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Semi-hollow modules

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Abstract

The unitary left R-module F is defined in this article as a unity-commutative ring R. We introduce and discuss the idea of semi-hollow modules in this work. The notion of semi-hollow refers to a module that has semi-small proper submodules. We examine this concept of a module and offer some details.

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1. Introduction

An R-module F is known as hollow module if each of its proper submodules is small [1]. A submodule A of F is referred to as small (as shown by $A \ll F$) if, for any proper submodule B of F, $A + B \neq F$ [2]. Mijbass and Abdullah in [3] as a overview of the small submodule, the notion of a semi-small submodule was presented. If $K + U \neq F$, then a proper submodule K is referred to as semi-small of an R-module F, for every primary submodule U of F and $ab \in U$ for $a \in R$ and $b \in F$, implies either $b \in U$ or $a^n \in [U{:}\, F]$, for some positive integer n, where $[U{:}\, F] = \{a \in R{:}\, aF \subseteq U\}$, [4]. In the present paper, we provide the idea of a semi-hollow module. An R-module F can be called to be proper submodule B of F as primary if and only if it is semi- hollow and all of its proper submodule are semi-small. We also provide the basic characteristics of such modules. Furthermore, we give up particular relationships between this concept and other related modules, including hollow module. In Section 2 introduces semi-hollow modules, which are an extension of hollow modules.

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2. Semi-Hollow Modules

This section presents and examines the idea of semi-hollow modules, which are a generalization of hollow modules. We start with this definition:

Remember that an R-module is known as hollow, if all proper submodule is small [2].

Definition 2.1: Let F be an R-module is known as semi-hollow, if all proper submodule of F is semi-small. Where U is a proper submodule of an R-module is said to be semi-small denoted by ($U \ll_s F$) if ($F = B + U$), for some primary submodule U of F, implies that $B = F$. Equivalently; ($F \neq B + U$) for each primary submodule B of F [3] When in R-module F a proper submodule B is seen as primary if $rb \in B$, $r \in R$, and $b \in F$ implies that, for any positive number n, either $b \in B$ or $rn \in [B\colon F]$, [4, 8] where $[B\colon F] = \{r \in R\colon rF \subseteq B\}$.

Remarks and Examples 2.2:

1. Each hollow module is semi-hollow but we don't know the convers is true or not in general.
2. The Z12 is not semi-hollow as Z-module since 3Z12 is not semi-small because 3Z12+ 4Z12= Z12, but 4Z12 $\neq$ Z12 and 4Z12 is a primary submodule.
3. The semi-simple module F does not have to be semi-hollow, for example Z6 is not semi-hollow as Z-module, since $(\bar{0})$ is the only semi-small submodule in F.
4. The direct summand is not be necessary semi-hollow, for example 2Z10 $\oplus$ 5Z10 = Z10, since 2Z10 is not semi-small because 2Z10 + 5Z10= Z10, but 5Z10 $\neq$ Z10 and 5Z10 is a primary submodule.

Proposition 2.3: Consider F and F′ be are modules and let $g\colon F \to F'$ be an R-epimorphism, if F is semi-hollow, then F′ is also semi-hollow.

Proof: Assume that F′ has a proper submodule U. It is clear that the proper submodule of F is $f^{-1}(U)$. All of F is proper submodule, because F is semi-hollow. So, $f^{-1}(U)$ of F is semi-small. Therefore by [3], $f(f^{-1}(U))$ of F′ is semi-small. That is, U is semi-small of F′.

Corollary 2.4: *F/K is semi-hollow for any proper submodule K of F if F is a semi-hollow R-module.*

Proof: It's clear by proposition (2.3), since there exist a natural epimorphisim

$$\pi : F \longrightarrow F/K$$

Remark 2.5: Is not true in general, the converse of corollary (2.4), such as; $Z/4Z \cong Z_4$ is semi-hollow. Yet a module Z is not semi-hollow, because 2Z in Z is not semi-small, since $2Z + 3Z = Z$ and 3Z in Z is primary submodule.

