Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 4-B, pp. 1343–1353
DOI : 10.47974/JDMSC-2273

Pu–supplement sub-modules

Halah Qahtan Hamdi *
Nuhad Salim Al-Mothafar †
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

Abstract

In this article, we present a notion such as purely supplement submodule, also consider some properties of this concept, and research some relationships between purely supplement submodules and some other related ideas. Furthermore, we give amply purely supplemented modules and weakly purely supplemented modules.

Subject Classification: *16P70, 16P10.*

Keywords: *Supplement submodules, Pure submodules, Purely-supplement submodules, Amply purely supplement submodules, Weakly purely supplement submodules, Purely-supplemented modules, Amply purely supplemented modules, Weakly purely supplemented modules.*

1. Introduction

In our work, R has the identity and is commutative. M unitary left R-modul. Its established that, for ideal I in R, a sub-module N in M is considered pure (for short $N \leq_P M$) if $IM \cap N = IN$ [1]. If N a proper sub module, for the module M named small. In short $(N \ll M)$ if sub module K fora module M, $N + K = M$ implies that K equals M, [2]. A new submodule was created by Ahmed, Dhari, and Atiya [3] and it is a generalization of a small, which is named a purely small submodule (Pu-small) submodule, as A sub-module N named purely small sub-module $N \ll_{Pu} M$, when $N + K \neq M$, when pure proper sub - module K for M. Equivalently; $N \ll_{Pu} M$, when the pure sub module K for the module M with $N + K = M$ so $K = M$.

* *E-mail:* **hala.hamdi@sc.uobaghdad.edu.iq (Corresponding Author)**
† *E-mail:* **nuhad.salim@sc.uobaghdad.edu.iq**

In this search as in [4], [5], [6], and [7], we will use purely- small submodules to present a new generalization of supplement, weakly supplement, ample supplement, supplemented modules, weakly supplemented modules, and amply supplemented. Also, we will practice purely- small submodules to present a new generalization of purely-supplemented modules, called (amply, weakly) purely-supplemented modules [8]. We will prove the main properties of these concepts. We give some of their elementary properties and give some remarks and propositions of these concepts.

Now, we need to list the basic properties of the concept of Pu-small submodule.

Lemma 1: [14] *If D is R -module & P, $S \leq D$ s.t $P \leq S \leq D$, suppose $S \ll_{Pu} D$, so $P \ll_{Pu} D$.*

Lemma 2: [14] *If P, S are submodule inD . If $P + S \ll_{Pu} D$, so $P \ll_{Pu} D$ &$S \ll_{Pu} D$.*

Lemma 3: [11]: *If $\pi: M \to \frac{M}{A}$ be the natural epimorphism & $Ker\,\pi = A$ pure. assume $B \ll_{Pu} M$,so $\pi(B) = \frac{B}{A} \ll_{Pu} \frac{M}{A}$.*

2. Purely Supplement Sub-modules

This paper presents and studies purely-supplement submodules with some kinds of purely-supplement, such as weakly purely-supplement also ample purely-supplement sub-modules. So, purely-supplemented, weakly purely-supplemented, and amply purely-supplemented modules are studied, with examples and basic properties.

Suppose M is R-module & $\mathrm{Q}, \mathrm{G} \leq \mathrm{M}$, if $\mathrm{M} = \mathrm{Q} + \mathrm{G}$ & G be minimal relative to the property $\mathrm{M} = \mathrm{Q} + \mathrm{G}$, and $\mathrm{Q} \cap \mathrm{G} \ll \mathrm{G}$, then G named supplement submodule for U,[9, 1] . A module M a supplemented when each sub - module have supplement. See [10,12].

Definition 2.1: The sub - module N named purely-supplement for short (Pu-supplement) of V , when $\mathrm{N} + \mathrm{V} = \mathrm{M}$ & $\mathrm{N} \cap \mathrm{V} \ll_{\mathrm{Pu}} \mathrm{N}$.

Remarks and Examples 2.2:

