Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 4-B, pp. 1369–1373
DOI : 10.47974/JDMSC-2276

Outer derivations of low-dimensional diassociative algebras

N. F. Mohammed *
Hanan F. Qasim †
Z. H. Maibed §
Department of Mathematics
College of Education for Pure Sciences (Ibn-Al-Haithem)
University of Baghdad
Baghdad
Iraq

Abstract

The work of outer derivations is extremely extensive research that occurs in many sorts of mathematics fields. This work is concerned with the description of outer derivation for diassociative algebras which is isomorphic to the first cohomology group. The inner derivation (resp. the derivation) interpreted as 1-coboundaries (resp. 1-cocycles). In this regard, we propose an algebraic method to get an important specific case of derivations, known as inner derivation. The procedure is used to determine the inner derivatives of diassociative algebras.

Subject Classification: *13D03, 16-04, 16S99.*

Keywords: *Outer derivation, Inner derivation, Diassociative algebra.*

1. Introduction

In 2001, Loday et al. [1] introduced a new class of algebra so-called diassociative algebras. Loday has also demonstrated the connection of two dialgebras and algebras of Leibniz. On the other hand, the studying of outer derivation groups is among the most active research in several sciences. It can be interpreted as elements of the first cohomology group that has essential techniques of algebraic topology. This study provides a solution to some cohomological problems that are most widely used by researchers in [2–4]. The dimension of outer derivations is a particularly valuable invariant that is critical for geometrically classifying diassociative algebras and it has been intensively treated by many authors, see for example [5-10].

* *E-mail:* **nadia.f.m@ihcoedu.uobaghdad.edu.iq (Corresponding Author)**
† *E-mail:* **Hanan.f.k@ihcoedu.uobaghdad.edu.iq**
§ *E-mail:* **zena.h.m@ihcoedu.uobaghdad.edu.iq**

© Taru Publications

In all of this, the current research emphasizes on describing outer derivation spaces of dissociative algebras. We devised an approach for representing the inner derivation mapping. Then, this method is used to complex dissociative algebras of dimension two.

2. Building Foundations

Definition 2.1: Suppose D is a linear space provided with two linear associative of binary operations designated by $\dashv$ together with $\vdash$, called respectively left and right products, achieving each of the axioms:

$$\alpha \dashv (\beta \vdash \delta) = \alpha \dashv (\beta \dashv \delta),$$
$$\alpha \vdash (\beta \dashv \delta) = (\alpha \vdash \beta) \dashv z,$$
$$(\alpha \vdash \beta) \vdash \delta) = (\alpha \dashv \beta) \vdash \delta$$

$\forall \alpha, \beta, \delta \in V$. The $(D, \dashv, \vdash)$ with the utilized axioms is considered a diassociative algebra.

By providing the basis $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_n\}$ in D, elements α and β of D are able expressed as $\alpha = \sum_{i=1}^{n} \alpha^{i} \mathcal{G}_i$ and $\beta = \sum_{j=1}^{n} \beta^{j} \mathcal{G}_j$. Notice that the products.

$\alpha \dashv \beta$ and $\alpha \vdash \beta$ of any α and β in D, is completely determined by the products $\mathcal{G}_i \dashv \mathcal{G}_j$ and $\mathcal{G}_i \vdash \mathcal{G}_j$, respectively, of pairs of basis elements, that is:

$\mathcal{G}_i \dashv \mathcal{G}_j = \sum_{k=1}^{n} \gamma_{ij}^{k} \mathcal{G}_k$ and $\mathcal{G}_i \vdash \mathcal{G}_j = \sum_{k=1}^{n} \delta_{ij}^{k} \mathcal{G}_k$

For $i, j, k = 1, 2, 3, \ldots n$.

Definition 2.2: Homomorphism of diassociative algebras D_1 and D_1 is a linear operator $f: D_1 \longrightarrow D_2$ so that $f(\alpha \vdash \beta) = f(\alpha) \vdash f(\beta)$ and $f(\alpha \dashv \beta) = f(\alpha) \dashv f(\beta), \forall \alpha, \beta \in D_1$.