Proposition 2.6: Each of the proper submodules of an R-module F is primary. Then F is semi-hollow if and only if F is hollow.

Proof: By [3], it is clear.

Proposition 2.7: Consider F be a module that is finitely generated. If and only if F is hollow, then F is semi-hollow.

Proof: Assume that E is a proper submodule of F. Because F is a semi-hollow module, thus E is semi-small. But F is finitely generated, hence by [3] E is small in F. Therefor F is a hollow R-module. Each small submodule is semi-small by [3], hence the converse is clear.

If a ring R is semi-hollow, we will refer to it as an R-module. Multiplication of an R-module F is defined as follows: $U = IF = [U: F]F$, [5] if each submodule U of the R-module F $\exists$ an ideal I of R.

Proposition 2.8: Consider F be a faithful multiplication of an R-module that is finitely generated. If and only if R is a semi-hollow ring, then F is semi-hollow.

Proof: Suppose that I is ideal of a ring R, so $I = [U: F]$ where U is a submodule of F, thus $[U: F]$ F is a semi-small submodule of F. Since F is multiplication finitely generated faithful, thus by [3] $[U: F] = I$ is a semi-small ideal of R. Therefore, a semi-hollow ring is R.

Conversely, suppose that R is a semi-hollow ring and U is a proper submodule of F. So, by [3], $[U: F]$ is a semi-small ideal of R. Because F is multiplication, so by [3] U is a semi-small submodule of F. Hence F is semi-hollow.

Corollary 2.9: *A cyclic R-module is F. If and only if F is hollow, then F is semi-hollow.*

Proof: Clear, by proposition (2.8) Considering that all cyclic R-modules are multiplication.

Remember that F in R-module is known as a cancelation module; if U and W are two ideals of R and $UF = WF$, then $W = I$. Also, F is known as weak cancelation; if of R, U and W are two ideals, then $U + \text{ann } F = W + \text{ann } F$.

Mijbass shown in [6, p.32] that if F is a cancelation and multiplication module, so F is finitely generated and faithful, resulting in the results that follows.

Corollary 2.10: *Consider F be a multiplication and cancelation module over a semi-hollow ring, then F is semi-hollow module.*

Proof: F is a semi-hollow module by proposition (2.8), and it is faithful and finitely generated since it is a multiplication and cancelation module [6, P.52].

Corollary 2.11: *If F is a semi-hollow ring multiplication module, then F has a cancelation submodule. F is then a semi-hollow module.*

Proof: Assume that w is a submodule of F that cancellations. F is a cancellation module since it is a multiplication module [6, P.61]. Furthermore, F is a semi-hollow module according to proposition (2.7).

Proposition 2.12: *A finitely generated primary submodule is contained in a multiplication module F. A module is considered hollow if it is semi-hollow.*

Proof: Proposition (2.7) gives the result since F is a multiplication module that has a primary submodule that is finitely generated [6, P.58].

Corollary 2.13: *A primary annihilator multiplication module is given by F. F, if semi-hollow, a module is hollow.*

Proof: Because F is a multiplication module with a primary annihilator and is therefore finitely generated, it is a hollow module [6, P.56]

Corollary 2.14: *For multiplication and cancellation, let F be a weak module. A hollow module is F if it is semi-hollow.*

Proof: Since F is a weak cancelation and multiplication module and is a hollow module as shown to the Proposition (2.14), it is finitely generated [6, p.60].

Corollary 2.15: *Let F be a weak cancellation and multiplication module. F is a cyclic module (hence a hollow module) if it is semi-hollow.*

Proof: Corollary (2.15) and corollary (2.9) are clear.

Proposition 2.16: Consider a module F in R-module is hollow module with a faithful submodule E of F that is finitely generated if F is a semi-hollow module.

Proof: Because the submodule E of the multiplication module F is faithfully finitely generated, produced [6, p.56] F is similarly finitely, resulting in a hollow module.

Proposition 2.17*:* Over integral domain R let F be a faithful multiplication. A module F is hollow if it is semi-hollow.