1. Clearly each supplement submodule be Pu -supplement, but reverse not true. For example;- Let $\mathrm{N} = \langle\bar{2}\rangle, \mathrm{V} = \langle\bar{3}\rangle$ in Z-module Z_{24}, $\mathrm{N} + \mathrm{V} = \mathrm{Z}_{24}$, $\mathrm{N} \cap \mathrm{V} = \langle\bar{6}\rangle \ll_{\mathrm{Pu}} \langle\bar{2}\rangle$, since $\langle\bar{6}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$, hence $\langle\bar{2}\rangle$ is Pu - supplement of $\langle\bar{3}\rangle$ in Z_{24}, but not supplement since $\mathrm{N} \cap \mathrm{V} = \langle\bar{6}\rangle$, $\langle\bar{6}\rangle$ not small in $\langle\bar{2}\rangle$, since $\langle\bar{6}\rangle + \langle\bar{4}\rangle = \langle\bar{2}\rangle$ and $\langle\bar{4}\rangle \neq \langle\bar{2}\rangle$.
2. If $\mathrm{N} = \langle\bar{2}\rangle$ and $\mathrm{V} = \langle\bar{3}\rangle$, $\mathrm{N} + \mathrm{V} = \mathrm{Z}_{12}$, $\mathrm{N} \cap \mathrm{V} = \langle\bar{6}\rangle$, $\langle\bar{6}\rangle$ not Pu -small in $\langle\bar{2}\rangle$ Since $\langle\bar{6}\rangle + \langle\bar{4}\rangle = \langle\bar{2}\rangle$ and $\langle\bar{4}\rangle$ is pure in Z_{12} , hence $\langle\bar{4}\rangle \leq_{\mathrm{P}} \langle\bar{2}\rangle$ and $\langle\bar{4}\rangle \neq \langle\bar{2}\rangle$, so $\langle\bar{2}\rangle$ is not Pu -supplement of $\langle\bar{3}\rangle$ is Z_{12}.

Proposition 2.3: Assume that P , S are sub module, so we have statement equivalents.

1- P be Pu -supplement forS in M.

2- $M = P + S$ & $P \cap S \ll_{Pu} M$.

Remark 2.4: For any module, we have the following:

M is amply Pu -supplemented ⇒ M is Pu–supplemented ⟹ M is weakly Pu-supplemented.

Proof: Clear.

Proposition 2.5: If M is Pu supplemented, weakly Pu – supplemented& amply Pu-supplemented are closed under isomorphisms.

Proof: Assume $f: M_1 \longrightarrow M_2$ & A , B sub modules in M_2 with $A + B = M_2$. Then$f^{-1}(A) + f^{-1}(B) = M$, Since for $x \in M_1$ $f(x) \in M_2$ so there exists a belonging to A and b belonging to B s.t f (x) = a +b. But f isomorphism. Thus there exist $a_1 \in M_1, b_1 \in M_1$ with $f(a_1) = a$ and $f(b_1) = b$. Therefore f (x)=$f(a_1) + f(b_1) = (a_1 + b_1)$.This implies that $X - (a_1 + b_1) \in \operatorname{Ker} f$. so $X = a_1 + b_1 + K$ for$k \in \operatorname{Ker} f$. Thus $X \in f^{-1}(A) + f^{-1}(B)$. Therefore $M_1 = f^{-1}(A) + f^{-1}(B)$. But M_1 amply supplemented, then $f^{-1}(B)$ contains a Pu-supplemented W of $f^{-1}(A)$ in M_1, i.e $f^{-1}(A) + W = M_1$ and $f^{-1}(A) \cap W \ll_{Pu} W$. Now, it is clear that $A + f(W) = M_2$. claim $A \cap f(W) \ll_{Pu} f(W)$ where $f(W) \leq_P B$, since $f^{-1}(A) \cap W \ll_{Pu} W$ then it is clear, $A + f(W) = M_2$.Thus M_2 is an amply Pu - supplement.

Proposition 2.6: If P & S are sub - modules s.t $S \leq P \leq M$. If S bePu-supplement, so S must be Pu -supplement in P.

Proof: Since S is Pu -supplement in M, ∃ sub - module K s.t $S + K = M$ &$S \cap K \ll_{Pu} S$. now $P = P \cap M = P \cap (S + K)$ so $P = S + (P \cap K)$. Hence S be Pu-supplement for $P \cap K$ in A since $S \cap (P \cap K) = S \cap K \ll_{Pu} S$. Therefore, S is Pu-supplement in P .

Proposition 2.7: If P & S sub - modules, where P pure in M s.t $P \leq S \leq M$. If S be Pu-supplement , so $\frac{S}{P}$ is Pu -supplement in $\frac{M}{P}$.

Proof: Since S is Pu -supplement, a sub -module K exists s.t $S + K = M$ and $S \cap K \ll_{Pu} S$. Now $\frac{M}{P} = \frac{S+K}{P} = \frac{S}{P} + \frac{K+P}{P}$ & $\frac{S}{P} \cap \frac{K+P}{P} = \frac{S \cap (K+P)}{P} = \frac{P+(S \cap K)}{P}$, $S \cap K \ll_{Pu} S$ by [Lemma 3], we have $\frac{P+(S \cap K)}{P} \ll_{Pu} \frac{S}{P}$. Thus $\frac{S}{P}$ be Pu-supplement in $\frac{M}{P}$.

Proposition 2.8: Assume M is there a module, & A be Pu -hollow submodule of M. Then A be Pu-supplement for A_1 in M s.t $M = A + A_1$.