Definition 2.3: A derivation of D is operator $d: D \to D$ satisfying $d(\alpha \dashv \beta) = d(\alpha) \dashv \beta + \alpha \dashv d(\beta)$ and $d(\alpha \vdash \beta) = d(\alpha) \vdash \beta + \alpha \vdash d(\beta)$, for all $\forall \alpha, \beta \in D$.

Definition 2.4: Suppose D represents diassociative algebra on a field F. and z denotes a vector in D. Let $ad_z: D \to D$ be a linear operator satisfying $ad_z(\alpha) = \alpha \dashv z - z \vdash \alpha$. The function ad_z is defined as inner derivation of D.

The whole collection of inner derivations(resp. derivations) of D refers to $Inn(D)$ (resp. $Der(D)$) which $Inn(D)$will be an ideal for $Der(D)$. The quotient space $Out(D, D) = Der(D)/Inn(D)$ is referred to as a space of outer derivatives. It can be considered as the dialgebra's one-dimensional Hochschild cohomology. In other words $H^1(D, D) \approx Out(D, D)$.

3. Main Results

Firstly, we roughly characterize the theoretical basis of the algorithm to obtain the inner derivations.

3.1 *The Algorithm for Finding Inner Derivations*

Suppose D is n-dimensional diassociative algebra while $\{\mathcal{G}_1,\cdots,\mathcal{G}_n\}$ is a basis of D. Fix a vector Z in D. Let ad_z to arise an inner derivation of D. Specifically,

$$ad_z(\mathcal{G}_i) = L^{\dashv}_{\mathcal{G}_i} z - R^{\vdash}_{\mathcal{G}_i} z \tag{1}$$

Where $L^{\dashv}_{\mathcal{G}_i} z = \mathcal{G}_i \dashv z$ and $R^{\vdash}_{\mathcal{G}_i} z = z \vdash \mathcal{G}_i$ are left and right multiplication operations, respectively. The ad_z can be a linear in a matrix structure

$$ad_z(\mathcal{G}_i) = \sum_{j=1}^{n} d_{ji}\, \mathcal{G}_j \tag{2}$$

Thus, we have

$$\sum_{j=1}^{n} d_{ji}\, \mathcal{G}_j = L^{\dashv}_{\mathcal{G}_i} z - R^{\vdash}_{\mathcal{G}_i} z \tag{3}$$

Equated the coefficient in the basis vectors resulted in the system of equations:

$$d_{ji} = \sum_{t=1}^{n} \mu_t \gamma_{it}^{j} - \sum_{t=1}^{n} \mu_t \delta_{ti}^{j} \tag{4}$$

The set of solutions provides a matrix-based of the inner derivation.

In the follow-up, D will be a 2-dimensional complex diassociative algebra and $Dias_n^q$ indicates qth the class of diassociative algebra in n-dimension. Next, we apply the algorithm to the algebraic classifying that created by [11].

Theorem 3.2: *If D is diassociative algebra. Following that, D will be isomorphic to any of these pair of non-isomorphic diassociative algebras:*

$$Dias_2^1\colon \mathcal{G}_1 \vdash \mathcal{G}_1 = \mathcal{C}_1. \quad \mathcal{G}_i \dashv \mathcal{G}_1 = \mathcal{G}_i,$$
$$Dias_2^2\colon \mathcal{G}_1 \vdash \mathcal{G}_1 = \mathcal{C}_1. \quad \mathcal{G}_1 \dashv \mathcal{G}_i = \mathcal{G}_i,$$
$$Dias_2^3\colon \mathcal{G}_1 \vdash \mathcal{G}_1 = \mathcal{G}_2, \quad \mathcal{G}_1 \dashv \mathcal{G}_1 = \alpha\mathcal{G}_2, \quad \alpha \in \mathbb{C}.$$
$$Dias_2^4\colon \mathcal{G}_1 \vdash \mathcal{G}_i = \mathcal{G}_i, \quad \mathcal{G}_i \dashv \mathcal{G}_1 = \mathcal{G}_i,$$
$$\forall i = 1,2.$$

