll$W. Since V is S- 2-P sub-mod, so either $rx \in$V or $a^2 \in [V:_D W]$. If $rx \in$V, then $x \in [\ V:_W (r)\ ]$. If $a^2 \in [\ V:_D M\ ]$. Thus $a^2$W $\subseteq$V. Hence $a^2Wr \subseteq Vr \subseteq$V, so $a^2Wr \subseteq$V. This implies that $a^2$W $\subseteq [\ V:_W \langle r \rangle\ ]$. Thus $a^2[[V:_W \langle r \rangle]:_D W]$. Therefore $[V:_W \langle r \rangle\ ]$ is a small 2-prime sub-mod.

Conversely, let ax $\in V$ ,$< x > \ll W$ where $a \in D$, $x \in$W so $axr \in Vr \subseteq$V and hence $axr \in$V. then $ax \in [V:_W \langle r \rangle]$ and $\langle ax \rangle \ll$W since $\langle ax \rangle \leq \langle x \rangle$. But $[V:_W \langle r \rangle]$ is small 2-prime sub-mod, so either $x \in [V:_W \langle r \rangle]$ or $a^2 \in \left[[V:_W \langle r \rangle]:_D W\right]$. If $x \in [V:_W \langle r \rangle]$, take $r = 1$, so $x \in$V and if $a^2 \in \left[[V:_W \langle 1 \rangle]\text{e}:_D W\right] = [V:_D W]$.

**Proposition 3.2:** Let $V, K$ be S- 2-P sub-mods of an D-mod W such that e$[V:_D W] = [K:_D W]$. Then V $\cap K$ is a S- 2-P sub-mod of W.

***Proof:*** Let $rx \in$ V $\cap K$, withe$\langle x \rangle \ll$W, whereer $\in D, x \in$W. Thus e$rx \in$V and $rx \in K$. Since each of eV and $K$ are S- 2-P. So, either $x \in$V or $r^2 \in [V:_D W]$ and either $x \in K$ or $r^2 \in [K:_D W]$. Which means thate $x \in$ V $\cap K$ or $r^2 \in$ [V $\cap K:_D W]$. Therefore V $\cap K$ is S- 2-P sub-mod.

**Remark 3.3:** A homomorphic image of a small 2-prime sub-mod is not in general S- 2-P sub-mod. For example, let $f: \mathbb{Z}_{24} \longrightarrow \mathbb{Z}_{24}$, defined bye$f(\overline{x}) = 2\overline{x}$, for all $\overline{x} \in \mathbb{Z}_{24}$. Thus $f(\langle \overline{6} \rangle) = \langle \overline{12} \rangle$. Since $\langle \overline{6} \rangle$ is small prime, so be Remarks and Examples 2.2 (1), $\langle \overline{6} \rangle$ is S- 2-P sub-mod of $\mathbb{Z}_{24}$. But $\langle \overline{12} \rangle$ is not S- 2-P as we shown in Remark 2.7.

**Definition 3.4:** [12, 13].

1. An D-mod W is said to be compressibleeifeiteW can be embedded in each of its non-zero sub-mods.
2. An D-mod W is said to beesmallecompressible if it W can be embedded in each of its non-zero small sub-mods.

**Theorem 3.5:** [2] *If V was appropriate sub-mod of an D-mod W and $\frac{W}{V}$ is small compressible, then V is small prime sub-mod of W.*

**Corollary 3.6:** *If V is appropriate sub-mod of an D-mod W and $\frac{W}{V}$ is small compressible then V is S- 2-P sub-mod.*

***Proof:*** It is clear by Theorem 3.5 and Remarks and Examples 2.2 (1).

**Lemma 3.7:** [2]. *Every compressible module is also small-compressible.*

**Corollary 3.8:** *Let V be appropriate sub-mod of an D-mod W. If $\frac{W}{V}$ is compressible module, then V is S- 2-P sub-mod.*

**Corollary 3.6:** *If V is appropriate sub-mod of an D-mod W and $\frac{W}{V}$ is small compressible then V is S- 2-P sub-mod.*

***Proof:*** It is clear by Theorem 3.5 and Remarks and Examples 2.2 (1).

**Lemma 3.7:** [2]. *Every compressible module is also small-compressible.*

**Corollary 3.8:** *Let V be appropriate sub-mod of an D-mod W. If* $\frac{W}{V}$ *is compressible module, then V is S- 2-P sub-mod.*

## Conclusion

We successfully introduce the idea of small 2-prime sub-mod in this paper and demonstrate how it is an extension of the 2-prime sub-mod concept. We also introduced some significant connections between Z-small prime sub-mods and other module types.

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