Proof: Assume A_1 be proper sub - module s.t $M = A + A_1$. So $A \cap A_1$ be proper submodule for A if $A \cap A_1 = A$. Hence $A \leq A_1$ and $M = A_1$, which is a contradiction. Now, since A is Pu-hollow, thus $A \cap A_1$ is Pu-small in A . Therefore, A is a Pu -supplement of A.

Lemma 2.9: *Let C a R-module with PIP property & P, S sub-module for C with S pure in C. If $P \ll_{Pu} S$ then $P \ll_{Pu} C$.*

Proof: Assume K pure s.t $P + K = C$. To show K = C . Now, since $S \cap (P + K) = S \cap C$, implies that $P + (K \cap S) = B$. To show $K \cap S$ is pure in S. K pure in C & S pure, C has PIP property, so $K \cap B$ is pure in C, but $K \cap B \leq B$ by [12, Remark(2.1.18)]. Then $K \cap S$ is pure in S. Now $P \ll_{Pu} S$, then $K \cap S = S$, i.e $S \leq K$ so $P \leq K$,hence $K = P + K = C$. Thus $K = C$, thus $P \ll_{Pu} C$.

Recall that: If intersection for two pure is also a pure, so R-module M has PIP Property, [13].

Proposition 2.10: Let A and A_1 is Pu-supplement of A in M with M(PIP Property). Then

1) If $M = X + A_1$, for some sub-module X of A , then A_1 is Pu-supplement for X in M .
2) For $A_{2 \ll_{Pu}} M$, so A_1 is Pu-supplement of $A + A_2$ in M (under the condition $A_1 \ll_{Pu} M$).

3. Weakly Pu –Supplement and Amply Pu –Supplement Modules

Here, one generalization for Pu-supplement, weakly Pu-supplement, and ample -Pu-supplement submodules are introduced with some properties.

Definition 3.1: M named weakly Pu-supplemented module if, for all sub-module P in M, there is sub-module S in M s.t, $M = P + S$ also $P \cap S \ll_{Pu} M$.

Examples and Remarks 3.2:

1- Each Pu-supplement sub-module exists a weakly Pu-supplement, however, the covers need not be correct. In Z_{12}as Z- module the sub-module $< \overline{2} >$ is a weak Pu-supplement of $< \overline{3} >$ while $< \overline{2} >$ is not Pu-supplement of $< \overline{3} >$ in Z_{12}. Since $< \overline{2} > + < \overline{3} >= Z_{12}and< \overline{2} > \cap < \overline{3} > = < \overline{6} > \ll_{Pu} Z_{12}$, while $< \overline{6} > + < \overline{4} > =< \overline{2} >$, and $< \overline{4} ><_P< \overline{2} >$ but $< \overline{4} >\neq< \overline{2} >$, so $< \overline{6} >$ is not Pu-small in $< \overline{2} >$.

2- In Z_{12} as Z-module the submodule $< \overline{4} >$ is not a weakly Pu-supplement of $< \overline{2} >$ since $< \overline{4} > + < \overline{2} >\neq Z_{12}$.

3- Consider Z_{24} as Z-module the submodule $<\overline{3}>$ has an ample Pu-supplement in Z_{24} , since $<\overline{3}>+<\overline{2}>= Z_{24}$ and $<\overline{4}>\subset<\overline{2}>$ where $<\overline{4}>$ is Pu-supplement of $<\overline{3}>$ in Z_{24} .

4- In Z_{24} as Z-module , the submodule $<\overline{2}>$ has no ample Pu-supplement in Z_{24}, since $<\overline{3}> +<\overline{2}> = Z_{24}$, but $<\overline{3}>$ has no proper submodule which is a Pu-supplement of $<\overline{2}>$ in Z_{24}. By Remark (2.2) (2).

The following proposition explains the relationship between the weakly Pu-supplement and Pu-co essential sub-module.

Assume M a R-module, & A belongs to M as a sub-module and is called a purely co-essential submodule of B for short (Pu-coessential submodule) if $\frac{B}{A} \ll_{Pu} \frac{M}{A}$ ($A \leq_{Pu.ce} B$).

Proposition 3.3: Assume that T & G sub module using (PSP) Property & are $T \leq_P M$. If T and G has the same weak Pu-supplement submodule, so T is Pu-co essential submodule for G.

Proof: Assume K be weakly Pu-supplement submodule for T & G. so, we have $T + K = G + K = M$,& $T\cap K \ll_{Pu} M$ & $G\cap K \ll_{Pu} M$.