Theorem 3.3: *The space of all inner derivations of diassociative algebras is provided in below.*

$$\mathrm{ad_z\ of}\ Dias_2^1 = \begin{bmatrix} 0 & 0 \\ 0 & \mu_1 \end{bmatrix}, \quad \mathrm{ad_z\ of}\ Dias_2^3 = \begin{bmatrix} 0 & 0 \\ (\alpha-1)\mu_1 & 0 \end{bmatrix},$$
$$\mathrm{ad_z\ of}\ Dias_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & -\mu_1 \end{bmatrix}, \ \mathrm{ad_z\ of}\ Dias_2^4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: Let us consider $Dias_2^1$. The structure constants are offered as: $\gamma_{11}^1 = \gamma_{21}^2 = \delta_{11}^1 = 1$ and all the rest are zeros. By equation (4), we have

$$d_{11} = d_{12} = d_{21} = 0 \text{ and } d_{22} = \mu_1$$

Hence, $$ad_z = d_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & \mu_1 \end{bmatrix}$$

The remaining cases may be handled similarly with a slight adjustment.

Using the derivation that obtained from [12] and Theorems 3.3, the span bases of outer derivation will be determined as stated in the next corollary:

Corollary 3.4: *The space of all outer derivation of 2-dimensional complex diassociative algebras can be found below:*

$$Out(Dias_2^1, Dias_2^1) = \{0\}, \quad Out(Dias_2^3, Dias_2^3) = span_{\mathbb{C}}\{\overline{E_{11}}\},$$
$$Out(Dias_2^2, Dias_2^2) = \{0\}, \quad Out(Dias_2^4, Dias_2^4) = span_{\mathbb{C}}\{\overline{E_{21}}, \overline{E_{22}}\}.$$

4. Conclusion

The current study is on the applications of low-dimensional outer derivation spaces which can be as elements of the 1- cohomology group. For 2-dimensional associative algebras, this space has dimensions ranging from zero to two.

References

[1] J.-L. Loday, A. Frabetti, F. Chapoton, and F. Goichot, Dialgebras and Related Operads, Lecture Notes in Mathematics, vol. 1763. Berlin, Germany: Springer (2001).

[2] A. Frabetti, Dialgebra (co)homology with coefficients, Preprint No. ULP-IRMA-1999-39, SCAN-0004048 (1999).

[3] N. F. Mohammed, S. G. Gasim, and A. S. Mohammed, "On cohomology groups of four-dimensional nilpotent associative algebras," *Baghdad Science Journal*, vol. 19, no. 2, p. 0329 (2022).

[4] H. Sadeq and T. Majeed, "On tensor product of representation for Lie algebra," in AIP Conference Proceedings, vol. 2834, no. 1. Melville, NY, USA: AIP Publishing (Dec. 2023).

[5] P. D. Beites, A. F. Ouaridi, and I. Kaygorodov, "The algebraic and geometric classification of transposed Poisson algebras," Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, vol. 117, no. 2, p. 55 (2023).

[6] H. Abdelwahab, A. J. Calderón, and I. Kaygorodov, "The algebraic and geometric classification of nilpotent binary Lie algebras," *International Journal of Algebra and Computation*, vol. 29, no. 6, pp. 1113–1129 (2019).

[7] N. F. Mohammed, I. S. Rakhimov, and S. K. S. Husain, "Contractions of low dimensional complex associative algebras," in AIP Conference Proceedings, vol. 1795, no. 1. Melville, NY, USA: AIP Publishing (Jan. 2017).

[8] M. Nesterenko, "Contractions of representations and realizations of Lie algebras," *Algebraic and Geometric Methods of Analysis*, vol. 73 (2023).

[9] H. Sadeq and T. Majeed, "Some results of representation for Lie algebras," in AIP Conference Proceedings, vol. 2834, no. 1. Melville, NY, USA: AIP Publishing (2023).

[10] S. Tôgô, "Outer derivations of Lie algebras," *Transactions of the American Mathematical Society*, vol. 128, no. 2, pp. 264–276 (1967).

[11] I. M. Rikhsiboev, I. S. Rakhimov, and W. Basri, "Diassociative algebras and their derivations," in Journal of Physics: Conference Series, vol. 553, no. 1, p. 012006. Bristol, UK: IOP Publishing (Nov. 2014).

[12] M. Abubakar, I. Rakhimov, and I. Rikhsiboev, "On derivations of low dimensional associative dialgebras," in Proceedings of the 3rd International Conference on Mathematical Sciences, vol. 1602, pp. 730–735 (Jun. 2014).

Received February, 2025