Proof: We are done by Proposition (2.7) since F is finitely generated because it is a faithful multiplication over integral domain R [6, p. 54].

Remember that if a submodule E is unique maximal and includes all proper submodules in F, then F is referred as local.[7, p.352]. The intersection every maximal submodules of F it is defined Rad F if F has maximal submodule, Rad(F) = ∩ { B : B is a maximal submodule in F} [2, Th. 9.1.1, p.123].

Proposition 2.18: If F a module of R is semi-hollow and has a submodule P that is maximal if and only if it is local.

Proof: Let F be semi-hollow and has a maximal submodule say P, hence E is primary. Suppose N is another maximal submodule, so $F = N + P$ and this implies that P is not semi-small in F which is a contradiction to proposition (2.6). Therefore, Rad $F = P$ since F has a unique maximum submodule P. Now let $A \subset F$. If $A \not\subset P$, then there exist $a \in A$, and $a \notin P$. Hence, Ra is not small in F by [2, lemma 5.1.4], and by remarks(2.2) Ra is not semi-small in F which is a contradiction. Therefor F is local. And so $A \subseteq P$.

Conversely, because F is local, it must be hollow; hence, F is semi-hollow.

Corollary 2.19: *Consider F be an R-module. Thus, the following propositions are equivalent:*

1. *F is hollow and has a maximal submodule.*
2. *F is semi-hollow and has a maximal submodule.*
3. *F is local*

Proof: $(1) \Longrightarrow (2)$ clear by remark and examples $(2.2,1)$

$(2) \Leftrightarrow (3)$ clear by proposition (2.18)

$(3) \Longrightarrow (1)$ by proposition (2.18) and [7, 41.2-2, p.352] we are done.

Corollary 2.20: *Consider F be an R-module. Thus, the following propositions are equivalent:*

1. *F is semi-hollow and Rad $F \neq F$.*
2. *F is local*
3. *F is hollow and $Rad\ F \neq F$*
4. *F is semi-hollow and cyclic module.*
5. *F is hollow and cyclic module.*

Proof: (1) ⇔ (2) as a corollary (2.19), it follows

(2) ⇔ (3) ⇔ (4) see [7, 41.4-2, p.352].

(4) ⇔ (5) it follows by proposition (2.7).

Remember that if $IF \cap E = IE$ [5], then of an R-module F a submodule E is pure. Thus, the following corollary can be provided.

Corollary 2.21: *Consider F be a multiplication R-module with a pure weak cancelation submodule E, so that $ann(F) = ann\ (E)$. If F is semi-hollow, so it is a module is hollow.*

Proof: It is assumed that F is weak cancelation [6, p. 62]. The result is as follows corollary (2.15)

Proposition 2.22: Let F_1 and F_2 be R-module with $ann\ F_1 + ann\ F_2 = R$. If $F = F_1 \oplus F_2$, then F is semi-hollow if and only if a semi-hollow is F_1 and F_2.

Proof: Let P_1:F $\longrightarrow F_1$, P_2: F $\longrightarrow F_2$ be the natural projections then by proposition (2.3) F_1 , F_2 are semi-hollow.

Conversely, let $N \subset F$. Then by [6, prop.4.2, ch.1], there exist $N_1 \subseteq F_1$, $N_2 \subseteq F_2$, such that $N = N_1 \oplus N_2$. Since F_1, F_2 are semi-hollow, N_1 is a semi-small in F_1 , N_2 is semi-small in F_2. Thus by [7] $N = N_1 \oplus N_2$ is semi-small in F. Therefore, F is semi-hollow.

Conclusion

The class of hollow modules has been expanded in this study to include a novel concept known as semi-hollow modules. We study a few properties and provide some similar conditions for a module to be hollow.

1. Each hollow modules are semi-hollow modules.
2. Epimorphic image of a semi-hollow modules is semi-hollow modules.
3. If a module is finitely generated semi-hollow, then it is hollow.
4. A finitely generated multiplication and faithful is semi-hollow if and only if it is hollow.
5. A cyclic R-module is hollow if and only if it is semi-hollow.

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