So $G = G\cap(T + K)$ by (Modular Law) $G= T + (G\cap K)$. Now let $\frac{M}{T} = \frac{G}{T} + \frac{X}{T}$. Where $\frac{X}{T} \leq_{PU} \frac{M}{T}$. Hence by [15, Remark (3.1.5), P.56], $X \leq_P M$. $M = G + X = \left(T + (G\cap K)\right) + X = T + X + (G\cap K)$. Since M has PSP Property $T + X \leq_P M$, & $G\cap K \ll_{Pu} M$, So $M = T + X$ with $T \leq X$. Thus, $M = X$, so T is Pu-co essential sub-module of G.

Recall: If M be R- module. The sub - module N named Purely–co closed in summary (Pu-co closed) denoted by($N \leq_{Pu.cc} M$).If $K \leq N$, with $\frac{N}{K} \ll_{Pu} \frac{M}{K}$, then $N = K$.

Proposition 3.4: If N submodule. Consider the following conditions.

1- N is Pu- co closed in M.

2- For all $P \leq N, P \ll_{Pu} M$ implies that $P \ll_{Pu} N$.

Then (1) $\Longrightarrow$ (2). It M is a weakly Pu- supplement, then (2) $\Longrightarrow$ (1).

Proof: (1) $\Longrightarrow$ (2) if N is Pu -co closed submodule & $P \leq N$ & $P \ll_{Pu} M$. let $N = P + S$ for some pure submodule S of N. We claim that $\frac{N}{S} \ll_{Pu} \frac{M}{S}$ To prove this claim. Let $\frac{N}{S} + \frac{H}{S} = \frac{M}{S}$, where $\frac{H}{S} \leq_P \frac{M}{S}$ so we have $M = N + H = P + S + H = P + H$ and by [14, Remark 4, P.14] $H <_P M$ .But $P \ll_{Pu} M$ thus $M = H$ & hence $\frac{M}{S} = \frac{H}{S}$. Now, since N is Pu-co closed also. $\frac{N}{S} \ll_{Pu} \frac{M}{S}$, therefore $N = S$ i.e. $P \ll_{Pu} N$.

(2) $\Rightarrow$ (1)Assume that M is a weakly Pu-supplemented module. So, exist sub module L in M s.t $\mathrm{M{=}N+L}$ & $\mathrm{N\cap L \ll_{Pu} M}$.Since $\mathrm{N\cap L \leq N}$ & $\mathrm{N\cap L \ll_{Pu} M}$, so by (2), $\mathrm{N\cap L \ll_{Pu} N}$. This complete proof.

Corollary 3.5: *Assume M be module also $N \leq K$ be submodules of M.If N a Pu -co closed also K Pu-coclosed ,so N Pu-co closed.*

Proof: Assume A is submodule such that $\frac{\mathrm{N}}{\mathrm{A}} \ll_{\mathrm{Pu}} \frac{\mathrm{M}}{\mathrm{A}}$ by [11, Remark (2.2) (1)], $\frac{\mathrm{K}}{\mathrm{A}}$ is Pu-co closed in $\frac{\mathrm{M}}{\mathrm{A}}$. Therefore $\frac{\mathrm{N}}{\mathrm{A}} \ll_{\mathrm{Pu}} \frac{\mathrm{K}}{\mathrm{A}}$ By Proposition (3.4). Since N bePu-co closed , thus $\mathrm{N = A}$. Hence N Pu-co closed.

Recall A submodule W has ample supplement for submodule $\mathrm{J \subset M}$ & $\mathrm{W + J = M}$, this supplement J f or W & $\mathrm{J_1 \subset J}$ .See [16].

Definition 3.6: P be sub - module S has an ample Pu- Supplement in M if $\mathrm{P \subset M}$ & $\mathrm{P + S = M}$ $\exists$Pu-supplement $\mathrm{P_1}$ & $\mathrm{P_1 \subset P}$.

Examples and Remarks 3.7:

1- Consider Z_{24} as Z-module, the submodule $\langle\bar{3}\rangle$ has an ample Pu -supplement in Z_{24} since $\langle\bar{3}\rangle + \langle\bar{2}\rangle = Z_{24}$ and$\langle\bar{4}\rangle \subset \langle\bar{2}\rangle$, where $\langle\bar{4}\rangle$ Is an Pu-supplement of $\langle\bar{3}\rangle$ in Z_{24}.

2- In Z_{24} as Z-module, the submodule $\langle\bar{2}\rangle$ has no ample Pu-supplement in Z_{24} Since $\langle\bar{3}\rangle + \langle\bar{2}\rangle = Z_{24}$, but has no proper submodule which is an Pu -supplement of $\langle\bar{2}\rangle$ in Z_{24}.

4. Pu – Supplemented, Weakly Pu – Supplemented and Amply Pu – Supplemented Modules

Definition 4.1: The R-module M named Pu-Supplemented when each sub-module of M has an Pu- Supplemented in M.

Examples and remarks 4.2:

1- Each supplemented module is an Pu -supplemented. A reverse need not accurate since Pu -small pure must not small submodule. In Z_{12} as Z-module is the Pu-supplemented, while it is not supplemented.

2- where M be F-regular R-module, so the supplemented and Pu-supplemented modules coincide, since the small sub-module and Pu-small sub-module coincide if M is F-regular module, see [3].

3- Z-module Q not supplemented module. So, its not Pu-supplemented Z-module.

4- If M semi-simple module, Then M Pu-supplemented module.

5- Z-module Z_{24} is Pu-supplemented since $\langle\bar{0}\rangle$ has Pu-supplement Z_{24}, $\langle\bar{2}\rangle$ has Pu-supplement $\langle\bar{3}\rangle$,$\langle\bar{3}\rangle$ has Pu-supplement $\langle\bar{4}\rangle$,$\langle\bar{4}\rangle$has Pu-supplement $\langle\bar{3}\rangle$, $\langle\bar{6}\rangle$ has Pu-supplement Z_{24}, and Z_{24} Pu–supplement $\langle\bar{0}\rangle$.

6- Z-module Z not Pu–supplemented since the submodule $2Z$ has no an Pu–supplement submodule. Since the only Pu-small submodule of Z is $\langle\bar{0}\rangle$.

Proposition 4.3: If A & B are sub module, where A Pu–supplemented module. If $\mathrm{A}+\mathrm{B}$ have Pu-supplemented sub-module, so B has Pu–supplemented submodule.

Proposition 4.4: If $\mathrm{M}=\mathrm{M}_1+\mathrm{M}_2$, where M_1 & M_2 a Pu-supplemented module. Implies that M is a Pu -supplemented module.

Proof: Suppose U a submodule. Then $\mathrm{M}_1+\mathrm{M}_2+\mathrm{U}=\mathrm{M}$. Since $(\mathrm{M}_2+\mathrm{U})\cap\mathrm{M}_1$ is a sub-module of the Pu-supplemented M_1, there is submodule K of M_1 which is the Pu-supplement of $(\mathrm{M}_2+\mathrm{U})\cap\mathrm{M}_1$. Which assures that $(\mathrm{M}_2+\mathrm{U})\cap\mathrm{M}_1+\mathrm{K}=(\mathrm{M}_2+\mathrm{U})\cap\mathrm{M}_1=\mathrm{M}_1$ and $(\mathrm{M}_2+\mathrm{U})\cap\mathrm{M}_1\cap\mathrm{K}=(\mathrm{M}_2+\mathrm{U})\cap\mathrm{K}\ll_{\mathrm{Pu}}\mathrm{K}$. Since $\mathrm{M}_1$is a submodule of $\mathrm{M}_2+\mathrm{U}$, it follows that $\mathrm{M}=\mathrm{M}_1+\mathrm{M}_2+\mathrm{U}$ and $\mathrm{M}_2+\mathrm{U}+\mathrm{K}=\mathrm{M}$, thus $\mathrm{M}_2+\mathrm{U}$ has an Pu–supplemented submodule, therefore, by Proposition (4.3) U has an Pu-supplemented submodule.

Corollary 4.5: *Let $M=\sum_{i=1}^{n}M_i$, where $M_1,M_2,\dots,M_n$ Are Pu-supplemented modules, implies that M is a Pu-supplemented module.*

Proposition 4.6: Let $\varphi\colon \mathrm{M}\to\mathrm{N}$ be an isomorphism. If $\mathrm{A}\ll_{\mathrm{Pu}}\mathrm{M}$, implies $\varphi(\mathrm{A})\ll_{\mathrm{Pu}}\mathrm{N}$.

Proof: If L pure submodule s.t $\varphi(\mathrm{A})+\mathrm{L}=\mathrm{N}$ since φ isomorphism, then by 15, Remark (3.1.5), p56] φ^{-1} (L) is pure in M. Hence A+φ^{-1} (L) = M. Since $\mathrm{A}\ll_{\mathrm{Pu}}\mathrm{M}$, hence $\varphi^{-1}(\mathrm{L})=\mathrm{M}$ & $\varphi(\mathrm{M})\leq\mathrm{L}$ so that $\varphi(\mathrm{A})\leq\mathrm{L}$. Thus $\mathrm{L}=\varphi(\mathrm{A})+\mathrm{L}=\mathrm{N}$ and $\varphi(\mathrm{A})\ll_{\mathrm{Pu}}\mathrm{N}$.

Proposition 4.7: If M is Pu-supplemented, weakly Pu -supplemented and amply Pu-supplemented. so isomorphic image for M be Pu–supplemented module, weakly Pu -supplemented and amply Pu-supplemented.

Proof: Assume $\mathrm{f}\colon\mathrm{M}\to\mathrm{M}$ a R-isomorphic &B is submodule in $\mathrm{f(M)}$.We have f^{-1} (B) is a submodule of M, since M is Pu–supplemented $\exists$ submodule Ks.t K be Pu–supplemented submodel of f^{-1} (B), which assures $\mathrm{f}^{-1}(\mathrm{B})+\mathrm{K}=\mathrm{M}$ and $\mathrm{f}\big(\mathrm{f}^{-1}(\mathrm{B})\big)\cap\mathrm{f(K)}\ll_{\mathrm{Pu}}\mathrm{K}$, $\mathrm{f}\big(\mathrm{f}^{-1}(\mathrm{B})\big)\cap\mathrm{f(K)}=\mathrm{B}\cap\mathrm{f(M)}\cap\mathrm{f(K)}=\mathrm{B}\cap\mathrm{f(K)}$. Hence, $\mathrm{B}\cap\mathrm{f(K)}\ll_{\mathrm{Pu}}\mathrm{f(K)}$ and $\mathrm{B}+\mathrm{f(K)}=\mathrm{f(M)}$. To see that let $\mathrm{y}\in\mathrm{f(M)}$, $\exists\mathrm{x}\in\mathrm{M}$ s.t $\mathrm{y}=\mathrm{f(x)}$. $\mathrm{M}=\mathrm{f}^{-1}(\mathrm{B})+\mathrm{K}$, $\mathrm{x}=\mathrm{a}+\mathrm{k}$, $\mathrm{a}\in\mathrm{f}^{-1}(\mathrm{B})$ & $\mathrm{k}\in\mathrm{K}$. Thus, $\mathrm{f(a)}\in\mathrm{f(B)}$,$\mathrm{f(x)}\in\mathrm{f(K)}$, and $\mathrm{y}=\mathrm{f(x)}=\mathrm{f(a)}+\mathrm{f(K)}\in\mathrm{B}+\mathrm{f(K)}$. Therefore,f(M)be Pu–supplemented module.

Corollary 4.8: *If M is Pu–supplemented & N be pure sub-, implies that $\frac{M}{N}$ be Pu–supplemented module.*

Proof: Assume M is Pu–supplemented module also N sub module. There is the natural epimorphism $\pi = M \longrightarrow \frac{M}{N}$ .Hence $\frac{M}{N}$ is an epimorphic image of M, Proposition (4.7) assure that $\frac{M}{N}$ is an Pu -supplemented module.

Corollary 4.9: *Let $M = \oplus_{i=1}^{n} M_i$, M a Pu–supplemented module iff $M_1, M_2, \dots, M_n$ Are Pu-supplemented modules.*

Remark 4.10**:** The converse of corollary (4.8) must not accurate. For example $\frac{Z}{6Z} \simeq Z_6$ such as Z-module is an Pu-supplemented module. Since $\langle 0 \rangle$ is Pu-supplement Z_6, $\langle 2 \rangle$ is Pu–supplemented $\langle 3 \rangle$,$\langle 3 \rangle$ Pu–supplemented $\langle 2 \rangle$, and Z_6 is Pu-supplement $\langle 0 \rangle$. But the Z as Z-module is not Pu–supplemented module. For example:- the Z- module Z is not Pu–supplemented. Since $2Z \leq Z$ and 2Z has no Pu–supplemented sub-module the only Pu–small submodule of Z is $\langle 0 \rangle$.

Now, we will present the relationship between Pu–supplemented modules, semisimple modules, and $\mathrm{Rad}_{\mathrm{Pu}}(\mathrm{M})$ Sub-module.

Recall: A module M be semisimple when any sub - module for M is direct summand, [1],[16], [17-19].

Lemma 4.11**:** *Assume M is R-module with PIP property also let $A, B, \&\ N$ submodule with N pure in M such that $A \leq B \leq N \leq M$. If $B \ll_{Pu} N$ implies that $A \ll_{Pu} M$.*

Recall that: If M is an R-module then sum for each purely small submodule of M is named purely radical (for short, Pu-radical module) of an R-module M represented by $Rad_{Pu}(M)$={$\sum(N: N \ll_{Pu} M)$}. Clearly every radical of R-module M contained in Pu-radical of M i.e. $Rad(M) \leq Rad_{Pu}M$. Where $Rad(M)$= {The sum of all small submodules of M},[9]. Details in, [10,17].

Lemma 4.12**:** *Let M be a Pu-supplemented module using (PIP) property, also $Rad_{Pu}(M)$ is a pure sub-module Then.*

a : $\frac{M}{Rad_{Pu}(M)}$ is semi - simple module.

b : If A be sub-module & $A \cap Rad_{Pu}(M) = \{0\}$, so A semi-simple.

Proof a : Let $\frac{N}{Rad_{Pu}(M)}$ sub-module of $\frac{M}{Rad_{Pu}(M)}$. now M a Pu-supplemented module, there is submodul K which a Pu-supplemented submodule of N. Thus $K + N = M$ & $N \cap K \ll_{Pu} K \leq M$, by Lemma (4.11). Hence $N \cap K \ll_{pu} M$, assures $N \cap K \leq Rad_{Pu}(M)$. Also $\frac{M}{Rad_{Pu}(M)} = \frac{N+K}{Rad_{Pu}(M)} = \frac{N}{Rad_{Pu}(M)} + \frac{K+Rad_{Pu}(M)}{Rad_{Pu}(M)}$ and $\frac{N}{Rad_{Pu}(M)} \cap \frac{K+Rad_{Pu}(M)}{Rad_{Pu}(M)} = \frac{N \cap (K+Rad_{Pu}(M))}{Rad_{Pu}(M)} = 0$.Thus , $\frac{N}{Rad_{Pu}(M)}$ direct summand of $\frac{M}{Rad_{Pu}(M)}$ & $\frac{M}{Rad_{Pu}(M)}$ is semisimple.

Proof b : Let A be a submodule of M with $A \cap Rad_{Pu}(M) =< 0 >$. From (a), $\frac{M}{Rad_{Pu}(M)}$ is the semi-simple module, since every submodule of semi-simple is semi-simple, then by [1], $\frac{A+Rad_{Pu}(M)}{Rad_{Pu}(M)}$ is semisimple .(the first isomorphism theorem [1]) , $\frac{A+Rad_{Pu}(M)}{Rad_{Pu}(M)} = \frac{A}{A \cap Rad_{Pu}(M)} \simeq A$. Therefore, A is semisimple.

Remark 4.13**:** If M is an Pu–supplemented module, then $\frac{M}{Rad_{Pu}(M)}$ Need not have only a trivial Pu-small sub-module. For example the Z_6 is not Pu-supplemented, Since every sub-module of Z_6 is $\langle \bar{0} \rangle$, so $Rad_{Pu}(Z_6) = \langle \bar{0} \rangle$. $\frac{\langle \bar{2} \rangle}{Rad_{Pu}(Z_6)} \simeq \langle \bar{2} \rangle$ is not Pu–small in $\frac{Z_6}{Rad_{Pu}(Z_6)} \simeq Z_6$. therefore, $\frac{Z_6}{Rad_{Pu}(Z_6)}$ it does not have a proper a Pu–small submodule.

The following explains the relationship between Pu–supplemented modules and Pu -hollow modules.

Recall that: An R-module D named hollow if every proper sub - module is Pu-small,[14].

Proposition 4.14**:** Every a Pu-hollow module is Pu–supplemented.

Proof: Assume M a Pu-hollow module, also if A a submodule for M, If $A = M$, So A has Pu–supplemented $\langle 0 \rangle$, if A is the proper submodule of M. Hence A is Pu-small submodule of M. Since $A + M = M$ and $A \cap M = A \ll_{Pu} M$, so A has Pu–supplemented. Therefore M is Pu–supplemented.

Remark 4.15**:** The conversion of Proposition (4.14) must not accurate. For example Z_6- module is Pu–supplemented. See Remark (4.13), but it is not a Pu–hollow module. For example the Z6 as Z6-module, since $\langle \bar{2} \rangle \leq Z_6$, $\langle \bar{2} \rangle$ not Pu-small submodule in Z_6 , $Z_6$is not Pu-hollow.

5. Weakly Supplemented and Amply Supplemented Modules.

We now introduce the generalization of weakly and amply supplemented modules with some properties.

Recall that: An R-module M a weakly supplemented (amply supplemented) if a submodule has the weak supplement (ample supplement) submodule in M. See [16].

Definition 5.1**:** Module M is named weakly Pu- supplemented. (amply Pu-supplemented) if each submodule of M has weak Pu-supplement (ample Pu-supplement) sub-module in M.

Examples and Remarks 5.2**:**

1- Every amply Pu-supplemented is an Pu-supplemented module. However, the converse must not be correct in general. For example the Z-module Z_{24}

Pu-supplemented. See example and remarks (4.2)(5) but not amply Pu-supplemented, since $\langle\bar{2}\rangle$ Is not an ample Pu-supplement submodule, See example and remarks (3.4) (2).

2- If M is F-regular R-module, then weakly supplemented (amply supplemented) and weakly Pu - supplemented (amply Pu - supplemented) modules coincide.

3- Every Pu -supplemented is a weakly Pu -supplemented module. However, the converse must not correct in for example, the Z-module Q is F-regular which is weakly supplemented $[16, P, 238]$ from (2), Q as Z-module is weakly Pu-supplemented, However, the Z-module Q is not Pu - supplemented.

4- The Z-module Z is not amply Pu-supplemented (not weakly Pu-supplemented) since $2Z$ has an Pu-supplemented (a weakly Pu-supplement) submodule. See examples and remarks (4.2) (6).

The following shows the properties of amply Pu-supplemented and weakly Pu -supplemented.

Proposition 5.3**:** An isomorphic image of amply Pu-supplemented is an amply Pu-supplemented module.

Proof: Assume $f\colon M \longrightarrow \acute{M}$ be an isomorphism also let M the amply Pu-supplemented module let L & N a sub - module for $\acute{M}$ s.t $L + N = \acute{M}$. Hence $f^{-1}(L) + f^{-1}(N) = M$, since M is amply Pu-supplemented, so there exists Pu-supplement X for $f^{-1}(N)$ s.t $X \subset f^{-1}(N)$. Hence $f^{-1}(L) + X = M$ and $f^{-1}(L) \cap X \ll_{Pu} f(X)$ By Proposition (4.6). Therefore $\acute{M}$ Is an amply Pu - supplemented module.

Proposition 5.4**:** An isomorphic image of weakly Pu–supplemented is a weakly Pu-supplemented module.

Proof: It follows from the proposition(4.7).

References

[1] F. Kasch, Modules and Rings, vol. 17. Academic Press (1982).

[2] A. Kabban and W. Khalid, "On Jacobson-Small Submodules," *Iraqi J. Sci.*, vol. 60, no. 7, pp. 1584–1591 (2019), doi: 10.24996/ijs.2019.60.7.18.

[3] M Ahmed, I. A. Dhari, and Z. A. Atiya. A., "Purely Small and Purely Hollow Modules," *Iraqi J. Sci.*, vol. 63, no. 12, pp. 5487–5495 (2022), doi: 10.24996/ijs.2022.63.12.34.

[4] G. F. Birkenmeier, F. T. Mutlu, C. Nebiyev, N. Sokmez, and A. Tercan, "Goldie*-Supplemented Modules – Corrigendum," *Glasg. Math. J.*, vol. 54, no. 2, pp. 479–480 (2012).

[5] B. Koşar, C. Nebiyev, and N. Sökmez, "G-Supplemented Modules," *Ukr. Math. J.*, vol. 67, no. 6, pp. 975–980 (2015).

[6] B. Kosar, C. Nebiyev, and A. Pekin, "A Generalization of G-Supplemented Modules," *Miskolc Math. Notes*, vol. 20, no. 1, p. 345 (2019).

[7] C. Nebiyev and H. H. Ökten, "Weakly G-Supplemented Modules," *Eur. J. Pure Appl. Math.*, vol. 10, no. 3, pp. 521–528 (2017).

[8] E. M. Kamil and W. Khalid, "On H-µ-Supplemented Modules," *Iraqi J. Sci.*, vol. 60, no. 5, pp. 1129–1137 (2019), doi: 10.24996/ijs.2019.60.5.21.

[9] H. H. Nuha, The Radicals of Modules (1996).

[10] R. Wisbauer, Foundations of Module and Ring Theory (1991).

[11] H. K. Hamdi and N. S. Al-Mothafar, "Purely Co-Closed and Purely Co-Essential Submodules," Taylor and Francis, under publication (2024).

[12] N. S. Al-Mothafer, Sums and Intersections of Submodules (2002).

[13] B. H. Al-Bahraany, Modules with the Pure Intersection Property (2000).

[14] H. Q. Hamdi and N. S. Al-Mothafar, "Study on Purely Small Submodules," *J. Discrete Math. Sci. Cryptogr.*, vol. 27, no. 5, pp. 1723–1728 (2024), doi: 10.47974/JDMSC-2017.

[15] I. H. Muslem, "Some Types of Retractable and Compressible Modules," M.Sc. thesis, College of Education for Pure Science, Univ. of Baghdad (2016).

[16] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, Lifting Modules: Supplements and Projectivity in Module Theory. Frontiers in Mathematics, Boston: Birkhäuser (2006).

[17] C. Lomp, "On Dual Goldie Dimension," Diploma thesis, Univ. of Düsseldorf (1996).

[18] I. S. Ahmed, A. Al-Fayadh, H. H. Ebrahim, and L. S. Abdalbaqi, "Implementation of the Neutrosophic Sets in Measurable Space with Respect to Neutrosophic Ring," *Int. J. Neutrosophic Sci.*, vol. 25, no. 3, pp. 187–193 (2025).

[19] S. H. Asaad, I. S. Ahmed, and A. S. Mohammed, "Generalization Fuzzy Completely Intra G-Ideals in G-Semi Groups," *J. Phys. Conf. Ser.*, vol. 1664, no. 1 (2020), doi: 10.1088/1742-6596/1664/1/012028.

Received February, 2025